MAT351 Partial Differential Equations
Lecture 22

December 28, 2020

Maximum Principle for harmonic functions

Laplace Equation

$u \in C^2(U)$ satisfies the **Laplace equation** in U if

$$\Delta u = 0 \text{ in } U.$$

A function $u \in C^2(U)$ for some open ~~connected~~ set U with $\Delta u = 0$ is called *harmonic*.

Theorem

Let $U \subset \mathbb{R}^n$ be open and let $u : \overline{U} \to \mathbb{R}$ be a function such that $u \in C^2(U) \cap C^0(\overline{U})$ be harmonic.

- **Weak Maximum Principle:** *The maximum and the minimum value of u are attained on ∂U:*

$$\max_{x \in \overline{U}} u(x) = \max_{x \in \partial U} u(x) \quad \textit{and} \quad \min_{x \in \overline{U}} u(x) = \min_{x \in \partial U} u(x)$$

- **Strong Maximum Principle:** *If U is connected and there exists $x_0 \in U$ such that*

$$u(x_0) = \max_{x \in \overline{U}} u(x) \quad \textit{or} \quad u(x_0) = \min_{x \in \overline{U}} u(x)$$

  then $u \equiv const \equiv u(x_0)$.

Connected sets

A subset $W \subset \mathbb{R}^n$ is connected if for any pair of open and disjoint sets $U_1, U_2 \subset \mathbb{R}^n$ such that $W \subset U_1 \cup U_2$ it follows that either $W \cap U_1 = \emptyset$ or $W \cap U_2 = \emptyset$.

Theorem (Mean Value Property)

Let $u \in C^2(U)$ be harmonic for $U \subset \mathbb{R}^n$ open. Then, it holds

$$u(\mathbf{x}_0) = \frac{1}{Vol(B_r(\mathbf{x}_0))} \int_{B_r(x_0)} u(\mathbf{x}) d\mathbf{x} \quad \forall \mathbf{x}_0 \in U \text{ and } \forall B_r(\mathbf{x}_0) \subset U.$$

and

$$u(\mathbf{x}_0) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r(\mathbf{x}_0)} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r(\mathbf{x}_0)} \quad \forall x_0 \in U \text{ and } \forall B_r(\mathbf{x}_0) \subset U.$$

where ω_{n-1} is the $(n-1)$-dimensional surface of $\partial B_1(0) = \mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$.

Proof of the strong maximum principle.

Let $u \in C^2(U) \cap C^0(\overline{U})$ be harmonic for a connected and open subset $U \subset \mathbb{R}^n$.
Assume there exists $\mathbf{x}_0 \in U$ such that $u(\mathbf{x}_0) = \max_{\mathbf{x} \in \overline{U}} u(\mathbf{x}) =: M$.
We define

$$V = \{\mathbf{x} \in U : u(\mathbf{x}) = M\} \neq \emptyset.$$

as well $W = U \backslash V$.

Claim: V is open.

Proof of the Claim. Pick $\mathbf{x} \in V$ and $r > 0$ such that $B_r(\mathbf{x}) \subset U$. By the mean value property

$$M = u(\mathbf{x}) = \frac{1}{v_n(r)} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y} \leq M.$$

Hence

$$0 = \frac{1}{v_n(r)} \int_{\partial B_r(\mathbf{x})} (u(\mathbf{y}) - M) d\mathbf{y} = 0$$

Since $u(\mathbf{y}) - M \leq 0$, it follows $u(\mathbf{y}) = M$ on $B_r(\mathbf{x})$. Hence $B_r(\mathbf{x}) \subset V$.

Claim: $W = U \backslash V$ is open.

Proof of the Claim. If $\mathbf{x} \in W$, then $|u(\mathbf{x}) - M| > 0$. Since u is continuous there exists δ such that $|\mathbf{y} - \mathbf{x}| < \delta$ implies $|u(\mathbf{y}) - u(\mathbf{x})| \leq \frac{|M - u(\mathbf{x})|}{2}$. Then it follows that
$|u(\mathbf{y}) - M| \geq |u(\mathbf{x}) - M| - |u(\mathbf{y}) - u(\mathbf{x})| \geq \frac{|M - u(\mathbf{x})|}{2} > 0$. Hence $B_\delta(\mathbf{x}) \subset W$ and therefore W is open.

Finally, since U is connected and since V and W are open, either $V = \emptyset$ or $W = \emptyset$. But since $\mathbf{x}_0 \in V$ and therefore $V \neq \emptyset$, it follows $W = \emptyset$ and $U = V$. □

Poisson Formula

Our goal is to solve the Dirichlet problem on a disk $\overline{B_a(0)} = \{x \in \mathbb{R}^2 : |x|_2 \le a\}$ in $\mathbb{R}^2$.

Let $B_a(0) = \{x \in \mathbb{R}^2 : |x|_2 < a\}$, and $\partial B_a(0) = \overline{B_a(0)} \backslash B_a(0)$.

Precisely

Dirichlet Problem for the Laplace equation on $B_a(0)$

Let $h \in C^0(\partial B_a(0))$.

