MAT351 Partial Differential Equations
Lecture 16

November 18, 2020

Last Lecture

The wave equation on $[0, l]$

$$\begin{array}{rcll} u_{t,t} & = & c^2 u_{x,x} & \text{on } (0,l)\times\mathbb{R} \\ u(x,0) & = & \phi(x) & \text{on } [0,l] \\ u_t(x,0) & = & \psi(x) & \text{on } [0,l] \end{array} \tag{1}$$

and with **Dirichlet boundary conditions (DC)**

$$u(0,t) = u(l,t) = 0 \quad \forall t \in \mathbb{R}$$

or with **Neumann boundary conditions (NC)**

$$u_x(0,t) = u_x(l,t) = 0 \quad \forall t \in \mathbb{R}.$$

Via separation of variable we found a family of special solutions.

For (1) with DC we found special solutions of the form $u_n(x,t) = T_n(t)X_n(t)$, $n \in \mathbb{N}$, where

$$T_n(t) = A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)$$

and the functions $X_n(x) := \sin\left(\frac{n\pi}{l}x\right)$ solve the following ODE boundary problem

$$X_n'' + \left(\frac{n\pi}{l}\right)^2 X_n = 0 \quad \text{with } X_n(0) = X_n(l) = 0, \quad n \in \mathbb{N}.$$

Superposition principle

Any finite linear combination of u_n is also a solution of (1) with DC:

$$u(x,t) := \sum_{i=1}^{k} \left(A_{n_i} \cos\left(\frac{n_i \pi}{l} ct\right) + B_{n_i} \sin\left(\frac{n_i \pi}{l} ct\right)\right) \sin\left(\frac{n_i \pi}{l} x\right) \quad \text{where } n_1, \dots, n_k \in \mathbb{N}.$$

u has initial conditions

$$\phi(x) = u(x,0) = \sum_{i=1}^{k} A_{n_i} \sin\left(\frac{n_i \pi}{l} x\right), \quad \psi(x) = u_t(x,0) = \sum_{i=1}^{k} \frac{n\pi c}{l} B_{n_i} \sin\left(\frac{n_i \pi}{l} x\right).$$

For (1) with NC we found that

$$\tilde{u}_n(x,t) = T_n(t)\tilde{X}_n(x) \ \forall n \in \mathbb{N} \cup \{0\}.$$

is a solution.

The functions $\tilde{X}_n(x) := \cos\left(\frac{n\pi}{l} x\right)$, $n \in \mathbb{N}$, solve the following ODE boundary problem

$$X_n'' + \left(\frac{n\pi}{l}\right)^2 X_n = 0 \quad \text{with } (X_n)_x(0) = (X_n)_x(l) = 0, \quad n \in \mathbb{N} \cup \{0\}.$$

where we set $X_0(x) = 1$.

Again we have that $T_n(t)$ solves $T'' + \left(\frac{n\pi}{l}\right)^2 cT = 0$. Therefore

$$T_n(t) = A_n \cos\left(\frac{n\pi}{l} ct\right) + B_n \sin\left(\frac{n\pi}{l} ct\right) \text{ for } n \in \mathbb{N} \text{ and for } A_n, \ B_n \in \mathbb{R}.$$

But also $T_0(t) = \frac{1}{2} A_0 + \frac{1}{2} B_0 t$ for $A_0, B_0 \in \mathbb{R}$.

Eigenvalues and Eigenfunction

The constants $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ are called eigenvalues.
The functions $X_n(x)$ are called eigenfunctions of the the differential operator

$$L : V \to C^0([0,l]), \quad L\phi = -\frac{\partial^2}{\partial x^2}\phi \quad \text{for } V = \left\{\phi \in C^2([0,l]) : \phi(0) = \phi(l) = 0\right\}.$$

The differential equality that determines X_n has the form of an **eigenvalue equation**

$$LX_n = \lambda_n X_n.$$

Similar, the functions $\tilde{X}_n(x)$ are called eigenfunctions for the differential operator

$$\tilde{L} : \tilde{V} \to C^0([0,l]), \quad \tilde{L}\phi = -\frac{\partial^2}{\partial x^2}\phi \quad \text{for } \tilde{V} = \left\{\phi \in C^2([0,l]) : \phi_x(0) = \phi_x(l) = 0\right\}.$$

The terminology is motivated from Linear Algebra:
Consider a matrix $A \in \mathbb{R}^{n\times n}$ we say $\lambda \in \mathbb{R}$ is an eigenvalue of A if there exists $v \neq 0$ such that

$$Av = \lambda v$$

Given an eigenvalue λ for A the set of eigenvectors E_λ is a vector space.
If we can find n different eigenvalue $\lambda_1, \ldots, \lambda_n$ then

$$E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n} = \mathbb{R}^n$$

Hence, for every vector w there are unique eigenvectors $v_i \in E_{\lambda_i}$ such that

$$W = v_1 + \cdots + v_n.$$

Let us go back to (1) with a DC. Consider an infinite serie of the form

$$u(x,t) := \lim_{N\to\infty} \sum_{n=1}^{N} u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right) \sin(\lambda_n x)\,. \tag{2}$$

When does such a serie converges uniformily?

