MAT351 Partial Differential Equations
Lecture 18

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Notions of convergence

Definition (Pointwise and uniform convergence)

We say an infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to a function f in (a, b) if

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as} \quad N \to \infty \text{ for all } x \in (a, b).$$

We say the series converges uniformly to f in $[a, b]$ if

$$\max_{x \in [a,b]} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as} \quad N \to \infty$$

Note that for the notion of uniform convergence we include a and b.

Definition (Mean square convergence)

The serie $\sum_{n=1}^{\infty} f_n(x)$ converges in mean square (or L^2) sense to f in (a, b) if

$$\int_a^b \left| f(x) - \sum_{n=1}^{N} f_n(x) \right|^2 dx \to 0 \quad \text{as} \quad N \to \infty.$$

Remark

We have: uniform convergence $\Rightarrow$ pointwise and mean square convergence.

But in general not the other way.

Example

Consider $f_n(x) = (1-x)x^{n-1}$ on $[0,1]$. Then

$$\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} \left(x^{n-1} - x^n\right) = 1 - x^N \to 1 \text{ as } N \to \infty \quad \text{for all } x \in [0,1].$$

But the convergence is not uniform because

$$\max_{x\in[0,1]} \left|1 - (1 - x^N)\right| = 1 \quad \text{for all} \quad N \in \mathbb{N}.$$

On the other hand, we still have mean square convergence because

$$\int_0^1 \left|1 - (1 - x^N)\right| dx = \int_0^1 x^{2N} dx = \frac{1}{2N+1} \to 0 \text{ as } N \to \infty.$$

Consider

$$f_n(x) = \frac{n}{1+n^2x^2} - \frac{n-1}{1+(n-1)^2x^2} \quad \text{on } (0,l)$$

$$\sum_{n=1}^{N} f_n(x) = \frac{N}{1+N^2x^2} = \frac{1}{N\left[\frac{1}{N^2}+x^2\right]} \to 0 \quad \text{as } N\to\infty \text{ if } x>0.$$

So the series converges pointwise to 0.

On the other hand

$$\int_0^l \frac{N^2}{(1+N^2x^2)^2}\,dx = N\int_0^{Nl}\frac{1}{(1+y^2)^2}\,dy \to \infty \quad \text{where } y = Nx$$

because

$$\int_0^{Nl}\frac{1}{(1+y^2)^2}\,dy \to \int_0^{\infty}\frac{1}{(1+y^2)^2}\,dy$$

Hence the series does not converge in mean square sense to 0.

Also it does not converge uniformily because

$$\max_{x\in(0,l)} \frac{N}{1+N^2x^2} = N \to \infty$$

Recall we have an inner product

$$(f,g) = \int_a^b f(x)g(x)dx \text{ for } f,g \in C^0([a,b])$$

and a norm given by $\|f\|_2 = \sqrt{(f,f)}$. Convergence of $\sum_{n=1}^N f_n(x)$ to $f(x)$ in L^2 sense means that

$$\left\| f - \sum_{n=1}^N f_n \right\|_2^2 \to 0$$

that is convergence w.r.t. the norm $\|\cdot\|_2$.

Theorem (Least Square Approximation)

Let X_n, $n \in \mathbb{N}$, be a set of eigenfunctions for the operator $-\frac{\partial^2}{\partial x^2}$ on $[a,b]$ with symmetric boundary condition. In particular, we have

$$(X_n, X_m) = \int_a^b X_n(x)X_m(x)dx = 0$$

Let $f : [a,b] \to \mathbb{R}$ be continuous and hence $\|f\|_2 < \infty$ and let $N \in \mathbb{N}$ be fixed.
Among all possible choices of N constants $c_1, c_2, \ldots, c_N \in \mathbb{R}$ the choice that minimizes

$$E_N := E_N(c_1,\ldots,c_N) := \left\| f - \sum_{n=1}^N c_n X_n \right\|_2^2 = \int_a^b \left(f(x) - \sum_{n=1}^N c_n X_n(x) \right)^2 dx$$

is $c_1 = A_1, \ldots, c_N = A_N$ where $A_n = \frac{1}{\|X_n\|_2^2}(f, X_n)$.

