MAT351 Partial Differential Equations

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Partial Derivatives

Consider a function $u$ of several variables:

$$ u = u(x, y, z) \quad \text{or more generally} \quad u = u(x_1, x_2, \ldots, x_n) $$

for $(x, y, z) \in U \subset \mathbb{R}^3$ or $(x_1, \ldots, x_n) \in U \subset \mathbb{R}^n$. We also write $x = \overrightarrow{x} = (x_1, \ldots, x_n)$. $U$ is a domain $\iff U$ connected, $U^\circ \neq \emptyset$ and $\partial U$ smooth.

$x, y, z$ (or $x_1, \ldots, x_n$) are called independent variables.

Notation

Let $u$ be sufficiently smooth (e.g. $u \in C^1(U)$). We denote the partial derivatives with

$$ \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} = \frac{\partial u}{\partial x_i}(x) = u_{x_i}(x) \quad i = 1, \ldots, n. $$

$e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. For partial derivatives of order $k \in \mathbb{N}$ we write

$$ \frac{\partial^k u}{\partial x_{i_1} \ldots \partial x_{i_k}}(x) = u_{x_{i_1},\ldots,x_{i_k}}(x) \quad i_1, \ldots, i_k \in \{1, \ldots, n\}. $$

For the collection of all partial derivatives of order $k \in \mathbb{N}$ we write

$$ \left\{ u_{x_{i_1},\ldots,x_{i_k}} : i_1, \ldots, i_k \in \{1, \ldots, n\} \right\} =: D^k u. $$
Common Differential Operators

Gradient: The vector \((u_x^1, \ldots, u_x^n) =: \nabla u\) is called the gradient of \(u\).

Directional Derivative: Given a vector \(v = (v_1, \ldots, v_n)\)

\[
\nabla u \cdot v = \sum_{i=1}^{n} u_{x_i} v_i = \frac{\partial u}{\partial v} \text{ derivative of } u \text{ in direction } v.
\]

In particular \(\nabla u \cdot (0, \ldots, 0, 1, 0, \ldots, 0) = u_{x_i}\)

Differential of a vectorvalued map, Divergence:

For \(V : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n, \ V(x) = (V^1(x), \ldots, V^n(x))\) one defines

\[
DV = \begin{pmatrix}
V^1_{x_1} & \cdots & V^1_{x_n} \\
\vdots & \ddots & \vdots \\
V^n_{x_1} & \cdots & V^n_{x_n}
\end{pmatrix} \quad \text{and} \quad \text{trace } DV = \sum_{i=1}^{n} V^i_{x_i} =: \nabla \cdot V =: \text{Div } V
\]

Hessian and Laplace operator: \(u(x) = u(x_1, \ldots, x_n)\) smooth, \(x \in U\). Then

\[
\nabla u : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{and} \quad D\nabla u = \begin{pmatrix}
u_{x_1,x_1} & \cdots & u_{x_1,x_n} \\
\vdots & \ddots & \vdots \\
u_{x_n,x_1} & \cdots & u_{x_n,x_n}
\end{pmatrix} \quad \text{and} \quad \text{trace } D^2 u = \sum_{i=1}^{n} u_{x_i,x_i} =: \Delta u.
\]
What is a Partial Differential Equation (PDE)?

Definition

A PDE is an equation which relates an unknown function $u$, its partial derivatives and its independent variables.

A general PDE on a domain $U \subset \mathbb{R}^n$ can be written as

$$F(x, u, D^1 u, \ldots, D^k u) = F(x, u(x), D^1 u(x), \ldots, D^k u(x)) = g(x), \quad x \in U \quad (1)$$

for functions

$$g(x) \text{ and } F(x, \theta, \theta^1, \ldots, \theta^k)$$

where $(x_1, \ldots, x_n) = x \in U$, $\theta \in \mathbb{R}$ and $\theta^i = (\theta^i_1, \ldots, \theta^i_{n^i}) \in \mathbb{R}^{n^i}$ and $i = 0, \ldots, k$.

$u$ and $D^1 u, \ldots, D^k u$ are also called dependent variables.

When we study a PDE often the domain $U$ is not specified yet in the beginning.

Definition

The order of a PDE is the highest order of a partial derivative that appears in the equation.

The most general form of a first order PDE for 2 independent variables is

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = g(x, y).$$
Linear PDEs

**Definition**

A PDE of the form

\[ F(x, u, D^1 u, \ldots, D^k u) = g(x) \]  \hspace{1cm} (2)

is called **linear** if the function

\[ (\theta, \theta^1, \ldots, \theta^k) \in \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^k} \mapsto F(x, \theta, \theta^1, \ldots, \theta^k) \in \mathbb{R} \]

is linear.

A linear PDE of order 2 in \( n \) independent variables can always be written in the form

\[ \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i,x_j} + \sum_{k=1}^{n} b_k(x) u_{x_k} + c(x) u = g(x) \]

with coefficients \((a_{i,j}(x))_{i,j=1,\ldots,n}, (b_k(x))_{k=1,\ldots,n}, c(x)\) that are functions in \( x \).

