MAT351 Partial Differential Equations
Lecture 23

January 11, 2021

Dirichlet Problem on a Disk

$$B_a(0) = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |\mathbf{x}| < a\} = B_a, \quad \overline{B_a(0)} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |\mathbf{x}| \leq a\} = \overline{B_a}.$$

Moreover $\overline{B_a} \backslash B_a =: \partial B_a$.

Dirichlet Problem for the Laplace equation on B_a

Let $h \in C^0(\partial B_a)$. Find $u \in C^2(B_a) \cap C^0(\overline{B_a})$ such that

$$\Delta u = u_{x_1,x_1} + u_{x_2,x_2} = 0 \text{ in } B_a \text{ \& } u = h \text{ in } \partial B_a. \tag{1}$$

Polar coordinates: $(x_1, x_2) = (r\cos\theta, r\sin\theta),\ r \in (0,\infty), \theta \in \mathbb{R}$.

Dirichlet Problem for the Laplace equation on B_a in polar coordinates

Let $\tilde{h}(\theta) \in C^0(\mathbb{R})$ with $\tilde{h}(\theta + 2\pi) = \tilde{h}(\theta)$ (for isnstance we choose $\tilde{h}(\theta) = h(a\cos\theta, a\sin\theta)$).
Find $\tilde{u} \in C^2((0,a) \times \mathbb{R}) \cap C^0((0,a] \times \mathbb{R})$ such that

$$\tilde{\Delta}\tilde{u} = \tilde{u}_{r,r} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta,\theta} = 0 \text{ in } (0,a) \times \mathbb{R} \text{ \& } \tilde{u}(a,\theta) = \tilde{h}(\theta) \text{ for } \theta \in \mathbb{R}. \tag{2}$$

and

$$\tilde{u}(r,\theta) = \tilde{u}(r, \theta + 2\pi) \text{ for } r \in (0,\infty), \theta \in \mathbb{R} \text{ \& } \limsup_{r \to 0} |\tilde{u}(r,\theta)| < \infty.$$

Theorem (Poisson Formula in Polar coordinates)

The unique solution of the Dirichlet Problem (2) *is given by the Poisson formula*

$$\tilde{u}(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\tilde{h}(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\, d\phi = \int_0^{2\pi} P(r,\theta-\phi)\tilde{h}(\phi)\frac{d\phi}{2\pi}.$$

where

$$P(r,\theta) = \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta)}$$

is called the **Poisson kernel***.*

Theorem (Poisson Formula in Euclidean coordinates)

The unique solution $u \in C^2(B_a) \cap C^0(\overline{B_a})$ *of* (1) *is given by*

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{\partial B_a} \frac{h(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\, ds(\mathbf{y}).$$

In particular, we showed that

$$\tilde{u}(r,\theta) \to \tilde{h}(\theta_0) \text{ if } (r,\theta) \to (a,\theta_0).$$

In fact, from the proof we can see that this convergence is uniform w.r.t. θ:

$$\sup_{\theta_0} |\tilde{u}(r,\theta_0) - \tilde{h}(\theta_0)| \to 0 \text{ if } r \to a.$$

Consequences of the Poisson formula in polar coordinates

Recall the L^2 or **mean square norm**

$$\left\|\tilde{h}\right\|_2 = \sqrt{\int_0^{2\pi} (\tilde{h}(\theta))^2 d\theta}.$$

for $\tilde{h} \in C^0(\mathbb{R})$ that is 2π-periodic.

Theorem (Mean square convergence of the Full Fourier series)

Let $\tilde{h} \in C^0(\mathbb{R})$ that is 2π- periodic. Then, the full Fourier serie of $\tilde{h}$ converges in L^2 or mean square sense to $\tilde{h}$. More precisely

$$\left\|\tilde{h} - \mathcal{S}^N\right\|_2 \to 0 \text{ for } N \to \infty.$$

where $\mathcal{S}^N$ are the partial sums of the Fourier serie of $\tilde{h}$.

Proof.

In the proof of the Poisson formula we saw that

$$P(r, \theta - \phi) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (\sin(n\theta)\sin(n\phi) + \cos(n\theta)\cos(n\phi))$$

uniformly in $\phi, \theta \in \mathbb{R}$ and $r \in [c, d] \subset (0, a)$.

We insert this back into Poisson's formula. By uniform convergence we can exchange integration w.r.t. ϕ and summation w.r.t. $n \in \mathbb{N}$. It follows

$$\tilde{u}(r,\theta) = \underbrace{\int_0^{2\pi} h(\phi)\frac{d\phi}{2\pi}}_{=A_0(r)} + \sum_{n=1}^{\infty} \underbrace{\left(\left(\frac{r}{a}\right)^n \int_0^{2\pi} \tilde{h}(\phi)\cos(n\phi)\frac{d\phi}{\pi}\right)}_{=:A_n(r)} \cos(n\theta)$$

$$+ \sum_{n=1}^{\infty} \underbrace{\left(\left(\frac{r}{a}\right)^n \int_0^{\pi} \tilde{h}(\phi)\sin(n\phi)\frac{d\phi}{2\pi}\right)}_{=:B_n(r)} \sin(n\theta). \tag{3}$$

The right hand side is still a uniformily converging series in $\theta \in \mathbb{R}$. Hence, after multiplying with $\sin(k\theta)$, $k \in \mathbb{N}$, $\cos(k\theta)$, $k \in \mathbb{N}$ or 1, and then integrating w.r.t. θ over $[0, 2\pi]$ we get that $A_n(r)$ and $B_n(r)$ are the Fourier coefficients of $\theta \mapsto \tilde{u}(r,\theta)$ and (3).

