MAT351 Partial Differential Equations
Lecture 20

December 7, 2020

Some Preliminaries

A subset $U \subset \mathbb{R}^n$ is open if and only if for all $x \in U$ we can find $\epsilon_x > 0$ such that

$$\{y \in \mathbb{R}^n : |x - y|_2 < \epsilon_x\} =: B_{\epsilon_x}(x) \subset U.$$

where $B_\epsilon(x) = \{y \in \mathbb{R}^n : |y - x|_2 < \epsilon\}$.
Let $U \subset \mathbb{R}^n$ be arbitrary, not necessarily open.

$$U^\circ = \{x \in U : \exists \epsilon_x > 0 \text{ s.t. } B_\epsilon(x) \subset U\}$$

denotes the *open interior* of U.
A subset $A \subset \mathbb{R}^n$ is closed if

$$(x_n)_{n\in\mathbb{N}} \subset A \text{ and } \lim_{n\to\infty} x_n = x \;\Rightarrow\; x \in A.$$

If $U \subset \mathbb{R}^n$ is again arbitrary

$$\overline{U} = \left\{x \in \mathbb{R}^n : \exists (x_n)_{n\in\mathbb{N}} \subset U \text{ s.t. } \lim_{n\to n} x_n = x\right\}$$

denotes the *closure* of U. The closure $\overline{U}$ of U is closed.

Fact

A subset $U \subset \mathbb{R}^n$ is open if and only if $\mathbb{R}^n \backslash U$ is closed.

Then, for U open we define $\overline{U} \backslash U = \partial U$, the boundary of U.

A set $W \subset \mathbb{R}^n$ is called connected if there don't exist sets U_1, U_2 open and disjoint such that $U_1 \cap W, U_2 \cap W \neq \emptyset$ and $W \subset U_1 \cup U_2$.

Or equivalently, W is connected if for any pair of open and disjoint sets $U_1, U_2 \subset \mathbb{R}^n$ such that $W \subset U_1 \cup U_2$ it follows that either $W \cap U_1 = \emptyset$ or $W \cap U_2 = \emptyset$.

$u \in C^2(U)$ if and only if all partial derivatives u_{x_i,x_j}, $i, j = 1 \ldots n$, exists and are continuous.

Given $u \in C^2(U)$ the Laplace operator is defined by

$$\sum_{i=1}^{n} u_{x_i,x_i} =: \Delta u$$

Δ is a map between $C^2(U)$ and $C^0(U)$.

Laplace Equation

$u \in C^2(U)$ satisfies the **Laplace equation** in U if

$$\Delta u = 0 \text{ in } U.$$

A function $u \in C^2(U)$ for some open **connected** set U with $\Delta u = 0$ is called *harmonic*.

In one dimension the Laplace equation becomes $\frac{d^2}{dx^2} u(x) = 0$ and open, connected sets are intervals of the form (a, b) with $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$.

Hence, harmonic functions are linear functions

$$u(x) = Ax + B, \ A, B \in \mathbb{R}.$$

Maximum Principle for harmonic functions

Theorem

Let $U \subset \mathbb{R}^n$ be open and let $u : \overline{U} \to \mathbb{R}$ be a function such that $u \in C^2(U) \cap C^0(\overline{U})$ be harmonic. Precisely, $u \in C^0(\overline{U})$ and $u|_U \in C^2(U)$.

- **Weak Maximum Principle:** *The maximum and the minimum value of u are attained on ∂U:*

$$\max_{x \in \overline{U}} u(x) = \max_{x \in \partial U} u(x) \quad \textit{and} \quad \min_{x \in \overline{U}} u(x) = \min_{x \in \partial U} u(x)$$

- **Strong Maximum Principle:** *If U is connected and there exists $x_0 \in U$ such that*

$$u(x_0) = \max_{x \in \overline{U}} u(x) \quad \textit{or} \quad u(x_0) = \min_{x \in \overline{U}} u(x)$$

  then $u \equiv const \equiv u(x_0)$.

Remark

The strong maximum principle implies the weak one. But we will prove both principles separately.

Proof of the weak maximum principle.

The proof is similar to the proof of the weak maximum principle for solutions of the diffusion equation.

Define $v(\mathbf{x}) = u(\mathbf{x}) + \epsilon\frac{1}{2n}|\mathbf{x}|^2$ for some $\epsilon > 0$. We have $v \in C^2(U) \cap C^0(\overline{U})$.

First, let us assume v attains its maximum value in $x_0 \in U$. Then, by the second derivative test in calculus the matrix $D^2 v(\mathbf{x}_0)$ is negative semi-definite. Or equivalently, all eigenvalues of $A = D^2 v(\mathbf{x}_0)$ are non-positive. It follows that the trace of the matrix A is also non-positive. Hence

$$\text{tr}(D^2 v(x_0)) = \sum_{i=1}^{n} v_{x_i,x_i}(\mathbf{x}) = \Delta v(\mathbf{x}_0) \leq 0.$$

On the other hand

$$\Delta v = \Delta u + \Delta(\epsilon\frac{1}{2n}|\mathbf{x}|^2) = 0 + \epsilon > 0$$

This is a contradiction.