Find $u \in C^2(B_a(0)) \cap C^0(\overline{B_a(0)})$ such that

$$\begin{aligned} u_{x,x} + u_{y,y} &= 0 \quad \text{in } B_a(0) \\ u &= h \quad \text{in } \partial B_a(0) \end{aligned}$$

For that we first consider $u_{x,x} + u_{y,y} = 0$ in polar cooardinates:

$$0 = \tilde{u}_{r,r} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta,\theta}$$

where $\tilde{u}(r,\theta) = u(r\cos\theta, r\sin\theta)$ and $r > 0$ and $\theta \in \mathbb{R}$.

Similar, we can rewrite the boundary data h as $\tilde{h}(\theta) = h(a\cos\theta, a\sin\theta)$.

Note that $\tilde{h}(\theta)$, $\theta \in \mathbb{R}$, is 2π-periodic.

We apply the method of separation of variables to the Laplace equation in polar coordinates (compare with exercise): Assume $\tilde{u}(r,\theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) = -\frac{1}{r^2}R(r)\Theta''(\theta)$$

Hence

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda = const.$$

The general solution for Θ is

$$\Theta(\theta) = \begin{cases} A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta), & \lambda > 0, \\ A + Bx, & \lambda = 0, \\ A\cosh(\sqrt{-\lambda}\theta) + B\sinh(\sqrt{-\lambda}), & \lambda < 0 \end{cases}$$

Since Θ is periodic, we only have to consider the cases $\lambda > 0$ and $\lambda = 0$ for $B = 0$.

Moreover, by evaluation of the function for the points 0 and 2π we see that $\lambda = n^2$, $n \in \mathbb{N} \cup \{0\}$.

The equation for R becomes

$$r^2 R''(r) + rR'(r) - n^2 R(0) = 0.$$

Solutions for $n \in \mathbb{N}$ are r^n, r^{-n} and $Cr^n + Dr^{-n}$ for $C, D \in \mathbb{R}$, and $\log r$, C and $C + D \log r$ for $n = 0$.

Since we are looking for smooth solutions u on $B_a(0)$ that are continuous we can assume that $D = 0$.

Now we consider infinite sums of the form

$$\tilde{u}(r, \theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta) \right). \tag{1}$$

Finally, let us bring the boundary condition into play. At $r = a$ we require

$$\tilde{h}(\theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos(n\theta) + B_n \sin(n\theta) \right). \tag{2}$$

So, assuming that $\tilde{h} \in C^1(\mathbb{R})$ (and 2π-periodic) this is the full Fourier series that converges uniformily and the Fourier coefficients are uniquely determined by the formulas

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{h}(\phi) d\phi, \ A_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} \tilde{h}(\theta) \sin(n\phi) d\phi, \ B_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} \tilde{h}(\phi) \cos(n\phi) d\phi.$$

Uniform convergence of (2) implies uniform convergence of (1).

By replacing A_n and B_n with the Fourier coefficients of h we can rewrite the formula for u as

$$\begin{aligned}\tilde{u}(r,\theta) &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n(\sin(n\theta)\sin(n\phi)+\cos(n\phi)\cos(n\theta))\right]h(\theta)d\theta \\ &= \int_{-\pi}^{\pi}\left[1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n(\theta-\phi))\right]\frac{d\pi}{2\pi}\end{aligned}$$

Recall $\cos\theta = \frac{e^{i\theta}+e^{i\theta}}{2}$ and the formula $\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$ for $z \in \mathbb{C}$ with $|z| < 1$. Hence

$$\begin{aligned}1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n(\theta-\phi)) &= 1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} \\ &= 1+\frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}+\frac{re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}} \\ &= 1+\frac{re^{i(\theta-\phi)}(a-re^{-i(\theta-\phi)})-re^{-i(\theta-\phi)}(a-re^{i(\theta-\phi)})}{(a-re^{i(\theta-\phi)})(a-re^{-i(\theta-\phi)})} \\ &= \frac{a^2-r^2}{a^2-ar2\cos(\theta-\phi)+r^2}.\end{aligned}$$

We get

Poisson solution formula for the Laplace equation on the disk

$$\tilde{u}(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h(\phi)(a^2-r^2)}{a^2-2ar\cos(\theta-\phi)+r^2}d\phi$$

We can also write this formula again in Euclidean coordinates.

For that note that an infinitesimal length segement of the boundary $\partial B_a(0)$ is given by $ds = a d\phi$ where $d\phi$ is the infinitesimal angle of the segment ds.

Also note that for $\mathbf{x} = (r, \theta)$ and $\mathbf{y} = (s, \phi)$ we have

$$|\mathbf{x} - \mathbf{y}|^2 = r^2 + s^2 - 2rs\cos(\theta - \phi)$$

by the cosine rule. It follows that

Poisson formula, second version

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{\partial B_a(0)} \frac{u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} ds(\mathbf{y}).$$

Theorem

Let $h \in C^0(\partial B_a(0))$ be given in polar coordinates by $h(a\cos\theta, a\sin\theta) = \tilde{h}(\theta)$ for $\tilde{h} \in C^0(\mathbb{R})$ that it 2π periodic. Then the Poisson formula provides the unique harmonic function on $B_a(0)$ for which

$$\lim_{x \to x_0} u(\mathbf{x}) = h(\mathbf{x}_0)\ \forall \mathbf{x}_0 \in \partial B_a(0).$$

Proof of the Theorem

Uniqueness follows by the weak maximum principle.