Proof

We expand E_N:

$$E_N = \int_a^b \left(f(x) - \sum_{n=1}^{N} c_n X_n(x) \right)^2 dx$$

$$= \int_a^b |f(x)|^2 dx - 2 \sum_{n=1}^{N} c_n \int_a^b f(x) X_n(x) dx + \sum_{n,m=1}^{N} c_n c_m \int_a^b X_n(x) X_m(x) dx.$$

By orthogonality of the eigenfunctions the last term reduces to $\sum_{n=1}^{N} c_n^2 \int_a^b |X_n(x)|^2 dx$. Hence

$$0 \le E_N = \|f\|_2^2 - \sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|_2^2} + \sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|_2^2} - 2 \sum_{n=1}^{N} c_n (f, X_n) + \sum_{n=1}^{N} c_n^2 \|X_n\|_2^2$$

$$= \|f\|_2^2 - \sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|_2^2} + \sum_{n=1}^{N} \|X_n\|_2^2 \left(\frac{(f, X_n)^2}{\|X_n\|_2^4} - 2 c_n \frac{(f, X_n)}{\|X_n\|_2^2} + c_n^2 \right)$$

$$= \|f\|_2^2 - \sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|_2^2} + \sum_{n=1}^{N} \|X_n\|_2^2 \left(\frac{(f, X_n)}{\|X_n\|_2^2} - c_n \right)^2$$

The coefficients appear only in one place and we see that the right hand side is minimal if

$$c_n = \frac{1}{\|X_n\|_2^2} (f, X_n) = A_n.$$

□

Corollary (Bessel's Inequality)

$$\sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|_2^2} = \sum_{n=1}^{N} A_n^2 \|X_n\|^2 \le \|f\|_2^2 .$$

In particular, if $\|f\|_2^2 = \int_a^b |f(x)|^2 dx$ *is finite then the series*

$$\sum_{n=1}^{\infty} A_n^2 \|X_n\|^2 = \sum_{n=1}^{\infty} A_n \int_a^b |X_n(x)|^2 dx \;\; \textit{converges absolutely.}$$

By the theorem we have for any collection $c_1, \dots, c_N \in \mathbb{R}$:

$$\left\| f - \sum_{n=1}^{N} A_n X_n \right\|_2 = E_N(A_1, \dots, A_N) \le E_N(c_1, \dots, c_N)$$

If we can find a sequence of finite linear combinations

$$g_i = \sum_{n=1}^{N_i} c_n^i X_n \text{ with } N_i \to \infty \text{ for } i \to \infty$$

such that $g_i \to f$ in L^2 sense, that is $\|g_i - f\| = E_N(c_1^i, \dots, c_{N_i}^i) \to 0$, then

$$\sum_{n=1}^{N} A_n X_n \to f \text{ in } L^2 \text{ sense}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(f, X_n)^2}{\|X_n\|_2^2} = \sum_{n=1}^{\infty} A_n^2 \|X_n\|_2^2 = \|f\|_2^2 .$$

We say eigenfunctions X_n, $n \in \mathbb{N}$, are complete if this holds for every function $f \in C^0([a, b])$.

Pointwise convergence

We will prove pointwise convergence of the full Fourier series on $[-l,l]=[-\pi,\pi]$.

That is we consider the set of eigenfunctions $\sin(nx),\ 1,\ \cos(nx)$ with periodic boundary condition on $[-\pi,\pi]$, that is the functions are periodic with period 2π: $X_n(x)=X_n(x+2\pi)$ for all $x\in\mathbb{R}$.

