MAT351 Partial Differential Equations
Lecture 19

November 30, 2020

Last Lecture

Theorem

Let $f \in C^1(\mathbb{R})$ *be periodic with period* 2π*.*

Then **the full Fourier serie of** f

$$\frac{1}{2}\tilde{A}_0 + \sum_{n=1}^{\infty}\left[A_n \sin(nx) + \tilde{A}_n \cos(nx)\right]$$

converges on $[-\pi, \pi]$ *uniformily to* f*. Recall that the coefficients are given by*

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx,\ \tilde{A}_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,\ \tilde{A}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx.$$

We also showed that

$$\frac{1}{2}|\tilde{A}_0| + \sum_{n=1}^{\infty}\left(|A_n| + |\tilde{A}_n|\right)$$

Note that

$$A_n = \frac{1}{\|X_n\|^2}(f, X_n), \qquad \tilde{A}_n = \frac{1}{\left\|\tilde{X}_n\right\|^2}(f, \tilde{X}_n) \quad \text{for } n \in \mathbb{N}$$

where $X_n(x) = \sin(nx)$, $\tilde{X}_n(x) = \cos(nx)$ and $(f, g) = \int_{-\pi}^{\pi} f(x)g(x)dx$ for $f, g \in C^0(\mathbb{R})$ periodic.

But for $\tilde{X}_0(x) = 1$ the definition says

Fourier sine and cosine serie

Consider $f \in C^1([0,\pi])$ with Dirichlet boundary conditions: $f(0) = f(\pi) = 0$.
Let f_{odd} be the odd periodic extension of f to $\mathbb{R}$. Then $f_{odd} \in C^1(\mathbb{R})$ because

$$\lim_{h\downarrow} \frac{f(h) - f(0)}{h} = \lim_{h\downarrow 0} \frac{f(h)}{h} = \lim_{h\downarrow 0} \frac{-f(h)}{-h} = \lim_{h\uparrow 0} \frac{f_{odd}(h)}{h}.$$

We know that the full Fourier series of f_{odd} has the form

$$\sum_{n=1}^{\infty} A_n \sin(nx) = \mathcal{F}(f)$$

and converges uniformily on $[-\pi,\pi]$ to f_{odd}.
Recall that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} \underbrace{f_{odd}(x)}_{-f(-x)}\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx$$

Hence $A_n = A_n^{sin}$ where A_n^{sin} is the Fourier sine coefficient.
Therefore, the full Fourier series $\mathcal{F}(f_{odd})$ coincides with the Fourier sine series $\mathcal{S}(f)$ of f on $[0,\pi]$ and we have the following

Corollary

Let $f \in C^1([0,\pi])$ with $f(0) = f(\pi) = 0$.

Then, the Fourier sine series converges uniformily to f on $[0,\pi]$.

For $f \in C^1([0,\pi])$ with Neumann boundary conditions ($f'(0) = f'(\pi) = 0$) we consider the even periodic extension f_{even}.

Then agan $f_{even} \in C^1(\mathbb{R})$, the full Fourier serie converges uniformily to f and $\tilde{A}_n = A_n^{cos}$ where A_n^{cos} are the coefficients of the Fourier cosine series.

We obtain

Corollary

For $f \in C^1([0,\pi])$ with Neumann boundary conditions, the Fourier cosine series of f converges uniformly to f on $[0,\pi]$.

Recall that more generally one has

Theorem

Let $f \in C^0(\mathbb{R})$ be periodic with period 2π.

Then the full Fourier serie of f converges uniformily on $[-\pi,\pi]$ to f.

Corollary

Let $f \in C^0([0,\pi])$ with $f(0) = f(\pi) = 0$. Then, the Fourier sine series converges uniformily to f on $[0,\pi]$.

For $f \in C^1([0,\pi])$ with Neumann boundary conditions, the Fourier cosine series of f converges uniformly to f on $[0,\pi]$.

Application: Heat equation with Dirichlet boundary conditions on $[0, \pi]$

Let $f \in C^1([0, \pi])$ with Dirichlet boundary conditions and let f_{odd} be the odd periodic extenstion. Then f has an expansion as Fourier sine serie:

$$\sum_{n=1}^{\infty} A_n \sin(nx) = f(x).$$

Theorem

The series

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-kn^2 t} \sin(nx)$$

converges uniformly on $[0, \pi] \times [0, \infty)$.

We have $u \in C^2([0, \pi] \times (0, \infty)) \cap C^0([0, \pi] \times [0, \infty))$ *and* u *solves*

$$\begin{aligned} u_t &= k u_{x,x} \quad \text{on } [0, \pi] \times (0, \infty) \\ u(x, 0) &= f(x) \quad \text{on } [0, \pi] \\ u(0, t) &= 0 \text{ and } u(\pi, t) = 0 \ \forall t > 0. \end{aligned}$$

Proof of the theorem

First, we have for $N, M \in \mathbb{N}$ and $M > N$

$$\max_{(x,t)\in[0,\pi]\times[0,\infty)} \left| \sum_{n=1}^{M} A_n e^{-ktn^2} \sin(nx) - \sum_{n=1}^{N} A_n e^{-kn^2 t} \sin(nx) \right|$$

$$= \max_{(x,t)\in[0,\pi]\times[0,\infty)} \left| \sum_{n=N+1}^{M} A_n e^{-kn^2 t} \sin(nx) \right|$$

$$\leq \sum_{n=N+1}^{M} |A_n| e^{-kn^2 t} |\sin(nx)| \to 0.$$

We use that $\sum_{n=1}^{\infty} |A_n| < \infty$ is finite, what was part of the proof of uniform convergence of the Fourier serie for $f \in C^1(\mathbb{R})$ periodic.

