Theorem

$h \in C^0(\mathbb{R})$ 2π-periodic

Then $\frac{1}{2} A_0 + \sum_{n=1}^{N} A_n \cos(nx) + B_n \sin(nx)$

$\longrightarrow h$ in L^2-sense

Corollary (Parseval equality)

$$\frac{\pi}{2} A_0^2 + \sum_{n=1}^{\infty} A_n^2 \int_0^{2\pi} \cos(nx)^2 dx + B_n^2 \int_0^{2\pi} \sin(nx)^2 dx$$

$$= \int_0^{2\pi} h(x)^2 dx$$

Proof: least square approximation

Application: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2$

$$h(x) = x \qquad x \in [-\pi, \pi]$$

$$\int_{-\pi}^{\pi} h^2(x)\,dx = \int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^3$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = 0$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cdot\sin(nx)\,dx = (-1)^{n+1}\,\frac{2}{n}$$

$$\frac{2}{3}\pi^3 = \sum_{n=1}^{\infty} B_n^2 \underbrace{\int_{-\pi}^{\pi} \sin(nx)^2\,dx}_{\pi} = \sum_{n=1}^{\infty} \frac{4}{n^2}\pi$$

Theorem (Liouville)

Let $u \in C^2(\mathbb{R}^2)$ u harmonic s.t. $|u| \leq c < \infty$

then $u = \text{const}$

Proof: $x, y \in \mathbb{R}^2$, $r > 0$, $R = r + |x-y|$

$\Longrightarrow B_r(x) \subset B_R(y) \subset \mathbb{R}^2$

$z \in B_r(x)$: $|z-y| \leq \underbrace{|x-z|}_{r} + |x-y| < R$

$\hat{u} = u + c \geq 0$

MVP: $\hat{u}(x) = \frac{1}{vol(B_r(x))} \int_{B_r(x)} \hat{u}(z)\, dz \leq \frac{1}{vol(B_r(x))} \int_{B_R(y)} \hat{u}(z)\, dz$

$= \underbrace{\frac{vol(B_R(y))}{vol(B_r(x))}}_{\left(\frac{r+|x-y|}{r}\right)^2 \longrightarrow 1} \hat{u}(y) \Longrightarrow \hat{u}(x) \leq \hat{u}(y)$

$u(x) \underset{=}{\leq} u(y)$ ☒

__Theorem__

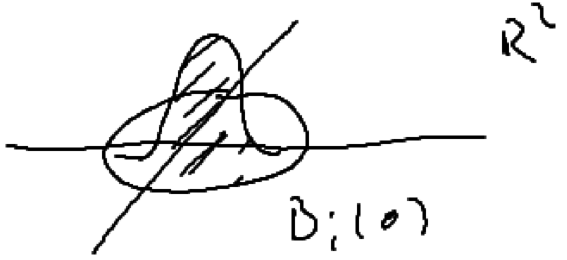
$u \in C^0(\Omega)$, Ω open in $\mathbb{R}^2$. u satisfies MVP

$\forall x \in \Omega$: $u(x) = \frac{1}{2\pi r}\int_{\partial B_r(x)} u(x)\,dx = \frac{1}{\mathrm{Vol}(B_r(x))}\int_{B_r(x)} u(x)\,dx$ $\forall r > 0$ s.t. $B_r(x) \subset \Omega$

Then $u \in C^\infty(\Omega)$ and u harmonic.

__Proof:__ $\varphi \in C_0^\infty(B_1)$ st. $\int_{B_1} \varphi(x)\,dx = 1$

and φ is radial $\varphi(x) = \psi(|x|)$



Transformation polar coord $x=(x_1,x_2)=(r\cos\Theta, r\sin\Theta)$

$$1 = \int_{B_1} \varphi(x)\,dx = \int_0^{2\pi}\int_0^1 \psi(r)\cdot r\,dr\,d\Theta$$

$$= 2\pi \int_0^1 \underbrace{\psi(r)\,r}\,dr$$

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2}\varphi\left(\frac{x}{\varepsilon}\right) \Rightarrow \varphi_\varepsilon \in C_0^\infty(B_\varepsilon(0))$$

$\varepsilon > 0$ $B_\varepsilon(y) \subset \Omega$ $y \in \Omega$

$$\int_{B_\varepsilon(y)} u(z)\,\varphi_\varepsilon(z-y)\,dz = \int_{B_\varepsilon(0)} \underbrace{u(z+y)}_{f(z)\ \rightsquigarrow\ \tilde{f}(r,\Theta)} \varphi_\varepsilon(z)\,dz$$

$$= \int_0^{2\pi}\int_0^{\varepsilon} \tilde{f}(r,\Theta)\cdot\frac{1}{\varepsilon^2}\psi\left(\frac{r}{\varepsilon}\right) r\,dr\,d\Theta$$

$$\int_0^\varepsilon \underbrace{r\int_0^{2\pi} \tilde{f}(r,\theta)\, d\theta}\ \frac{1}{\varepsilon^2}\psi\left(\frac{r}{\varepsilon}\right) dr$$