**Example (Poisson equation)**

\[ \Delta u = \sum_{i=1}^{n} u_{x_i,x_j} = g(x) \]

where \( a_{i,j} = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \)
Nonlinear PDEs

Definition

A PDE of the form

\[ F(x, u, D^1 u, \ldots, D^k u) = g(x) \]  

(3)

is called

- **semi linear** if we can write

  \[ F(x, \theta, \theta^1, \ldots, \theta^k) = L(x, \theta^k) + G(x, \theta, \theta^1, \ldots, \theta^{k-1}) \]

  and the function \( \theta^k \in \mathbb{R}^{n_k} \mapsto L(x, \theta^k) \) is linear.

- **quasi linear** if we can write

  \[ F(x, \theta, \theta^1, \ldots, \theta^k) = L(x, \theta, \theta^1, \ldots, \theta^{k-1}, \theta^k) + G(x, \theta, \theta^1, \ldots, \theta^{k-1}) \]

  and the function \( \theta^k \in \mathbb{R}^{n_k} \mapsto L(x, \theta, \theta^1, \ldots, \theta^{k-1}, \theta^k) \) is linear.

- **fully nonlinear** if the PDE is not \([\text{linear, semilinear or quasilinear}]\).

linear \(\Rightarrow\) semi-linear \(\Rightarrow\) quasi-linear \(\Rightarrow\) fully non-linear.
Consider a quasi linear PDE $F(x, u, D^1 u) = g(x)$. Hence $F$ has the form

$$F(x, \theta, \theta^1) = \sum_{i=1}^{n} a_i(x, \theta)\theta^1 + G(x, \theta).$$

The coefficients $(a_i)_{i=1,\ldots,n}$ are functions in $x$ and $\theta$. The PDE takes the form

$$\sum_{i=1}^{n} a_i(x, u) u_{x_i} + G(x, u) = g(x)$$

**Example (Inviscid (or Non-viscous) Burger’s equations)**

$$u_t + (u^2)_x = 0 \implies u_t + uu_x = 0$$

is a quasi-linear PDE of order 1 in 2 independent variables: $t = x_1$ and $x = x_2$. $a_1(x, u) = 1$, $a_2(x, u) = u$ and $G = g \equiv 0$.

Consider a PDE of order 2 $F(x, u, D^1 u, D^2 u) = g(x)$. If the PDE is quasi-linear, it can be written in the general form

$$\sum_{i,j=1}^{n} a_{i,j}(x, u, D^1 u) u_{x_i,x_j} + G(x, u, D^1 u) = g(x).$$

$(a_{i,j})_{i,j=1,\ldots,n}$, $G$ are functions in $x$, $\theta$ and $\theta^1$. 
Solutions

**Definition**

Consider a PDE of order $k$:

$$ F(x, u, D^1u, \ldots, D^k u) = g(x) \quad (4) $$

A classical solution of (4) on a domain $\Omega \subset \mathbb{R}^n$ where $n$ is the number of independent variables, is a sufficiently smooth function $u(x)$ that satisfies (4).

If $k \in \mathbb{N}$ is the order of the PDE, then, by sufficiently smooth, we mean that $u \in C^k(\Omega)$.

**Example**

The function $u(x, t) = \frac{x}{t}$ solves

$$ u_t + u \cdot u_x = 0 \quad \text{on} \quad \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2. \quad (5) $$
Homogeneous/Inhomogeneous Linear PDEs

Definition

Consider a linear PDE of order $k$:

$$L(x, u, D^1 u, \ldots, D^k u) = g(x)$$  \hspace{1cm} (6)

If $g(x) \equiv 0$, the PDE is called homogeneous.
Otherwise, the PDE is called inhomogeneous.

- If $u$ and $v$ solve the homogeneous linear PDE
  $$L(x, u, D^1 u, \ldots, D^k u) = 0$$  \hspace{1cm} (7)
  then also $\alpha u + \beta v$ solves the same homogeneous linear PDE on the domain $\Omega \subset \mathbb{R}^n$ for $\alpha, \beta \in \mathbb{R}$.  \textit{(Superposition Principle)}

- If $u$ solves the homogeneous linear PDE (7) and $w$ solves the inhomogeneous linear pde (6) then $v + w$ also solves the same inhomogeneous linear PDE.

- We can see the map
  $$u \mapsto \mathcal{L}u$$  where $(\mathcal{L}u)(x) = L(x, u, D^1 u, \ldots, D^k u)$

as a linear (differential) operator.

Hence, it makes sense to specify appropriate function vector spaces $V$ and $W$ such that $u \in V$ and $\mathcal{L}u \in W$.

For instance: For a PDE of order 2, we can choose $V = C^2(\Omega)$ and $W = C^0(\Omega)$. 

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