Since

$$\left|\sum_{n=1}^{N} u_n(x,t)\right| \le \sum_{n=1}^{N} (|A_n| + |B_n|) \le \sum_{n=1}^{\infty} |A_n| + \sum_{n=1}^{\infty} |B_n|$$

the series (2) converges uniformily provided $\sum_{n=1}^{\infty} |A_n|, \sum_{n=1}^{\infty} |B_n| < \infty$. Indeed

$$\max_{x\in[0,l]} \left|\sum_{n=1}^{M} u_n(x,t) - \sum_{n=1}^{N} u_n(x,t)\right| = \max_{x\in[0,l]} \left|\sum_{n=N+1}^{M} u_n(x,t)\right| \le \sum_{n=N+1}^{M} (|A_n| + |B_n|) \to 0$$

if $N < M \to \infty$. Now also recall the following theorem about differentiation of series

Theorem

Let $f_n(x)$ a sequence of functions on $[0,l]$ that are differentiable. Assume $\sum_{n=1}^{\infty} f_n(x)$ is converging uniformily.

If $\sum_{n=1}^{\infty} f_n'(x)$ is uniformily convergent then it follows that f is differentiable on $[0,l]$ and

$$f'(x) = \sum_{n=1}^{\infty} f'(x).$$

Hence, the partial derivatives u_x and u_t exist and satisfy

$$u_x(x,t) = \sum_{n=1}^{\infty} \lambda_n \left(A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right) \cos(\lambda_n x)$$
$$u_t(x,t) = \sum_{n=1}^{\infty} \lambda_n c \left(-A_n \sin(\lambda_n ct) + B_n \cos(\lambda_n ct)\right) \sin(\lambda_n x)$$

provided $\sum_{n=1}^{\infty} \lambda_n |A_n|, \sum_{n=1}^{\infty} \lambda_n |B_n| < \infty$.

Similar, the second partial derivatives $u_{x,x}$ and $u_{t,t}$ exist and satisfy

$$u_{x,x}(x,t) = \sum_{n=1}^{\infty} \lambda_n^2 \left(A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right) \cos(\lambda_n x)$$
$$u_{t,t}(x,t) = \sum_{n=1}^{\infty} \lambda_n^2 c^2 \left(-A_n \sin(\lambda_n ct) + B_n \cos(\lambda_n ct)\right) \sin(\lambda_n x)$$

provided $\sum_{n=1}^{\infty} \lambda_n^2 |A_n|, \sum_{n=1}^{\infty} \lambda_n^2 |B_n| < \infty$.

Consequently, since each function

$$u_n = (A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)) \sin(\lambda_n x)$$

satisfies the PDE $(u_n)_{t,t} = c^2 (u_n)_{x,x}$ with DC the series u satisfies the same PDE also with Dirichlet boundary condition $u(0,t) = u(l,t) = 0$.

In the same way we can construct solutions to the PDE with NC. We only have to replace

$$X_n, n \in \mathbb{N} \quad \text{with } \tilde{X}_n, n \in \mathbb{N} \cup \{0\}.$$

Moreover u satisfies the following initial condition

$$u(x,0) = \lim_{N\to\infty} \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi}{l}x\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right) =: \phi(x) \tag{3}$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin\left(\frac{n\pi}{l}x\right) =: \psi(x) \tag{4}$$

Note that these series are converge uniformily and hence are well-defined because we assumed

$$\sum_{n=1}^{\infty} |A_n|,\ \sum_{n=1}^{\infty} |B_n|,\ \sum_{n=1}^{\infty} \lambda_n |A_n|,\ \sum_{n=1}^{\infty} \lambda_n |B_n|,\ \sum_{n=1}^{\infty} \lambda_n^2 |A_n|,\ \sum_{n=1}^{\infty} \lambda_n^2 |B_n| < \infty.$$

Question

What kind of data pairs ϕ, ψ can be expanded as series for coefficients A_n and B_n as above?

In the same way we can find solutions for the PDE with NC:

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} (A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)) \cos(\lambda_n x).$$

The initial conditions are $\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda_n x)$ and $\frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \lambda_n c \cos(\lambda_n x)$.

