MAT351 Partial Differential Equations
Lecture 20

December 6, 2020

Some Preliminaries

Given $u \in C^2(U)$ for an open subset $U \subset \mathbb{R}^n$.

- Recall that $U \subset \mathbb{R}^n$ is open if and only if for all $x \in U$ we can find $\epsilon_x > 0$ such that
$$\{y \in \mathbb{R}^n : |x - y|_2 < \epsilon_x\} =: B_{\epsilon_x}(x) \subset U$$
Also recall that $\overline{U}$ denote the closure of U, that is
$$\overline{U} = \left\{x \in \mathbb{R}^n : \exists (x_n)_{n\in\mathbb{N}} \subset U \text{ s.t. } \lim_{n\to n} x_n = x\right\}$$
Then, we define $\overline{U}\backslash U = \partial U$.
- A set $W \subset \mathbb{R}^n$ is called connected if there don't exist sets U_1, U_2 open and disjoint such that $U_1 \cap W, U_2 \cap W \neq \emptyset$ and $W \subset U_1 \cup U_2$.

  Or in other words, W is connected if for any pair of open and disjoint sets U_1, U_2 such that $W \subset U_1 \cup U_2$ it follows that either $W \cap U_1 = \emptyset$ or $W \cap U_2 = \emptyset$.
- $u \in C^2(U)$ if and only if all partial derivatives u_{x_i,x_j}, $i, j = 1 \ldots n$, exists and are continuous.

  Given $u \in C^2(U)$ the Laplace operator is defined by
$$\sum_{i=1}^{n} u_{x_i,x_i} =: \Delta u$$
Δ is a map between $C^2(U)$ and $C^0(U)$.

Laplace and Poisson Equation

Laplace Equation

$u \in C^2(U)$ satisfies the **Laplace equation** in U if

$$\Delta u = 0 \text{ in } U.$$

A function $u \in C^2(U)$ for some open **connected** set U with $\Delta u = 0$ is called *harmonic*.

In one dimension the Laplace equation becomes

$$\frac{d^2}{dx^2} u(x) = 0$$

and open, connected sets are intervals of the form (a, b) with $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$.
Hence, harmonic functions are linear functions

$$u(x) = Ax + B, \;\; A, B \in \mathbb{R}.$$

Poisson Equation

Given $f \in C^0(U)$ the inhomogeneous version of the Laplace equation

$$\Delta u = f \text{ on } U$$

is called the **Poisson equation**.

If U is a domain with smooth boundary $\partial U \neq \emptyset$, then we usually require suplementary boundary conditions. For the Laplace equation this leads to the following boundary value problems.

Dirichlet Problem

Let $g \in C^0(\partial U)$, Does there exist $u \in C^2(U) \cap C^0(\overline{U})$ such that

$$\begin{aligned} \Delta u &= 0 && \text{in } U \\ u(x) &= g(x) && \text{for } x \in \partial U. \end{aligned}$$

Smooth boundary: ∂U is a smooth $(n-1)$-dimensional submanifold in $\mathbb{R}^n$.

In this case there is a unique tangent plan $T_x \partial U$ for every $x \in \partial U$ and a unique smooth normal vector field $N : \partial U \to \mathbb{R}^n$: $\langle N(x), v \rangle = 0$ for all $v \in T_x \partial U$ and for all $x \in \partial U$.

Recall $u \in C^1(\overline{U})$ can be defined by saying there exists an open set $\tilde{U}$ such that $U \subset \tilde{U}$ and there exists $\tilde{u} \in C^1(\tilde{U})$ such that $\tilde{u}|_{\overline{U}} = u$.

Neumann Problem

Let $g \in C^0(\partial U)$, Does there exist $u \in C^2(U) \cap C^1(\overline{U})$ such that

$$\begin{aligned} \Delta u &= 0 && \text{in } U \\ \frac{\partial}{\partial N} u(x) &= g(x) && \text{for } x \in \partial U. \end{aligned}$$

Physical interpretation of Laplace equation as diffusion in equilibrium

Physically, $u \in C^2(U)$ with $\Delta u = 0$ describes a distribution in equilibrium in U.

We can think of u as steady state solution of the diffusion equation for higher dimensions:

$$u_t = k\Delta u.$$

To see this imagine $u(x,t)$ as the distribution of some quantity in equilibrium in $U \subset \mathbb{R}^3$, that is there is no change over time. That is, for any subdomain $V \subset U$ we have

$$\int_V u_t(\mathbf{x},t)d\mathbf{x} = \frac{d}{dt}\int_V u(\mathbf{x},t)d\mathbf{x} = 0.$$

On the other, $\frac{d}{dt}\int_V u(\mathbf{x},t)d\mathbf{x}$ is equal to the total flux through the boundary of V

$$\int_{\partial V} \langle \mathbf{F}(\mathbf{x},t), N(\mathbf{x})\rangle d\mathbf{x}$$

where $\mathbf{F}(\mathbf{x},t) \in \mathbb{R}^3$, $x \in U$, is the flux density. As in a previous lecture we assume Fick's law.