Given $\phi\in C^0(\mathbb{R})$ that is periodic with period 2π, its full Fourier serie is

$$\frac{1}{2}\tilde{A}_0+\sum_{n=1}^{\infty}\left(\tilde{A}_n\cos(nx)+A_n\sin(nx)\right),\ x\in[-\pi,\pi]$$

with Fourier coefficients

$$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x)\sin(nx)dx,\ \tilde{A}_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x)dx,\ \tilde{A}_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x)\cos(nx)dx,\ n\in\mathbb{N}.$$

We denote

$$S_N(x)=\frac{1}{2}\tilde{A}_0+\sum_{n=1}^{N}\left(\tilde{A}_n\cos(nx)+A_n\sin(nx)\right),\ x\in[-\pi,\pi],\ N\in\mathbb{N}$$

the Nth partial sum. We can rewrite this as

$$S_N(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1+2\sum_{n=1}^{N}\left(\cos(ny)\cos(nx)+\sin(ny)\sin(nx)\right)\right]\phi(y)dy$$

This simplifies as

$$S_N(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\left[1+2\sum_{n=1}^{N}\cos(n(x-y))\right]}_{=:K_N(x-y)}\phi(y)dy$$

Lemma

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)d\theta = 1 \quad \textit{and} \quad K_N(\theta) = 1 + 2\sum_{n=1}^{N}\cos(n\theta) = \frac{\sin\left[(N+\frac{1}{2})\theta\right]}{\sin\left(\frac{1}{2}\theta\right)}.$$

Proof of the Lemma.

$$\int_{-\pi}^{\pi} K_N(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1d\theta + \sum_{n=1}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(n\theta)d\theta = 1.$$

This proves the first claim.

$$1 + 2\sum_{n=1}^{N}\cos(n\theta) = 1 + \sum_{n=1}^{N}\left(e^{in\theta} + e^{-in\theta}\right) = \sum_{n=-N}^{N} e^{iN\theta}.$$

Now consider for some $x \in \mathbb{C}$

$$\left(x^{-N} + x^{-(N-1)} + \cdots + 1 + \cdots + x^{N-1} + x^{N}\right)(1-x) = x^{-N} + \ldots x^{N} - \left(x^{-(N-1)} + \ldots x^{N+1}\right).$$

Hence

$$x^{-N} + \cdots + x^{N} = \frac{x^{-N} - x^{N+1}}{1-x} = \frac{x^{-N-\frac{1}{2}} - x^{N+\frac{1}{2}}}{x^{\frac{1}{2}} - x^{\frac{1}{2}}}.$$

If we set $x = e^{i}$, it follows $K_N(\theta) = \dfrac{e^{i(N+\frac{1}{2})\theta} - e^{-i(N+\frac{1}{2})\theta}}{e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}} = \dfrac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}$. □

Theorem

If $\phi \in C^0(\mathbb{R})$ with periodic boundary condition with period 2π, that is $\phi(x+2\pi) = \phi(x)$ for all $x \in \mathbb{R}$ and if ϕ is differentiable (not necessarily $\phi \in C^1(\mathbb{R})$) then

$$\frac{1}{2}\tilde{A}_0 + \sum_{n=1}^{\infty} \Big(A_n \sin(nx) + \tilde{A}_n \cos(nx) \Big) = \phi(x) \text{ for all } x \in \mathbb{R}.$$

Proof of pointwise convergence.

We want to show that $S_N(x) \to \phi(x)$ for all $x \in \mathbb{R}$. We write

$$S_N(x) - \phi(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(y-x)\,(\phi(y) - \phi(x))\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin((N+\frac{1}{2})(y-x))\, \frac{\phi(y)-\phi(x)}{\sin(\frac{1}{2}(y-x))}\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\sin((N+\frac{1}{2})\theta)}_{=Y_n(\theta)}\, \underbrace{\frac{\phi(x+\theta)-\phi(x)}{\sin(\frac{1}{2}\theta)}}_{=g(\theta)}\, d\theta$$

The functions Y_n, $n \in \mathbb{N}$, are eigenfunction for $-\frac{\partial^2}{\partial x^2}$ on $[0, \pi]$ with mixed boundary conditions $Y_n(0) = 0$ and $\frac{d}{d\theta} Y_n(\pi) = 0$.