Hence $v \in C^0(\overline{U})$ attains its maximum value on ∂U. We obtain the following chain of inequalities

$$u(\mathbf{x}) \leq u(\mathbf{x}) + \epsilon\frac{1}{2n}|\mathbf{x}|^2 \leq v(\mathbf{x}) \leq \max_{x\in\partial U} v(\mathbf{x}) \leq \max_{x\in\partial U} u(\mathbf{x}) + \epsilon\frac{1}{2n}\max_{x\in\partial U}|\mathbf{x}|_2^2 \;\; \forall x \in \overline{U}.$$

Since $\epsilon > 0$ was arbitrary, let $\epsilon \to 0$ and it follows

$$u(\mathbf{x}) \leq \max_{x\in\partial U} u(\mathbf{x}) \; \forall \mathbf{x} \in \overline{U}.$$

□

Mean Value property.

The *mean value property* for harmonic functions states that the value of a harmonic function at any point equals its average on a ball or on sphere (spherical mean) centered at the given point. More precisely:

Theorem (Mean Value Property)

Let $u \in C^2(U)$ be harmonic for $U \subset \mathbb{R}^n$ open. Then, it holds

$$u(\mathbf{x}_0) = \frac{1}{Vol(B_r(\mathbf{x}_0))} \int_{B_r(x_0)} u(\mathbf{x}) d\mathbf{x} \quad \forall \mathbf{x}_0 \in U \text{ and } \forall B_r(\mathbf{x}_0) \subset U.$$

and

$$u(\mathbf{x}_0) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r(\mathbf{x}_0)} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r(\mathbf{x}_0)} \quad \forall x_0 \in U \text{ and } \forall B_r(\mathbf{x}_0) \subset U.$$

where ω_{n-1} is the $(n-1)$-dimensional surface of $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$.

Proof of the mean value property

Let $x_0 \in U$ and $r > 0$ such that $B_r(x_0) \subset U$. W.l.o.g. we can assume $x_0 = 0$ by replacing $u(\mathbf{x})$ with $u(\mathbf{x} - \mathbf{x}_0)$. We set $B_r = B_r(0)$. Consider

$$\phi(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r(0)} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r}$$

We will show that $\phi'(r) = 0$. Then $\phi(r)$ is constant. On the other hand

$$\lim_{r \to 0} \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r}$$

by continuity. Indeed, for $\epsilon > 0$ there exists $\delta > 0$ such that $\forall r \in (0, \delta)$ it holds $|u(0) - u(\mathbf{x})| \leq \epsilon$ if $|\mathbf{x}| = r$. Hence

$$\left| \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r} \right| \leq \lim_{r \to 0} \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r} |u(\mathbf{x}) - u(0)| d\mathcal{S}^{n-1}_{\partial B_r} \leq \epsilon.$$

Let us show that $\phi'(r) = 0$. For that we first observe that

$$\frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r}(\mathbf{x}) = \frac{1}{\omega_{n-1}} \int_{\partial B_1} u(\mathbf{x}/r) d\mathcal{S}^{n-1}_{\partial B_1}(\mathbf{x})$$

Then

$$\frac{d}{dr} \int_{\partial B_1} u(r\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_1}(\mathbf{x}) = \int_{\partial B_1} \langle \nabla u(r\mathbf{x}), \mathbf{x} \rangle d\mathcal{S}^{n-1}_{\partial B_1}(\mathbf{x}) = \frac{1}{r^{n-1}} \int_{\partial B_r} \langle \nabla u(r\mathbf{x}), \frac{\mathbf{x}}{r} \rangle d\mathcal{S}^{n-1}_{\partial B_r}(\mathbf{x})$$

By the divergence theorem the right hand side is equal to $\int_{B_r} \Delta u(\mathbf{x}) d\mathbf{x} = 0$. □

Proof of the strong maximum principle.

Let $u \in C^2(U) \cap C^0(\overline{U})$ be harmonic for a connected and open subset $U \subset \mathbb{R}^n$.

Assume there exists $\mathbf{x}_0 \in U$ such that $u(\mathbf{x}_0) = \max_{\mathbf{x} \in \overline{U}} u(\mathbf{x}) =: M$.

Now, we define

$$V = \{\mathbf{x} \in U : u(\mathbf{x}) = M\} \neq \emptyset.$$

as well $W = U \backslash V$.

Claim: V is open.

Proof of the Claim. Pick $\mathbf{x} \in V$ and let $r > 0$ such that $B_r(\mathbf{x}) \subset U$.
By the mean value property we have

$$M = u(\mathbf{x}) = \frac{1}{v_n(r)} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y}$$

Hence

$$0 \geq \frac{1}{v_n(r)} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) - M d\mathbf{y} \geq 0$$

Since $u(\mathbf{y}) - M \leq 0$, it follows $u(\mathbf{y}) = M$ on $B_r(\mathbf{x})$. Hence $B_r(\mathbf{x}) \subset V$.

Claim: V is also closed and therefore $W \backslash V$ is open.

Let $\mathbf{x}_n \in V$ such that $\mathbf{x}_n \to x$. Since $u \in C^0(\overline{U})$, it holds $M = \lim_{n\to\infty} u(\mathbf{x}_n) = u(\mathbf{x})$.

Hence $\mathbf{x} \in V$.

Finally, since U is connected and since V and W are open, either $V = \emptyset$ or $W = \emptyset$. But since $\mathbf{x}_0 \in V$ and therefore $V \neq \emptyset$, it follows $W = \emptyset$ and $U = V$. □