MAT351 Partial Differential Equations

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Partial Derivatives

Consider a function u of several variables:

$$u = u(x, y, z) \text{ or more generally } u = u(x_1, x_2, \ldots, x_n)$$

for $(x, y, z) \in U \subset \mathbb{R}^3$ or $(x_1, \ldots, x_n) \in U \subset \mathbb{R}^n$. We also write $\mathbf{x} = \overrightarrow{x} = (x_1, \ldots, x_n)$.
U is a domain $\Leftrightarrow$ U connected, $U^\circ \neq \emptyset$ and ∂U smooth.
x, y, z (or $x_1, \ldots, x_n$) are called **independent variables**.

Notation

Let u be sufficiently smooth (e.g. $u \in C^1(U)$). We denote the partial derivatives with

$$\lim_{h \to 0} \frac{u(\mathbf{x} + he_i) - u(\mathbf{x})}{h} = \frac{\partial u}{\partial x_i}(\mathbf{x}) = u_{x_i}(\mathbf{x}) \quad i = 1, \ldots, n.$$

$e_i = (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0)$. For partial derivatives of order $k \in \mathbb{N}$ we write

$$\frac{\partial^k u}{\partial x_{i_1} \ldots \partial x_{i_k}}(\mathbf{x}) = u_{x_{i_1}, \ldots, x_{i_k}}(\mathbf{x}) \quad i_1, \ldots, i_k \in \{1, \ldots, n\}.$$

For the collection of all partial derivatives of order $k \in \mathbb{N}$ we write

$$\left\{ u_{x_{i_1}, \ldots, x_{i_k}} : i_1, \cdots i_k \in \{1, \cdots, n\} \right\} =: D^k u.$$

Common Differential Operators

Gradient: The vector $(u_{x_1}, \dots, u_{x_n}) =: \nabla u$ is called the gradient of u.

Directional Derivative: Given a vector $v = (v_1, \dots, v_n)$

$$\nabla u \cdot v = \sum_{i=1}^{n} u_{x_i} v_i = \frac{\partial u}{\partial v} \quad \text{derivative of } u \text{ in direction } v.$$

In particular $\nabla u \cdot (0, \dots, 0, \underbrace{1}_{i}, 0, \dots, 0) = u_{x_i}$

Differential of a vectorvalued map, Divergence:
For $V : U \subset \mathbb{R}^n \to \mathbb{R}^n$, $V(\mathbf{x}) = (V^1(\mathbf{x}), \dots, V^n(\mathbf{x}))$ one defines

$$DV = \begin{pmatrix} V^1_{x_1} & \dots & V^1_{x_n} \\ \dots & \dots & \dots \\ V^n_{x_1} & \dots & V^n_{x_n} \end{pmatrix} \quad \text{and} \quad \operatorname{trace} DV = \sum_{i=1}^{n} V^i_{x_i} =: \nabla \cdot V =: \operatorname{Div} V$$

Hessian and Laplace operator: $u(\mathbf{x}) = u(x_1, \dots, x_n)$ smooth, $\mathbf{x} \in U$. Then
$\nabla u : U \subset \mathbb{R}^n \to \mathbb{R}^n$ and

$$D\nabla u = \begin{pmatrix} u_{x_1,x_1} & \dots & u_{x_1,x_n} \\ \dots & \dots & \dots \\ u_{x_n,x_1} & \dots & u_{x_n,x_n} \end{pmatrix} \quad \text{and} \quad \operatorname{trace} D^2 u = \sum_{i=1}^{n} u_{x_i,x_i} =: \Delta u.$$

What is a Partial Differential Equation (PDE)?

Definition

A PDE is an equation which relates an unknown function u, its partial derivatives and its independent variables.

A general PDE on a domain $U \subset \mathbb{R}^n$ can be written as

$$F(\mathbf{x}, u, D^1 u, \dots, D^k u) = F(\mathbf{x}, u(\mathbf{x}), D^1 u(\mathbf{x}), \dots, D^k u(\mathbf{x})) = g(\mathbf{x}), \quad \mathbf{x} \in U \tag{1}$$

for functions

$$g(\mathbf{x}) \text{ and } F(\mathbf{x}, \theta, \theta^1, \dots, \theta^k)$$

where $(x_1, \dots, x_n) = \mathbf{x} \in U$, $\theta \in \mathbb{R}$ and $\theta^i = (\theta^i_1, \dots, \theta^i_{n^i}) \in \mathbb{R}^{n^i}$ and $i = 0, \dots, k$.
u and $D^1 u, \dots, D^k u$ are also called dependent variables.

When we study a PDE often the domain U is not specified yet in the beginning.

Definition

The order of a PDE is the highest order of a partial derivative that appears in the equation.

The most general form of a first order PDE for 2 independent variables is

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = g(x, y).$$

Linear PDEs

Definition

A PDE of the form

$$F(\mathbf{x}, u, D^1 u, \ldots, D^k u) = g(\mathbf{x}) \tag{2}$$

is called **linear** if the function

$$(\theta, \theta^1, \ldots, \theta^k) \in \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^k} \mapsto F(\mathbf{x}, \theta, \theta^1, \ldots, \theta^k) \in \mathbb{R}$$

is linear.

