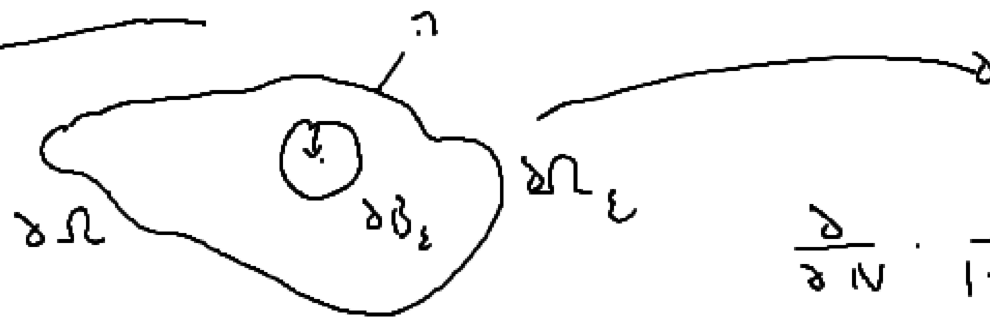
Last time: Representation formula $n=3$

If $u \in C^2(\bar\Omega)$ solves $\Delta u = 0$ on Ω, $x \in \Omega$

$$\Longrightarrow \quad u(x) = \int_{\partial\Omega} \left[u \frac{\partial \Phi(\cdot - x)}{\partial N} - \Phi(\cdot - x) \frac{\partial u}{\partial N} \right] dS$$

with $\Phi(x) = -\frac{1}{4\pi}\frac{1}{|x|}$ and Ω open, bounded with smooth bdy.

Important



$$\frac{\partial}{\partial N} \frac{1}{|\cdot - x|} = -\frac{d}{dr}\frac{1}{r}$$

Def

$$x \in \mathbb{R}^n \setminus \{0\} \longmapsto \Phi_n(x) = \begin{cases} \frac{1}{2\pi} \log|x| & n = 2 \\ \frac{1}{n(n-2)} \frac{1}{\mathrm{vol}(B_1(0))} \cdot \frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

is called the fundamental solution of the Laplace equation.

$$\Phi_3(x) = \frac{1}{3 \cdot 1} \frac{1}{\frac{4}{3}\pi} \cdot \frac{1}{|x|} = \Phi(x)$$

Remark: The Representation formula holds $\forall n \geq 2$ with Φ_n replacing Φ

For $n = 2$ it is important $\varepsilon \cdot C_n(\varepsilon) \to 0$ $\varepsilon \to 0$

Recall Dirichlet Problem: $\Delta u = 0$ on Ω
$u|_{\partial\Omega} = g$

Goal: Improve Repres. for. st RHS does not depend on $\frac{\partial u}{\partial N}$

Def: (Green fct)
The Green fct $G(x)$ for the operator Δ and the domain Ω at a point $x_0 \in \Omega$ is a fct $G : \bar{\Omega} \setminus \{x_0\} \to \mathbb{R}$ s.t.

(i) $G \in C^2(\bar{\Omega} \setminus \{x_0\})$ and $\Delta G = 0$ on $\bar{\Omega} \setminus \{x_0\}$

(ii) $G|_{\partial\Omega} = 0$

(iii) $\underbrace{G(x) - \Phi_n(x - x_0)}_{= H}$ is finite at $x_0 \in \Omega$
st. $H \in C^2(\bar{\Omega})$ and $\Delta H = 0$ on Ω.

Remark: It can be shown the Green fct ex. and is unique.

