Last Lecture: Green Identities

$\Omega \subset \mathbb{R}^n$ open, smooth edge $\partial\Omega$

$u \in C^2(\bar{\Omega})$, $v \in C^1(\bar{\Omega})$: $\nabla \cdot (v \nabla u) = \langle \nabla v . \nabla u \rangle + v \Delta u$

(1) $$\int_\Omega \langle \nabla v . \nabla u \rangle \, dx + \int_\Omega v \Delta u \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial N} \, ds$$

$\langle N . \nabla u \rangle = \dfrac{\partial u}{\partial N}$ on $\partial\Omega$.

N Normal vector field along $\partial\Omega$

(2) $u, v \in C^2(\bar{\Omega})$

$$\int_\Omega (v \Delta u - u \Delta v) \, dx = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds$$

Theorem (Dirichlet Principle)

Ω as above. $u \in C^2(\bar{\Omega})$ is the unique harmonic fct on Ω with $u|_{\partial\Omega} = h$ iff $E(u) \leq E(w)$ $\forall w \in \{ w \in C^1(\bar{\Omega}) : w|_{\partial\Omega} = h \} =: \mathcal{E}$

Proof: "$\Longleftarrow$" $\varphi \in C^1_c(\Omega) = \{\varphi \in C^1(\Omega) : \overline{\{x \in \Omega : \varphi(x) \neq 0\}} \subset \Omega \text{ bounded}\}$

$u + t\varphi \in \mathcal{E}$ $\quad (u + t\varphi)|_{\partial\Omega} = g \quad \forall t \in \mathbb{R}$

$$E(u + t\varphi) = \frac{1}{2}\int_\Omega |\nabla(u + t\varphi)|^2 \, dx$$

$E(u) \leq E(w)$ $\forall w \in \mathcal{E}$

$$= \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}t^2 \int_\Omega |\nabla \varphi|^2 \, dx + \frac{1}{2} t \int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dx$$

$$0 = \frac{d}{dt}\Big|_0 E(u + t\varphi) = \frac{1}{2}\int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dx$$

$$= -\frac{1}{2}\int_\Omega \varphi \cdot \Delta u \, dx + \underbrace{\int_{\partial\Omega} \varphi \cdot \frac{\partial u}{\partial N} \, ds}_{=0} \qquad \forall \varphi \in C^1_c$$

FTCV $\Longrightarrow$ $\Delta u = 0$ on Ω ∎

Remark: $n = 1$; $\Omega = (0, c)$

$$\frac{1}{i}\int_0^c u' v' \, dx = \int_0^c v \, u'' \, dx - [v(c) u'(c) - v(0) u'(0)]$$

Theorem (Representation Formula)

$n = 3$, $\Omega \subset \mathbb{R}^3$ open, smooth bdy

Let $u \in C^2(\bar{\Omega})$ harmonic. $x \in \Omega$

Then $$u(x) = \int_{\partial\Omega} \left(-u(y) \frac{\partial}{\partial N}\Big|_y \left(\frac{1}{|\cdot - x|} \right) + \frac{1}{|y-x|} \frac{\partial u}{\partial N}(y) \right) \frac{ds(y)}{4\pi}$$

Proof $\Phi(x) = -\frac{1}{4\pi} \frac{1}{|x|}$ harmonic on $\mathbb{R}^3 \setminus \{0\}$

$\Longrightarrow$ $v(y) = \Phi(y - x)$ " " $\mathbb{R}^3 \setminus \{x\}$

$\varepsilon > 0 : \; B_\varepsilon(x) \subset \Omega \, , \; \Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon(x)}$

Green 2nd Id:

$$0 = \int_{\Omega_\varepsilon} (v \Delta u - u \Delta v)\, dx = \int_{\partial \Omega_\varepsilon} \left(v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds$$

$$= \underbrace{\int_{\partial \Omega} \left(v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds}_{\text{RHS in Repres F.}} + \underbrace{\int_{\partial B_\varepsilon(x)} \left(v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds}$$

$$(x) = \int_{\partial B_\varepsilon(0)} \left(-\frac{1}{4\pi |y|} \underbrace{\frac{\partial u}{\partial N}(y+x)}_{f(y)} + \underbrace{u(y+x)}_{g(y)} \underbrace{\frac{\partial^{(x)}}{\partial N}\left(\frac{1}{4\pi |y|} \right)}_{\frac{d}{dr}\frac{1}{4\pi r} = \frac{-1}{4\pi r^2}} \right) ds$$

$(r\cos\theta \sin\phi , \; r \sin\theta \sin\phi , \; r\cos\phi)$

$\tilde{f}(r, \theta, \phi) \quad \tilde{g}(r, \theta, \phi)$

$$= \underbrace{\int_0^{2\pi}\int_0^{\pi}\left(-\frac{1}{4\pi\varepsilon}\tilde{g}(\varepsilon,\theta,\phi)\right.}_{A} + \underbrace{\left.\tilde{g}'(\varepsilon,\theta,\phi)\frac{-1}{4\pi\varepsilon^2}\right)\varepsilon^2\sin\phi \, d\phi \, d\theta}_{B}$$

$$|A| \leq \int_0^{2\pi}\int_0^{\pi}\frac{\varepsilon}{4\pi}\left|\tilde{g}(\varepsilon,\theta,\phi)\right|\sin\phi \, d\phi \, d\theta = (\times\times)$$

$$|g(y)| = |\langle N(y+x), \nabla u(y+x)\rangle| \leq |\nabla u|(y+x)$$

∇u is continuous $\exists \, \varepsilon_0 > 0$ $\quad |\nabla u|(y+x) \leq |\nabla u|(x) + C$

$\forall y \in \partial B_\varepsilon(0) \quad \varepsilon \in (0, \varepsilon_0)$

$$(\times\times) \leq \frac{\varepsilon}{4\pi}\int_0^{2\pi}\int_0^{\pi}(|\nabla u|(x) + C)\sin\phi \, d\phi \, d\theta$$

$$= \varepsilon\,(|\nabla u|(x) + C) \longrightarrow 0$$

$$B = \underbrace{\frac{1}{4\pi\varepsilon^2}\int_0^{2\pi}\int_0^{\pi} g(\varepsilon, \theta, \varphi)\,\varepsilon^2 \sin\phi \, d\phi \, d\theta}_{\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon(x)} u(y)\, ds(y) \;=\; u(x)}$$

$\varepsilon \to 0$

$$\Longrightarrow \quad 0 = + \int_{\partial\Omega} \left(v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds \; - \; u(x)$$