Therefore, the partial sums $\sum_{n=1}^{N} A_n e^{-kn^2 t} \sin(nx)$ are a Cauchy sequence w.r.t. to uniform convergence, and hence, the uniform limit

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-ktn^2} \sin(nx)$$

exists and $u(x,t)$ is a continuous function on $[0,\pi] \times [0,\infty)$ and $u(x,t)$ satisfies $u(x,0) = f(x)$, $u(0,t) = u(\pi,t) = 0$. In particular, u is continuous on $[0,\pi] \times [0,\infty)$.

Moreover, each term $A_n e^{-ktn^2} \sin(nx)$, $n \in \mathbb{N}$, has partial derivatives w.r.t. t and x:

$$-A_n n^2 k e^{-ktn^2} \sin(nx) \quad \text{and} \quad A_n n e^{-ktn^2} \cos(nx)$$

and second derivatives w.r.t. x: $-A_n n^2 e^{-ktn^2}\sin(nx)$ and solves the heat equation by the separation of variable method. Recall the serie $\sum_{n=1}^{\infty} n^{\alpha} e^{-ktn^2}$ converges absolutely whenever $t > 0$ for all $\alpha \in \mathbb{N}$. Hence, it follows that the series

$$\sum_{n=1}^{\infty} A_n n e^{-ktn^2}\sin(nx), \quad \sum_{n=1}^{\infty} A_n n^2 e^{-ktn^2}\sin(nx)$$

converge uniformly on $[0,\pi]\times[t_0,\infty)$ for $t_0 > 0$. This follows because, for instance,

$$\max_{(x,t)\in[0,\pi]\times[t_0,\infty)}\left|\sum_{n=N+1}^{\infty} A_n n e^{-ktn^2}\sin(nx)\right| \leq \sum_{n=N+1}^{\infty} |A_n| n e^{-kt_0 n^2}$$

$$\text{(Cauchy-Schwartz inequality)} \leq \sqrt{\left(\sum_{n=N+1}^{\infty} A_n^2\right)\left(\sum_{n=N+1}^{\infty} n^2 e^{-2kt_0 n^2}\right)}$$

It follows that we can compute the first and second order derivatives of $u(x,t)$ w.r.t. x and t by computing the partial derivatives of the partial sums:

$$u_x(x,t) = \sum_{n=1}^{\infty} A_n e^{-ktn^2} n\cos(nx), \;\; u_t(x,t) = \sum_{n=1}^{\infty} A_n k n^2 e^{-ktn^2}\sin(nx)$$

on $[0,\pi]\times[t_0,\infty)$ for $t_0 > 0$ and

$$k u_{x,x}(x,t) = -k\sum_{n=1}^{\infty} A_n n^2 e^{-ktn^2}\sin(nx) = u_t(x,t) \text{ for } (x,t)\in[0,\pi]\times(0,\infty).$$

In particular, $u \in C^2([0,\pi]\times(0,\infty))$ and solves the heat equation. □

Complex form of the full Fourier serie

Recall that

$$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}, \quad \cos(nx) = \frac{e^{inx} + e^{-inx}}{2}.$$

Let $f \in C^0(\mathbb{R})$ be periodic. The full Fourier serie can be written in complex form

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \mathcal{F}(f)$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

To see this we introduce a *Hermitien inner product*.

$$(f, g) = \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx \text{ for } f, g \in C^0(\mathbb{R}, \mathbb{C}) \text{ periodic}$$

where $g(x) = \mathrm{Re} g(x) + \mathrm{Im} g(x)$ and $\overline{g(x)} = \mathrm{Re} g(x) - \mathrm{Im} g(x)$ is the complex conjugate of the complex number $g(x)$.

Note that $C^0(\mathbb{R}, \mathbb{C})$ is complex vector space. $X_n = e^{inx}$, $n \in \mathbb{Z}$, are orthogonal w.r.t. $(\cdot, \cdot)$ and

$$\left\| e^{inx} \right\|_2^2 = (e^{inx}, e^{inx}) = \int_{-\pi}^{\pi} e^{inx} e^{-inx} dx = 2\pi.$$

In particular, the general Fourier coefficients take the form

$$c_n = \frac{1}{\|X_n\|}(f, X_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

We can interpret a 2π periodic function f on $\mathbb{R}$ as a function $\hat{f}$ on the $1D$ circle $\mathbb{S}^1 \subset \mathbb{R}^2 = \mathbb{C}$:

$$f(x) = \hat{f}\left(e^{ix}\right).$$

We interpret the heat equation

$$\begin{aligned} u_t &= k u_{x,x} \quad \text{on } \mathbb{R} \\ u(x,0) &= f(x) \quad \text{on } \mathbb{R} \\ u(x,t) &= u(x+2\pi,t)\ \forall x \in \mathbb{R} \end{aligned}$$

with $f \in C^1(\mathbb{R})$ that is 2π periodic as heat equation on $\mathbb{S}^1$.

The solution is given by

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n e^{-ktn^2} e^{inx} = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} f(y) \sum_{n=-\infty}^{\infty} e^{-ktn^2} e^{in(x-y)} dy$$

where c_n are the complex Fourier coefficients of f.

We can write this formula as

$$u(x,t) = \int_{-\pi}^{\pi} f(y) K^{\mathbb{S}^1}(x-y,t) dy$$

with

$$K^{\mathbb{S}^1}(\theta,t) = \sum_{n=-\infty}^{\infty} e^{-ktn^2} e^{in\theta}$$

$K^{\mathbb{S}^1}(\theta,t)$ is called the fundamental solution for the heat equation or heat kernel on $\mathbb{S}^1$.