Let us consider the analogous problem for diffusion on $[0, l]$:

$$\begin{array}{rcll} u_t & = & k u_{x,x} & \text{on } (0,l) \times (0,\infty) \\ u(x,0) & = & \phi(x) & \text{on } (0,l) \end{array}$$

with Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0 \quad \forall t \in \mathbb{R}$$

or with **Neumann boundary conditions**

$$u_x(0,t) = u_x(l,t) = 0 \quad \forall t \in \mathbb{R}.$$

We can again apply the methode of Separation-of-Variables: We consider a solution of the form

$$u(x,t) = T(t)X(x).$$

This leads to

$$-\frac{T'}{kT} = -\frac{X''}{X} = \lambda.$$

Again we see easily that λ must be constant and T and X solve

$$T' + \lambda k T = 0 \quad \& \quad X'' + \lambda X = 0.$$

The general solution for X is the same as before. In particular, we can have $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ and the set of solutions $X_n(x) = \sin\left(\frac{n\pi}{l}x\right)$. The general solution for T in this case is

$$T(t) = Ae^{-\left(\frac{n\pi}{l}\right)^2 kt} \text{ for } A \in \mathbb{R}.$$

Hence, as before a family of special solutions of the diffuison equation with DC is given by

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \underbrace{\sin\left(\frac{n\pi}{l}x\right)}_{=:X_n(x)}, \ n \in \mathbb{N} \text{ and } A_n \in \mathbb{R}.$$

where X_n are as before. Then

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 kt} \sin(\lambda_n x)$$

solves the diffusion equation with DC and inital data $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)$ provided

$$\sum_{n=1}^{\infty} |A_n|, \ \ \sum_{n=1}^{\infty} \lambda_n |A_n|, \ \ \sum_{n=1}^{\infty} \lambda_n^2 |A_n| < \infty.$$

For NC we consider $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ with $n \in \mathbb{N} \cup \{0\}$ and the corresponding solutions $\tilde{X}_n$ of $\tilde{X}_n'' + \lambda X_n = 0$. Precisely, we set $\tilde{X}_n(x) = \cos(\lambda_n x)$ and $X_0(x) = 1$.

Again we also have to consider $\lambda_0 = 0$. In particular, for T we also consider the solutions of

$$T' = 0 \;\Leftrightarrow\; T_0(0) = \frac{1}{2} A_0 \in \mathbb{R}$$

Then, a family of special solutions of the diffusion equation with NC is given by

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \underbrace{\cos\left(\frac{n\pi}{l} x\right)}_{=:\tilde{X}_n(x)}, \; n \in \mathbb{N} \text{ and } A_n \in \mathbb{R}.$$

andn $u_0(x,t) = \frac{1}{2} A_0$.Again

$$u(x,t) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 kt} \cos(\lambda_n x)$$

is a solution to the diffusion equation with NC for the initial data

$$u(x,0) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda x).$$

provided

$$\sum_{n=1}^{\infty} |A_n|, \quad \sum_{n=1}^{\infty} \lambda_n |A_n|, \quad \sum_{n=1}^{\infty} \lambda_n^2 |A_n| < \infty.$$

Fourier Series

We encouter the following question

Question

Given a function ϕ on $[0, l]$ can we find a sequence $(A_n)_{n\in\mathbb{N}}$ such that

$$\phi(x) = \lim_{N\to\infty} \sum_{n=1}^{N} A_n \sin(\lambda_n x) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x),\ \ x \in [0, l] \text{ ?}$$

(where $\lambda_n = \left(\frac{n\pi}{l}\right)^2$). We call the series on the right hand side the Fourier sine series.
Or can we find a sequence $(B_n)_{n\in\mathbb{N}}$ such that

$$\phi(x) = \frac{1}{2}A_0 + \lim_{N\to\infty} \sum_{n=1}^{N} A_n \cos(\lambda_n x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda_n x), x \in [0, l] \text{ ?}$$

We call the series on the right hand side the Fourier cosine series.

How can we determine the coefficients A_n?

We perform the following forma calculations:

$$\int_0^l \phi(x)\sin(\lambda_m x)dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)\sin(\lambda_m x)dx = \sum_{n=1}^{\infty}\int_0^l A_n \sin(\lambda_n x)\sin(\lambda_m x)dx$$

Let consider a single term in the sum on the right hand side:

$$A_n \int_0^l \sin(\lambda_n x)\sin(\lambda_m x)dx = A_n \int_0^l \frac{1}{2}\left(\cos((\lambda_n - \lambda_m)x) - \cos((\lambda_n + \lambda_m)x)\right)dx$$

Here we used the first of the following identities

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y), \quad \sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y).$$

The first identity gives

$$\begin{aligned}\cos(x+y) - \cos(x-y) &= \cos(x)\cos(y) - \sin(x)\sin(y) - \cos(x)\cos(-y) + \sin(x)\sin(-y)\\ &= -2\sin(x)\sin(y)\end{aligned}$$