The the flux – or directional change of $u(x,t)$ in a point x at time t is proportional to the gradient $\nabla u(\mathbf{x},t)$

$$\mathbf{F}(\mathbf{x},t) = -k\nabla u(\mathbf{x},t).$$

Hence, by the divergence theorem

$$0 = \frac{d}{dt}\int_V u(\mathbf{x},t)d\mathbf{x} = \int u_t(\mathbf{x},t)d\mathbf{x} = k\int_V \Delta_{\mathbf{x}} u(\mathbf{x},t)d\mathbf{x}$$

implying that $0 = u_t(\mathbf{x},t) = k\Delta_x u(\mathbf{x},t)$. In particular $u(\mathbf{x},t) = u(\mathbf{x})$ does not depend on t.

Other Interpretations

- **Electrostatics.**

  Electric current is described by a vector field $\mathbf{E}$ in a domain $U \subset \mathbb{R}^3$ that satisfies

  $$\textit{Maxwell's equations} \quad \mathrm{Curl}(\mathbf{E}) = \left(\frac{\partial \mathbf{E}}{\partial x_3} - \frac{\partial \mathbf{E}}{\partial x_2}, \frac{\partial \mathbf{E}}{\partial x_2} - \frac{\partial \mathbf{E}}{\partial x_1}, \frac{\partial \mathbf{E}}{\partial x_1} - \frac{\partial \mathbf{E}}{\partial x_3}\right) = 0, \quad \mathrm{Div}\mathbf{E} = 4\pi\rho$$

  where ρ is the charge density of U.

  $\mathrm{Curl}\mathbf{E} = 0$ in $\mathbb{R}^n$ is equivalent to

  $$\int_a^b \langle \mathbf{E} \circ \gamma(t), \gamma'(t)\rangle dt = 0$$

  for any closed curve $\gamma \in C^1([a,b], \mathbb{R}^n)$ ($\gamma(a) = \gamma(b)$ and $\gamma'(a) = \gamma'(b)$).

  We know that in this case there exists a potential $\phi \in C^2(\mathbb{R}^n)$ such that $\mathbf{E} = \nabla\phi$.

  Hence, the vector field $\mathbf{E}$ is the gradient of a potential $-\phi$ that satisfies the Poisson equation:

  $$\mathbf{E}(\mathbf{x}) = -\nabla\phi(\mathbf{x}) \quad \Rightarrow \quad \Delta\phi = -4\pi\phi(x).$$

- **Classical Newtonian Gravity.**

  Let $\mathbf{g}$ be the gravitational force vector field in $\mathbb{R}^3$ according to a mass distribution ρ. Again one has the following laws

  $$\mathrm{Curl}\mathbf{g} = 0 \text{ in } \mathbb{R}^3 \quad \text{and} \quad \mathrm{Div}\mathbf{g} = -4\pi G\rho$$

  G is the gravitational constant.

  Hence, there exists a potential function ϕ such that $\nabla\phi = -\mathbf{g}$ and $\Delta\phi = -4\pi G\rho$.

- **Fluid dynamics.**

  Recall the transport equation

$$u_t + \langle V, \nabla u \rangle$$

  for a vector field $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. We studied this equation a some lectures ago and assume that

$$\mathrm{Div} V = 0$$

  which means the flow of V is incompressible and there are no sources and sinks.

  Now we also ssume V describes an irrotational flow. That means again that

$$\mathrm{Curl} V = 0 \text{ in } \mathbb{R}^n.$$

  We know that in this case there exists a potential $\phi \in C^2(\mathbb{R}^n)$ such that $V = \nabla\phi$.

  Hence, ϕ satisfies the Laplace equation:

$$\mathrm{Div} V = \Delta\phi = 0.$$

Polar Coordinates in $\mathbb{R}^2$

In $\mathbb{R}^2$ the Laplace operator is $\Delta u = u_{x,x} + u_{y,y} = 0$.