Given the $\tilde{h}$ as in the theorem the Poisson formula yields

$$\tilde{u}(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta-\phi)\tilde{h}(\phi)\frac{d\phi}{2\pi} = \int_{-\pi}^{\pi} P(r,\phi)\tilde{h}(\theta-\phi)\frac{d\phi}{2\pi} \tag{3}$$

where $P(r,\theta) = \frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}$ is the Poisson kernel.

We have 3 important facts

- $P(r,\theta) > 0$ because $0 < r < a$ and $a^2 - 2ar\cos\theta + r^1 \geq a^2 - 2ar + r^2 = (a-r)^2$.
- $\int_{-\pi}^{\pi} P(r,\theta)\frac{d\theta}{2\pi} = 1$ by piecewise integration of the previous series.
- $P(r,\theta)$ solve the Laplace equation on $B_a(0)$. Moreover $P(r,\theta) \in C^2([0,a)\times\mathbb{R})$.

The last fact allows us to differentiate under the integral in (3) and we can check that

$$\tilde{u}_{r,r} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta,\theta} = \int_{-\pi}^{\pi} \underbrace{\left[\frac{\partial^2}{\partial r^2}P(r,\theta-\phi) + \frac{1}{r}\frac{\partial}{\partial r}P(r,\theta-\phi) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}P(r,\theta-\phi)\right]}_{=0}\tilde{h}(\phi)\frac{d\phi}{2\pi}.$$

So $\tilde{u}$ is harmonic on $B_a(0)$.

It remains to prove that $\tilde{u}(r,\theta) \to h(\theta_0)$ if $(r,\theta) \to (a,\theta)$.

For that let us consider $r \in [0,a)$ such that $a - r < \delta$. We have

$$u(r,\theta) - h(\theta_0) = \int_{-\pi}^{\pi} P(r,\theta-\phi)[h(\phi) - h(\theta_0)]\frac{d\phi}{2\pi}$$

by the second fact.

But $P(r,\theta)$ is concentrated in $\theta = 0$ in the sense that for $\theta \in (\delta/2, 2\pi - \delta/2)$ we have

$$|P(r,\theta)| = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2} = \frac{a^2 - r^2}{(a-r)^2 + 4ar\sin^2(\theta/2)} < \epsilon. \tag{4}$$

for some $\delta > 0$ and $a - r$ small. (We used $1 - \cos\theta = \cos(\frac{\theta}{2} - \frac{\theta}{2}) - \cos(\frac{\theta}{2} + \frac{\theta}{2}) = -2\sin^2(\frac{\theta}{2})$)

Now we break the integral into two pieces:

$$|u(r,\theta) - h(\theta_0)| \leq \int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta-\phi)|h(\phi) - h(\theta_0)|\frac{d\phi}{2\pi} + \int_{|\phi-\theta_0|>\delta} P(r,\theta-\phi)|h(\phi) - h(\theta_0)|\frac{d\phi}{2\pi}$$

Given $\epsilon > 0$ we can choose $\delta > 0$ small such that $|h(\phi) - h(\theta_0)| < \epsilon$ for $|\phi - \theta_0| < \delta$.

Hence, the first integral can be estimated by

$$\int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta-\phi)\epsilon\frac{d\phi}{2\pi} \leq \int_{-\pi}^{\pi} P(r,\theta-\phi)\frac{d\phi}{2\pi} = \epsilon.$$

For the second integral we use (4) and that h is bounded on $\partial B_a(0)$ by a constant M:

$$\int_{|\phi-\theta_0|>\delta} P(r,\theta-\phi)2M\frac{d\phi}{2\pi} \leq \epsilon 2M$$

provided $|\theta - \theta_0| < \frac{\delta}{2}$.

Application: Mean Value Property, 2n Proof

Let u be harmonic on U and let $B_r(\mathbf{x}_0) \subset U$.
We replace $u(\mathbf{x})$ with $u(\mathbf{x} - \mathbf{x}_0)$ and $B_r(\mathbf{x}_0)$ and U with $B_r(0)$ with $U - \mathbf{x}_0$. By Poisson's formula

$$u(0) = \frac{r^2 - 0^2}{2\pi r} \int_{\partial B_r(0)} \frac{u(\mathbf{y})}{|\mathbf{y} - 0|^2} ds(\mathbf{y}) = \frac{r^2}{2\pi r} \int_{\partial B_r(0)} \frac{u(\mathbf{y})}{r^2} ds(\mathbf{y}) = \frac{1}{\omega_1 r} \int_{\partial B_r(0)} f(\mathbf{y}) ds(\mathbf{y}).$$