MAT351 Partial Differential Equations
Lecture 17

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Orthogonality and General Fourier Series

Consider two **continuous functions** $f, g : [a, b] \to \mathbb{R}$ that are **square integrable:**

$$\|f\|_2^2 = \int_a^b |f(x)|^2 dx, \quad \|g\|_2^2 = \int_a^b |g(x)|^2 dx < \infty$$

We define the **inner product** between f and g as the integral of their product:

$$(f, g) = \int_a^b f(x)g(x)dx \tag{1}$$

The product $g(x)f(x)$ is integrable because of the *Cauchy-Schwartz inequality*:

$$\int_a^b |f(x)g(x)|dx \le \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx} = \|f\|_2 \|g\|_2 < \infty.$$

We say that two square integrable functions f and g are **orthogonal** if $(f, g) = 0$.

Note that a real valued continuous function f is never orthogonal to itself unless $f = 0$.

Recall the case of an inner product (v, w) on $\mathbb{R}^n$, for instance $v_1 w_1 + \cdots + v_n w_n$.
The number $\|v\| = \sqrt{(v, v)}$.

A basis $v_1, \ldots, v_n$ of V is orthonormal if $\|v_i\| = 1$, $i = 1, \ldots, n$, and $(v_i, v_j) = 0$, $i \neq j$. Then

$$w = \sum_{i=1}^{n} (v_i, w) v_i \text{ and } \|w\|^2 = \sum_{i=1}^{n} |(v_i, w)|^2.$$

For instance, $v_1, \ldots, v_n$ can be the eigenvectors of a symmetric operator $A : \mathbb{R}^n \to \mathbb{R}^n$.

The theory of Fourier series translates this idea to an infinite dimensional context.

Let $[a, b] = [0, l]$. Let us go back to the operator

$$Lf = -\frac{\partial^2}{\partial x^2} f \quad \text{for} \quad f \in C^2([0, l]).$$

We saw that

$$\sin\left(\frac{n\pi}{l} x\right), \ n \in \mathbb{N}$$

was a set of eigenfunctions for the operator L with Dirichlet boundary conditions, and

$$1, \quad \cos\left(\frac{n\pi}{l} x\right), \ n \in \mathbb{N}$$

was a set of eigenfunctions for the same operator with Neumann boundary conditions.

To determine Fourier sine coefficients we computed that

$$\int_0^l \sin\left(\frac{n\pi}{l} x\right) \sin\left(\frac{m\pi}{l} x\right) dx = 0 \quad \text{for} \quad n \neq m \in \mathbb{N}.$$

Also we can compute that

$$\int_0^l \cos\left(\frac{n\pi}{l} x\right) \cdot 1 dx = \int_0^l \cos\left(\frac{n\pi}{l} x\right) \cos\left(\frac{m\pi}{l} x\right) dx = 0 \quad \text{for} \quad n \neq m \in \mathbb{N}.$$

Hence, these eigenvectors are orthogonal w.r.t. $(\cdot, \cdot)$.

General Fourier series

Let us consider two eigenfunctions X_1 and X_2 of $L = -\frac{d^2}{dx^2}$ on $[a, b]$ for eigenvalues $\lambda_1 \neq \lambda_2$.
We don't specify boundary conditions yet. We can compute the following

$$(-X_1'X_2 + X_1X_2')' = -X_1''X_2 + X_1X_2''$$

Integration over $[a, b]$ yields

$$\int_a^b \left[-X_1''(x)X_2(x) + X_1(x)X_2''(x)\right] dx = -X_1'(x)X_2(x) + X_1(x)X_2'(x)\Big|_a^b$$
$$= -X_1'(b)X_2(b) + X_1(b)X_2'(b) + X_1'(a)X_2(a) - X_1(a)X_2'(a).$$

If the right hand side is 0 we have that

$$0 = -\int_a^b X_1''(x)X_2(x)dx - \int_a^b X_1(x)X_2''(x)dx = (LX_1, X_2) - (X_1, LX_2) = (\lambda_1 - \lambda_2)(X_1, X_2)$$

Since $\lambda_1 \neq \lambda_2$, $(X_1, X_2) = 0$. Hence X_1 and X_2 are othogonal.

Question: When do we have

$$-X_1'(b)X_2(b) + X_1(b)X_2'(b) + X_1'(a)X_2(a) - X_1(a)X_2'(a) = 0 \ ?$$

For instance, for Dirichlet or Neumann boundary conditions on $[0, l] = [a, b]$.

But also for periodic boundary conditions: $f \in C^1(\mathbb{R})$ satisfies a periodic boundary conditions with period $l > 0$ if $f(x + nl) = f(x)$ for all $x \in \mathbb{R}$. Hence

$$f(a) = f(b) \ \& \ f'(a) = f'(b).$$

In general, we could consider boundary conditions of the form

$$\begin{Bmatrix} \alpha_1 f(a) + \beta_1 f(b) + \gamma_1 f'(a) + \delta_1 f'(b) = 0 \\ \alpha_2 f(a) + \beta_2 f(b) + \gamma_2 f'(a) + \delta_2 f'(b) = 0 \end{Bmatrix} \tag{2}$$

for 8 independent constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$.