Let us express Δ in polar coordinates

$$(x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta) \text{ for } (r,\theta) \in (0,\infty) \times [0, 2\pi).$$

The differential of the map $(r,\theta) \in (0,\infty) \times (0,2\pi) \mapsto (r\cos\theta, r\sin\theta)$ and its inverse is

$$D(x,y) = \begin{pmatrix} x_r & y_r \\ x_\theta & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \text{ and } [D(x,y)]^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}.$$

Consider u in the new coordinates, that is $\tilde{u} := u \circ (x,y)$. We compute

$$\begin{pmatrix} \tilde{u}_r \\ \tilde{u}_\theta \end{pmatrix} = \begin{pmatrix} -u_x \sin\theta + u_y \cos\theta \\ u_x \cos\theta + u_y \sin\theta \end{pmatrix} = D(x,y) \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

Here u_x and u_y is short for $u_x \circ (x,y)$ and $u_y \circ (x,y)$ respectively. Hence

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = [D(x,y)]^{-1} \begin{pmatrix} \tilde{u}_r \\ \tilde{u}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\,\tilde{u}_r - \frac{1}{r}\sin\theta\,\tilde{u}_\theta \\ \sin\theta\,\tilde{u}_r + \frac{1}{r}\cos\theta\,\tilde{u}_\theta \end{pmatrix} = \begin{pmatrix} (\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta})\tilde{u} \\ (\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta})\tilde{u} \end{pmatrix}.$$

Hence, the operator $\frac{\partial}{\partial x}$ transform under the coordinate change $x = r\cos\theta$, $y = r\sin\theta$ into

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$$

and similar for $\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}$.

Therefore, it follows

$$
\begin{aligned}
u_{x,x} \circ (x,y) &= \left(\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}\right)\tilde{u} \\
&= \Big(\cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial\theta\partial r} + \frac{1}{r}\sin^2\theta\frac{\partial}{\partial r} \\
&\qquad - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial\theta\partial r} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2}\Big)\tilde{u}.
\end{aligned}
$$

and also

$$
\begin{aligned}
u_{y,y} \circ (x,y) &= \left(\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}\right)\tilde{u} \\
&= \Big(\sin^2\theta \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial\theta\partial r} + \frac{1}{r}\cos^2\theta\frac{\partial}{\partial r} \\
&\qquad + \frac{1}{r}\cos\theta\sin\theta\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\cos\theta\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2}\cos^2\theta\frac{\partial^2}{\partial\theta^2}\Big)\tilde{u}.
\end{aligned}
$$

It follows that

$$
u_{x,x} \circ (x,y) + u_{y,y} \circ (x,y) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\tilde{u} = \tilde{u}_{r,r} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta,\theta}.
$$

Corollary

The Laplace operator is invariant w.r.t. rotations of $\mathbb{R}^2$ at the center.

Beweis. A rotation is linear transformation w.r.t. θ in polar coordinates. □

Spherical Coordinates in $\mathbb{R}^3$

Let us compute the Laplace operator $u_{x,x} + u_{y,y} + u_{z,z} = \Delta u$ in $\mathbb{R}^3$ in spherical coordintates

$$x = r \sin\theta \cos\phi, \; y = r \sin\theta \sin\phi, \; z = r \cos\theta.$$

For that we first consider cylindrical coordinates

$$x = s \cos\phi, \; y = s \sin\phi, \; z = z.$$

We set $\tilde{u} = u \circ (x, y, z)$. Similar as for polar coordinates we obtain

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \cos\phi \tilde{u}_s - \frac{1}{s} \sin\phi \tilde{u}_\phi \\ \sin\phi \tilde{u}_s + \frac{1}{s} \cos\phi \tilde{u}_\phi \\ \tilde{u}_z \end{pmatrix} \quad \text{and} \quad u_{x,x} + u_{y,y} = \tilde{u}_{s,s} + \frac{1}{s} \tilde{u}_s + \frac{1}{s^2} \tilde{u}_{\phi,\phi}.$$

In particular, we see $u_z = \tilde{u}_z$ and $u_{z,z} = \tilde{u}_{z,z}$.

Then, we apply cylindrical coordinates a second time setting

$$z = r \cos\theta, \; s = r \sin\theta, \; \phi = \phi,$$

and setting $\hat{u} = \tilde{u} \circ (s, \phi, z)$. As before we compute

$$\tilde{u}_s = \sin\theta \hat{u}_r + \frac{1}{r} \cos\theta \hat{u}_\theta$$

as well as

$$\tilde{u}_{s,s} + \tilde{u}_{z,z} = \hat{u}_{r,r} + \frac{1}{r} \hat{u}_r + \frac{1}{r^2} \hat{u}_{\theta,\theta} \quad \Rightarrow \quad u_{z,z} = \hat{u}_{r,r} + \frac{1}{r} \hat{u}_r + \frac{1}{r^2} \hat{u}_{\theta,\theta} - \tilde{u}_{s,s}$$

and in particular$\tilde{u}_\phi = \hat{u}_\phi$ and $\tilde{u}_{\phi,\phi} = \hat{u}_{\phi,\phi}$.