Mixed boundary conditions are symmetric. Hence, Y_n are orthogonal w.r.t. $(\cdot,\cdot)$ on $[0,\pi]$:

$$\int_0^\pi Y_n(\theta)Y_m(\theta)d\theta = 0, \quad \int_0^\pi (Y_n(\theta))^2 d\theta = \frac{\pi}{2}.$$

Since $Y(-\theta) = -Y(\theta)$, they are also orthogonal on $[-\pi,\pi]$:

$$\int_{-\pi}^\pi Y_n(\theta)Y_m(\theta)d\theta = 0, \quad \int_{-\pi}^\pi (Y_n(\theta))^2 d\theta = \pi.$$

Therefore

$$S_N(x) - \phi(x) = \frac{1}{2\pi}\int_{-\pi}^\pi Y_n(\theta)g(\theta)d\theta = \frac{1}{2}C_n \text{ for } g(\theta) = \frac{\phi(x+\theta)-\phi(x)}{\sin(\frac{1}{2}\theta)},$$

C_n in fact the Fourier coefficient of g w.r.t. the set of orthogonal eigenfunctions Y_n on $[-\pi,\pi]$. If we can show that $\int_{-\pi}^\pi |g(\theta)|^2 d\theta = \|g\|_2^2 < \infty$, then by the Bessel inequality the serie

$$0 \le \sum_{n=1}^\infty C_n^2 \underbrace{\|Y_n\|}_{=\pi} \le \|g\|_2^2 < \infty$$

converges and hence $C_n \to 0$. The claim is true, if g is continuous on $[-\pi,\pi]$. For that we only need continuity at $\theta = 0$ of

$$g(\theta) = \frac{\phi(x+\theta)-\phi(x)}{\sin(\frac{1}{2}\theta)} = \frac{\phi(x+\theta)-\phi(x)}{\theta}\frac{\theta}{\sin(\frac{1}{2}\theta)} \to 2\phi'(\theta).$$

□

Theorem

The full Fourier serie of $\phi \in C^1(\mathbb{R})$ periodic converges uniformily on $[-\pi, \pi]$.

Proof of uniform convergence.

Since we assume $\phi \in C^1(\mathbb{R})$ with periodic boundary condition, the function ϕ' is continuous and periodic. Hence, the full Fourier coefficients A'_n and $\tilde{A}'_n$ of ϕ are defined. By integration by parts

$$A_n = \int_{-\pi}^{\pi} \phi(x)\sin(nx)dx = -\frac{1}{n}\phi(x)\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} f'(x)\cos(nx)dx = \frac{1}{n}\tilde{A}'_n$$

Similar $\tilde{A}_n = -\frac{1}{n}A'_n$.

On the other hand we know that $\|\phi\|_2, \|\phi'\|_2 < \infty$ because ϕ and ϕ' are continuous functions on $[-\pi, \pi]$. In particular

$$\sum_{n=1}^{\infty}\left(|A'_n|^2 + |\tilde{A}'_n|^2\right) < \infty$$

It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(|A_n|+|\tilde{A}_n|\right) &= \sum_{n=1}^{\infty}\frac{1}{n}|A_n'|+\sum_{n=1}^{\infty}\frac{1}{n}|\tilde{A}_n'| \\
\text{(Cauchy-Schwartz)} \qquad &\le \sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty}\left(|A_n'|+|\tilde{A}_n'|\right)^2} \\
(a+b)^2 \le 2a^2+2b^2 \qquad &\le \sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty}2\left(|A_n'|^2+|\tilde{A}_n'|^2\right)}
\end{aligned}$$

Hence

$$\max_{x\in[-\pi,\pi]}|f(x)-S_N(x)| \le \sum_{n=N+1}^{\infty}|A_n\cos(nx)+\tilde{A}_n\sin(nx)| \le \sum_{n=N+1}^{\infty}|A_n|+|\tilde{A}_n| \to 0 \text{ as } N\to\infty.$$

□

In fact the following stronger theorems is true

Theorem

For every $f \in C^0(\mathbb{R})$ with periodic boundary conditions and period π its full Fourier serie converges uniformily to f on $[-\pi, \pi]$.