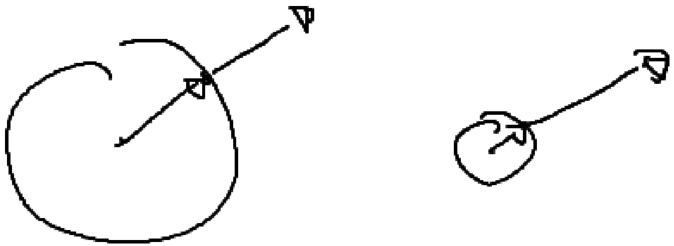
$$\int_{\partial B_r(y)} u(z)\, ds(y) = u(y)\cdot 2\pi\cdot r$$

$$= u(y)\, 2\pi \underbrace{\int_0^\varepsilon \frac{1}{\varepsilon^2}\psi\left(\frac{r}{\varepsilon}\right) r\, dr} \qquad = u(y)$$

$$\Rightarrow \quad u(y) = \int_{B_\varepsilon(y)} u(z)\, \varphi_\varepsilon(z-y)\, dz = \int_{\mathbb{R}^2} u(z)\, \varphi_\varepsilon(z-y)\, dz$$

$$\frac{\partial u}{\partial x_1}(y) = \int_{B_\varepsilon(y)} u(y) \frac{\partial}{\partial x_1} \varphi_\varepsilon(z-y)\, dz$$

<u>u is harmonic</u>



$x \in \Omega \quad \forall r > 0 \quad B_r(x) \subset \Omega$

$$u(x) = \frac{1}{2\pi r}\int_{\partial B_r(x)} u(y)\,dy = \frac{1}{2\pi \cancel{r}} \int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,\cancel{r}\,d\theta$$

$$0 = \frac{d}{dr}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,d\theta = \int_0^{2\pi} \Big\langle \underbrace{\nabla u}_{\nabla u(r\cos\theta,\, r\sin\theta)}, \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\Big\rangle\, d\theta$$

$$= \frac{1}{r}\int_0^{2\pi}\langle \nabla u, \underset{\substack{\uparrow \\ N(r\cos\theta,\, r\sin\theta)}}{N}\rangle\, r\, d\theta$$

$$= \frac{1}{r}\int_{\partial B_r}\langle \nabla u, N\rangle\, ds = \frac{1}{r}\int_{B_r(x)} \Delta u\, dx \Rightarrow \Delta u = 0 \quad \square$$

Green identities

① Let $u, v \in C^2(\Omega)$

$$\nabla\cdot(u\cdot\nabla v) = \mathrm{div}(u\nabla v) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(u \frac{\partial v}{\partial x_i}\right)$$

$$= \langle \nabla v, \nabla u\rangle + u\,\Delta v$$

$U \subset \Omega$ smooth bdy

$$\int_U \langle \nabla v, \nabla u\rangle\, dx + \int_U u\,\Delta v\, dx = \int_U \nabla(u\nabla v)\, dx$$

$$= \int_{\partial U} \langle N, u\nabla v\rangle\, ds = \int_{\partial U} u\langle N, \nabla v\rangle\, ds$$

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$$\int_{\Omega} v \cdot \Delta u \, dx - \int_{\Omega} \Delta v \cdot u \, dx = \int_{\partial \Omega} v \underbrace{\frac{\partial u}{\partial N}}_{\langle \nabla u , N \rangle} ds - \int_{\partial \Omega} u \frac{\partial v}{\partial N} ds$$

Dirichlet Principle

Physical idea: Among all fcts u on Ω that describe a possible state of a physical system, the preferred state is the one with minimal kinetic Energy.

States: $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with

$u|_{\partial\Omega} = h \qquad h \in C^0(\partial\Omega)$

kinetic Energy: $\quad E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx$

Theorem

u is the unique harmonic fct with $u|_{\partial\Omega} = h$ if and only if it minimizes E wrt all $w \in C^0(\bar\Omega) \cap C^2(\Omega)$ $w|_{\partial\Omega} = h$

Precisely $\quad E(u) \leq E(w) \quad \forall\, w \in C^0(\bar\Omega)\cap C^2(\Omega),\ w|_{\partial\Omega} = h \quad$ "$\mathcal{E}$"

Proof "$\Longrightarrow$" u harmonic, $w \in \mathcal{E}$

$\Longrightarrow \quad u - w = v \qquad v|_{\partial\Omega} = 0$

$$E(w) = \frac{1}{2}\int |\nabla(u-v)|^2 dx = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - 2\langle \nabla u, \nabla v\rangle + |\nabla v|^2\right) dx$$

$$= E(u) + \underbrace{E(v)}_{\geq 0} + \int_\Omega v \cdot \underbrace{\Delta u}_{=0}\, dx - \int_{\partial\Omega} v \cdot \underbrace{\frac{\partial u}{\partial N}}_{=0}\, ds$$