Definition

The set of boundary conditions (2) are called symmetric if

$$f'(x)g(x) - f(x)g'(x)\Big|_a^b = f'(b)g(b) - f(b)g'(b) - f'(a)g(a) + f(a)g'(a) = 0$$

for any pair of functions that satisfy (2).

Hence, we proved the following theorem.

Theorem

Eigenfunctions of $-\frac{\partial^2}{\partial x^2}$ with symmetric boundary conditions for eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal.

By explicite compuations we saw that this is true for L with Dirichlet boundary conditions on $[0, l]$ where the eigenfunctions are $\sin\left(\frac{n\pi}{l}x\right), \quad n \in \mathbb{N}$.

If there are 2 eigenfunctions X_1 and X_2 for the same eigenvalue λ, then either $X_1 = cX_2$ for some constant c, or they can be made orthogonal by the Gram-Schmidt orthogonalization procedure.

Considering $L = -\frac{\partial^2}{\partial x^2}$ with periodic boundary conditions on $[-l, l]$. There are eigenfunctions

$$\sin\left(\frac{n\pi}{l}x\right), \quad \cos\left(\frac{n\pi}{l}x\right)$$

for the same eigenvalue $\left(\frac{n\pi}{l}\right)^2$ that are orthogonal.

But also any linear combination is again an eigenfunction for the same eigenvalue. In particular

$$\sin\left(\frac{n\pi}{l}x\right), \quad \cos\left(\frac{n\pi}{l}x\right) + \sin\left(\frac{n\pi}{l}x\right).$$

But they are not orthogonal.

General Fourier coefficients

If a continuous and integrable function ϕ is given by an infinite converging serie $\sum_{n=1}^{\infty} A_n X_n$ for eigenfunctions X_n of $L = -\frac{\partial^2}{\partial x^2}$ on $[a, b]$ with symmetric boundary conditions, then the coefficients are determined by the formula

$$A_m = \frac{1}{\|X_m\|_2^2}(X_m, \phi) = \frac{1}{\int_a^b (X_m)^2(x)dx} \int_a^b \phi(x)X_m(x)dx.$$

Indeed

$$(\phi, X_m) = \left(\sum_{n=1}^{\infty} A_n X_n, X_m\right) = \sum_{n=1}^{\infty} A_n(X_n, X_m) = A_m(X_m, X_m) = A_m \|X_m\|_2^2 .$$

For instance, if we consider the set $\sin\left(\frac{n\pi}{l}x\right)$ of eigenfunctions $L = -\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions, we computed

$$A_m = \frac{1}{\int_0^l \left(\sin\left(\frac{m\pi}{l}x\right)\right)^2 dx} \int_0^l \phi(x) \sin\left(\frac{m\pi}{l}x\right) dx \text{ where } \int_0^l \left(\sin\left(\frac{m\pi}{l}x\right)\right)^2 dx = \frac{l}{2}.$$

For periodic boundary conditions on $[-l, l]$ the eigenfunctions are $1, \cos\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right)$ and the Fourier coefficients are

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x) \sin\left(\frac{n\pi}{l}x\right),\ n \in \mathbb{N},\ \tilde{A}_0 = \frac{1}{l}\int \phi(x)dx,\ \tilde{A}_n = \frac{1}{l}\int_{-l}^{l} \phi(x) \cos\left(\frac{n\pi}{l}x\right) dx,\ n \in \mathbb{N}.$$

Problem/Questions: In which sense does $\sum_{n=1}^{\infty} A_n X_n$ converge? And why does the second equality hold in the previous equation?

Notions of convergence

Definition (Pointwise and uniform convergence)

We say an infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to a function f in (a, b) if

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as} \quad N \to \infty \text{ for all } x \in (a, b).$$

We say the series converges uniformly to f in $[a, b]$ if

$$\max_{x \in [a,b]} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as} \quad N \to \infty$$

Note that for the notion of uniform convergence we include a and b.

Definition (Mean square convergence)

The serie $\sum_{n=1}^{\infty} f_n(x)$ converges in mean square (or L^2) sense to f in (a, b) if

$$\int_a^b \left| f(x) - \sum_{n=1}^{N} f_n(x) \right|^2 dx \to 0 \quad \text{as} \quad N \to \infty.$$

Remark

We have: uniform convergence $\Rightarrow$ pointwise and mean square convergence.

But in general not the other way.

Example

Consider $f_n(x) = (1-x)x^{n-1}$ on $[0,1]$. Then

$$\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} \left(x^{n-1} - x^n\right) = 1 - x^N \to 1 \text{ as } N \to \infty \quad \text{for all } x \in [0,1].$$

But the convergence is not uniform because

$$\max_{x\in[0,1]} \left|1 - (1 - x^N)\right| = 1 \quad \text{for all} \quad N \in \mathbb{N}.$$

On the other hand, we still have mean square convergence because

$$\int_0^1 \left|1 - (1-x^N)\right| dx = \int_0^1 x^{2N} dx = \frac{1}{2N+1} \to 0 \text{ as } N \to \infty.$$