It follows

$$
\begin{aligned}
u_{x,x} + u_{y,y} + u_{z,z} &= \hat{u}_{r,r} + \frac{1}{r}\hat{u}_r + + \frac{1}{r^2}\hat{u}_{\theta,\theta} + \frac{1}{s}\tilde{u}_s + \frac{1}{s^2}\tilde{u}_{\phi,\phi} \\
&= \hat{u}_{r,r} + \frac{1}{r}\hat{u}_r + \frac{1}{r^2}\hat{u}_{\theta,\theta} + \frac{1}{r\sin\theta}\left(\sin\theta\,\hat{u}_r + \frac{1}{r}\cos\theta\,\hat{u}_\theta\right) + \frac{1}{r^2\sin^2\theta}\hat{u}_{\phi,\phi} \\
&= \hat{u}_{r,r} + \frac{2}{r}\hat{u}_r + \frac{1}{r^2}\left(\hat{u}_{\theta,\theta} + \frac{\cos\theta}{\sin\theta}\hat{u}_\theta + \frac{1}{\sin^2\theta}\hat{u}_{\phi,\phi}\right).
\end{aligned}
$$

We can now look for special solutions of the Laplace equation in $\mathbb{R}^2$ or in $\mathbb{R}^3$ that only depend on $\sqrt{x^2+y^2}$ or $\sqrt{x^2+y^2+z^2}$, that is $r > 0$ in polar or spherical coordinates.

In $\mathbb{R}^2$ the Laplace equation for such functions reduces to

$$
0 = \tilde{u}_{r,r} + \frac{1}{r}\tilde{u}_r \;\Rightarrow\; 0 = r\tilde{u}_{r,r} + \tilde{u}_r = (r\tilde{u}_r)_r \;\Rightarrow\; c_1 = r\tilde{u}_r.
$$

Hence the solutions are $\tilde{u}(r) = c_1 \log r + c_2$. In Euclidean coordinates

$$
u(x,y) = c_1 \log\left(\sqrt{x^2+y^2}\right) + c_2.
$$

In $\mathbb{R}^3$ the Laplace equation for such functions reduces to

$$
0 = \hat{u}_{r,r} + \frac{2}{r}\hat{u}_r \;\Rightarrow\; 0 = r^2\hat{u}_{r,r} + 2r\hat{u}_r \;\Rightarrow\; c_1 = r^2\hat{u}_r.
$$

Hence, the solutions are $\hat{u}(r) = -\frac{c_1}{r} + c_2$. In Euclidean coordinates

$$
u(x,y,z) = -\frac{c_1}{\sqrt{x^2+y^2+z^2}} + c_2.
$$

Application to Newtonian Gravity

Imagine a star in an otherwise empty universe. We model the star as the point $0 \in \mathbb{R}^3$ with all its mass concentrated in 0.

What does **Classical Newtonian Gravity** tell us about the gravitational forces in the space around the star? We can write Newton's law as the following Poisson equation

$$\Delta\phi = 4\pi G\rho \text{ in } \mathbb{R}^3\backslash\{0\} \;\Rightarrow\; \frac{1}{4\pi G}\Delta\phi = \rho \text{ in } \mathbb{R}^3\backslash\{0\}$$

where ρ is the mass density in $\mathbb{R}^3\backslash\{0\}$.
By assumption the universe is empty in $\mathbb{R}^3\backslash\{0\}$. Hence, the mass density ρ is 0.

Moreover, we assume the Universe is fully isotrop and homogeneous. Hence, the gravitational forces are the same independent of the direction and hend only depend on the distance $r = \sqrt{x^2+y^2+z^2}$ to the star at 0. Hence, the Poisson equation becomes

$$\frac{1}{4\pi G}\left[\tilde{\phi}_{r,r} + \frac{2}{r}\tilde{\phi}_r\right] = 0.$$

A solution is

$$\tilde{\phi}(r) = -\frac{4\pi G m}{r} + c_2 \text{ for constants } m, c_2 > 0.$$

We also assume that very far away from the star there is almost no gravitational pull. Hence $c_2 = 0$ and therefore

$$\phi(x,y,z) = -\frac{4\pi G m}{\sqrt{x^2+y^2+z^2}} \text{ and } \mathbf{g}(x,y,z) = -\nabla\phi(x,y,z)$$

where m describes the mass of the star.