We denote the Fourier partial sums $\mathcal{S}^N(r,\theta) = A_0(r) + \sum_{n=1}^N (A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta))$.

$\mathcal{S}^N(r,\cdot)$ converges uniformly to $\tilde{u}(r,\cdot)$ if $0 < r < a$.

Note that $A_n(a)$, $B_n(a)$ become the Fourier coefficients of $\tilde{h}$ and that

$$A_n(r) = \left(\tfrac{r}{a}\right)^n A_n(a),\ B_n(r) = \left(\tfrac{r}{a}\right)^n B_n(a),\ n \in \mathbb{N}.$$

Hence $\mathcal{S}^N(a,\cdot) = \mathcal{S}^N(\cdot)$ are the partial sums of the Fourier serie of $\tilde{h}$.

But we don't know if $\mathcal{S}^N$ converges uniformly to $\tilde{h}$.

Recall

$$\left\|\tilde{h}\right\|_2 = \sqrt{\int_0^{2\pi} (\tilde{h}(\theta))^2 d\theta} \leq \sup_{\theta \in \mathbb{R}} |\tilde{h}(\theta)|.$$

It follows

$$0 < \left\|\tilde{h} - \mathcal{S}^N(\cdot)\right\|_2 \leq \underbrace{\left\|\tilde{h} - \tilde{u}(r,\cdot)\right\|_2}_{\leq \sup_{\theta\in\mathbb{R}}|\tilde{h}(\theta)-\tilde{u}(r,\theta)|} + \underbrace{\left\|\tilde{u}(r,\cdot) - \mathcal{S}^N(r,\cdot)\right\|_2}_{\leq \sup_{\theta\in\mathbb{R}}|\tilde{u}(r,\theta)-\mathcal{S}^N(r,\theta)|} + \underbrace{\left\|\mathcal{S}^N(r,\cdot) - \mathcal{S}^N(\cdot)\right\|_2}_{\leq |(\frac{r}{a})^n - 1| \|\mathcal{S}^N(\cdot)\|_2}.$$

Given $\eta > 0$ we pick $r \in (0, a)$ close to a such that

$$\sup_{\theta} \left|\tilde{h}(\theta) - u(r,\theta)\right| \leq \eta \quad \& \quad \left|\left(\frac{r}{a}\right)^n - 1\right| \leq \eta$$

Moreover, by Bessel's inequality, we have

$$\left\|\mathcal{S}^N(\cdot)\right\|_2^2 = A_0^2 + \sum_{n=1}^{N}(A_n^2 + B_n^2) \leq \left\|\tilde{h}\right\|_2^2.$$

Hence

$$\left\|\tilde{h} - \mathcal{S}^N(\cdot)\right\|_2 \leq \eta + \sup_{\theta\in\mathbb{R}} |\tilde{u}(r,\theta) - \mathcal{S}^N(r,\theta)| + \eta \left\|\tilde{h}\right\|_2$$

Therefore

$$\limsup_{N\to\infty} \left\|\tilde{h} - \mathcal{S}^N(\cdot)\right\|_2 \leq \limsup_{N\to\infty} \left(\eta + \sup_{\theta\in\mathbb{R}} |\tilde{u}(r,\theta) - \mathcal{S}^N(r,\theta)| + \eta \left\|\tilde{h}\right\|_2\right) \leq \eta(1 + \left\|\tilde{h}\right\|_2).$$

□

More Consequences

Let $\Omega \subset \mathbb{R}^2$ be open. $u \in C^2(\Omega)$ harmonic if and only if $\Delta u = 0$ on Ω.

Mean value property

Let $u \in C^2(\Omega)$ be harmonic. Then

$$u(\mathbf{x}) = \frac{1}{2\pi a} \int_{\partial B_a(\mathbf{x})} u(\mathbf{y}) ds(\mathbf{y}) = \frac{1}{\mathrm{vol}(B_a(\mathbf{x}))} \int_{B_a(\mathbf{x})} u(\mathbf{y}) d\mathbf{y}$$

for any $a > 0$ such that $B_a(\mathbf{x}) \subset \Omega$.

Liouville theorem

Let $u \in C^2(\mathbb{R}^2)$ be harmonic and $\sup |u| \leq C < \infty$.

Then $u(\mathbf{x}) \equiv const$.

Theorem

Let $\Omega \subset \mathbb{R}^2$ open and $u \in C^0(\Omega)$ such that the mean value property holds:

$$u(\mathbf{x}) = \frac{1}{2\pi a} \int_{\partial B_a(\mathbf{x})} u(\mathbf{y}) ds(\mathbf{y}) \quad \text{for } a > 0 \text{ whenever } B_a(\mathbf{x}) \subset \Omega.$$

It follows $u \in C^\infty(\Omega)$ and u is harmonic.