A linear PDE of order 2 in n indpendent variables can always be written in the form

$$\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}) u_{x_i,x_j} + \sum_{k=1}^{n} b_k(\mathbf{x}) u_{x_k} + c(\mathbf{x}) u = g(\mathbf{x})$$

with coefficients $(a_{i,j}(\mathbf{x}))_{i,j=1,\ldots,n}, (b_k(\mathbf{x}))_{k=1,\ldots,n}, c(\mathbf{x})$ that are functions in $\mathbf{x}$.

Example (Poisson equation)

$$\Delta u = \sum_{i=1}^{n} u_{x_i,x_j} = g(\mathbf{x}) \text{ where } a_{i,j} = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}.$$

Nonlinear PDEs

Definition

A PDE of the form

$$F(\mathbf{x}, u, D^1 u, \ldots, D^k u) = g(\mathbf{x}) \tag{3}$$

is called

- **semi linear** if we can write

  $$F(\mathbf{x}, \theta, \theta^1, \ldots, \theta^k) = L(\mathbf{x}, \theta^k) + G(\mathbf{x}, \theta, \theta^1, \ldots, \theta^{k-1})$$

  and the function $\theta^k \in \mathbb{R}^{n^k} \mapsto L(\mathbf{x}, \theta^k)$ is linear.
- **quasi linear** if we can write

  $$F(\mathbf{x}, \theta, \theta^1, \ldots, \theta^k) = L(\mathbf{x}, \theta, \theta^1, \ldots, \theta^{k-1}, \theta^k) + G(\mathbf{x}, \theta, \theta^1, \ldots, \theta^{k-1})$$

  and the function $\theta^k \in \mathbb{R}^{n^k} \mapsto L(\mathbf{x}, \theta, \theta^1, \ldots, \theta^{k-1}, \theta^k)$ is linear.
- **fully nonlinear** if the PDE is not $\Big[$ linear, semilinear or quasilinear $\Big]$.

linear $\implies$ semi-linear $\implies$ quasi-linear $\implies$ fully non-linear.

- Consider a quasi linear PDE $F(\mathbf{x}, u, D^1 u) = g(\mathbf{x})$. Hence F has the form

$$F(\mathbf{x}, \theta, \theta^1) = \sum_{i=1}^{n} a_i(\mathbf{x}, \theta)\theta^1 + G(\mathbf{x}, \theta).$$

The coefficients $(a_i)_{i=1,\dots,n}$ are functions in $\mathbf{x}$ and θ. The PDE takes the form

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u)u_{x_i} + G(\mathbf{x}, u) = g(\mathbf{x})$$

Example (Inviscid (or Non-viscous) Burger's equations)

$$u_t + (u^2)_x = 0 \implies u_t + uu_x = 0$$

is a quasi-linear PDE of order 1 in 2 independent variables: $t = x_1$ and $x = x_2$. $a_1(\mathbf{x}, u) = 1$, $a_2(\mathbf{x}, u) = u$ and $G = g \equiv 0$.

- Consider a PDE of order 2 $F(\mathbf{x}, u, D^1 u, D^2 u) = g(\mathbf{x})$. If the PDE is quasi-linear, it can be writen in the general form

$$\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}, u, D^1 u)u_{x_i,x_j} + G(\mathbf{x}, u, D^1 u) = g(\mathbf{x}).$$

$(a_{i,j})_{i,j=1,\dots,n}$, G are functions in $\mathbf{x}$, θ and θ^1.

Solutions

Definition

Consider a PDE of order k:

$$F(\mathbf{x}, u, D^1 u, \ldots, D^k u) = g(\mathbf{x}) \tag{4}$$

A classical solution of (4) on a domain $\Omega \subset \mathbb{R}^n$ where n is the number of independent variables, is a sufficiently smooth function $u(\mathbf{x})$ that satisfies (4).

If $k \in \mathbb{N}$ is the order of the PDE, then, by sufficiently smooth, we man that $u \in C^k(\Omega)$.

Example

The function $u(x, t) = \frac{x}{t}$ solves

$$u_t + u \cdot u_x = 0 \text{ on } \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2. \tag{5}$$

Homogeneous/Inhomogeneous Linear PDEs

Definition

Consider a linear PDE of order k:

$$L(\mathbf{x}, u, D^1 u, \ldots, D^k u) = g(\mathbf{x}) \tag{6}$$

If $g(\mathbf{x}) \equiv 0$, the PDE is called homogeneous.

Otherwise, the PDE is called inhomogeneous.

- If u and v solve the homogeneous linear PDE

$$L(\mathbf{x}, u, D^1 u, \ldots, D^k u) = 0 \text{ on a domain } \Omega \subset \mathbb{R}^n \tag{7}$$

  then also $\alpha u + \beta v$ solves the same homogeneous linear PDE on the domain Ω for $\alpha, \beta \in \mathbb{R}$. *(Superposition Principle)*
- If u solves the homogeneous linear PDE (7) and w solves the inhomogeneous linear pde (6) then $v + w$ also solves the same inhomogeneous linear PDE.
- We can see the map

$$u \in \mapsto \mathcal{L}u \text{ where } (\mathcal{L}u)(\mathbf{x}) = L(\mathbf{x}, u, D^1 u, \ldots, D^k u)$$

  as a linear (differential) operator.
  Hence, it makes sense to specify appropriate function vector spaces V and W such that $u \in V$ and $\mathcal{L}u \in W$.
  For instance: For a PDE of order 2, we can choose $V = C^2(\Omega)$ and $W = C^0(\Omega)$.