Then, if $n \neq m$, we compute

$$\begin{aligned}A_n \int_0^l \sin(\lambda_n x)\sin(\lambda_m x)dx &= A_m \frac{l}{2\pi}\int_0^l \frac{\pi}{l}\cos\left(\frac{(n-m)\pi}{l}x\right) - \cos\left(\frac{(n+m)\pi}{l}\right)dx\\ &= \frac{l}{2\pi}A_n\int_0^{\pi}\left(cos((n-m)x) - \cos((n+m)x)\right)dx\\ &= \frac{l}{2\pi}A_n\left[\frac{1}{n-m}\sin((n-m)x) - \frac{1}{n+m}\sin((n+m)\right]_0^{\pi} = 0\end{aligned}$$

If $n = m$, then

$$A_m \int_0^l \sin(\lambda_m x)^2 dx = A_m \frac{l}{2\pi} \int_0^\pi [1 - \cos(2mx)]\, dx = A_m \frac{l}{2} - A_m \frac{l}{2\pi} \left[\frac{1}{2m} \sin(2mx) \right]_0^\pi = \frac{l}{2} A_n$$

Hence

$$\begin{aligned} \int_0^l \phi(x) \sin(\lambda_m x) dx &= \int_0^l \sum_{n=1}^\infty A_n \sin(\lambda_n x) \sin(\lambda_m x) dx \\ &= \sum_{n=1}^\infty A_n \int_0^l \sin(\lambda_n x) \sin(\lambda_m x) dx = \frac{l}{2} A_m. \end{aligned}$$

$$A_m = \frac{2}{l} \int_0^l \phi(x) \sin(\lambda_m x) dx = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{m\pi}{l} x \right) dx.$$

This is the Fourier sine coefficient for ϕ.

By the same formal calculation we also compute the Fourier cosine coefficient for ϕ. Precisely:

$$\int_0^l \phi(x)\cos(\lambda_m x)dx = \int_0^l \sum_{n=1}^{\infty} \tilde{A}_n \cos(\lambda_n x)\cos(\lambda_m x)dx$$
$$= \sum_{n=1}^{\infty} \tilde{A}_n \int_0^l \cos(\lambda_n x)\cos(\lambda_m x)dx$$

Let us again consider a single term in the sum on the right hand side with $n, m \geq 1$:

$$\tilde{A}_n \int_0^l \cos(\lambda_n x)\cos(\lambda_m x)dx = \tilde{A}_n \frac{1}{2}\int_0^l \cos\left(\frac{(n+m)\pi}{l}x\right) + \cos\left(\frac{(n-m)\pi}{l}x\right)dx$$
$$= \tilde{A}_n \frac{l}{2\pi}\int_0^{\pi} \cos((n+m)x) + \cos((n-m)x)dx$$

For $n \neq m$ the right hand side in the last term is

$$= \tilde{A}_n \frac{l}{2\pi}\left[\frac{1}{n+m}\sin((n+m)x) + \frac{1}{n-m}\sin((n-m)x)\right]_0^{\pi} = 0.$$

For $n = m$ we obtain

$$= \frac{l}{\pi 2}\int_0^{\pi} [\cos(2mx) + 1]\, dx = \frac{l}{2\pi}\tilde{A}_m \left[\frac{1}{2n}\sin(2mx)\right]_0^{\pi} + \frac{l}{2}\tilde{A}_m = \frac{l}{2}\tilde{A}_m.$$

A computation yields the same conclusion even when $n = 0$ or $m = 0$.

We obtain that

$$\tilde{A}_0 = \frac{2}{l}\int_0^l \phi(x)dx, \quad \tilde{A}_m = \frac{2}{l}\int_0^l \phi(x)\cos(\lambda_m x)dx = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{m\pi}{l}x\right)dx.$$

Definition

The Fourier sine series of ϕ is defined

$$\sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi}{l}x\right)dx\right]\sin\left(\frac{n\pi}{l}x\right) =: \mathcal{S}(\phi)$$

Similar the Fourier cosine series of ϕ is defined

$$\sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi}{l}x\right)dx\right]\cos\left(\frac{n\pi}{l}x\right) =: \mathcal{C}(\phi)$$

Finally the **full** Fourier coefficients are (we abuse notation at this point)

$$B_m = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\left(\frac{m\pi}{l}x\right)dx.$$

$$A_0 = \frac{1}{l}\int_{-l}^{l}\phi(x)dx, \quad A_m = \frac{1}{l}\int_{-l}^{l}\phi(x)\cos\left(\frac{m\pi}{l}x\right)dx.$$

Definition

The Fourier series of ϕ is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n \sin(\frac{n\pi}{l}x)dx + B_n\cos(\frac{n\pi}{l}x)\right] = \mathcal{F}(\phi)$$