Idea: Solve $\Delta H^{x_0} = 0$ on Ω
$H^{x_0}|_{\partial\Omega} = -\Phi_n(\cdot - x_0)|_{\partial\Omega}$
and then define $G(x) = H^{x_0}(x) + \Phi_n(x - x_0) = G(x, x_0)$

Theorem: Let $u \in C^1(\overline{\Omega})$ be a solution of

$$\Delta u = 0 \text{ on } \Omega$$

$$u|_{\partial\Omega} = h$$

Then $u(x) = \displaystyle\int_{\partial\Omega} \underbrace{u(x)}_{h} \cdot \frac{\partial}{\partial N} G(x, x_0)\, dS(x)$. $\quad x \in \Omega$

Proof: $n = 3$

$$u(x) = \int_{\partial\Omega} \left[u \frac{\partial \Phi(\cdot - x)}{\partial N} - \Phi(\cdot - x) \frac{\partial u}{\partial N} \right] ds$$

$$= \int_{\partial\Omega} \left[u \frac{\partial G(\cdot, x)}{\partial N} - \underbrace{G(\cdot, x)}_{=0} \frac{\partial u}{\partial N} \right] ds - \underbrace{\int_{\partial\Omega} \left[u \frac{\partial H^x}{\partial N} - H^x \frac{\partial u}{\partial N} \right] ds}_{-\int_{\Omega} \left[u \Delta H^x - H^x \Delta u \right] dx = 0}$$

$\checkmark$

Theorem (Represent. formula for Poisson eqn.)

Ω as before. $f \in C^0(\bar{\Omega})$ Let $u \in C^2(\bar{\Omega})$ be a solution of $\Delta u = f$ on Ω

$$u|_{\partial\Omega} = 0$$

$$x_0 \in \Omega \Rightarrow u(x_0) = \int_\Omega G(x, x_0) f(x)\, dx$$

Proof: $\varepsilon > 0$. $B_\varepsilon(x_0) \subset \Omega$ $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon(x_0)}$

Green's 2nd Identity

$$\int_\Omega G(x, x_0) f(x)\, dx - \int_{B_\varepsilon(x_0)} G(x, x_0) f(x)\, dx = \int_{\Omega_\varepsilon} G(x, x_0)\, \Delta u(x)\, dx$$

$$= \int_{\partial\Omega_\varepsilon} \left[G(x, x_0) \cdot \frac{\partial u}{\partial N}(x) - u(x) \frac{\partial G(x, x_0)}{\partial N} \right] ds(x) = \int_{\partial\Omega} + \int_{\partial B_\varepsilon}$$

$$\int_{\partial\Omega} \underbrace{G(x,x_0)}_{=0} \frac{\partial u}{\partial N}(x) - \underbrace{u(x)}_{=0} \frac{\partial G(x,x_0)}{\partial N} \, ds(x) = 0$$

$$\underbrace{\int_{\partial B_\varepsilon(x_0)} \Phi(\cdot - x_0)\frac{\partial u}{\partial N} - u \cdot \frac{\partial \Phi(\cdot - x_0)}{\partial N} \, ds}_{\longrightarrow 0 + u(x_0) \quad \varepsilon \to 0} + \underbrace{\int_{\partial B_\varepsilon} H^{x_0}\frac{\partial u}{\partial N} - u\frac{\partial H^{x_0}}{\partial N} \, ds}_{= -\int_{B_\varepsilon(x_0)} H^{x_0}\underbrace{\Delta u}_{f} - \underbrace{u\Delta H^{x_0}}_{=0}}$$

exactly as in the proof the Represent formula

- f, H^{x_0} are continuous

$\exists\, C, \varepsilon_0 > 0$ s.t. $|f(x) \cdot H^{x_0}(x)| \leq C \quad |x - x_0| < \varepsilon$, $\varepsilon \in (0, \varepsilon_0)$

$$\left| \int_{B_\varepsilon(x_0)} H^{x_0} f \, dx \right| \leq \int_{B_\varepsilon(x_0)} |H^{x_0} f| \, dx \leq C \cdot \mathrm{vol}(B_\varepsilon(x_0)) \longrightarrow 0$$

$$\int_{B_\varepsilon(x_0)} g(x,x_0)\, f(x)\, dx = \underbrace{\int_{B_\varepsilon(x_0)} \Phi(x-x_0)\, f(x)\, dx}_{(*)} + \underbrace{\int_{B_\varepsilon(x_0)} H^{x}(x)\, f(x)\, dx}_{\to 0}$$

$$(*): \quad \left| \int_{B_\varepsilon(x_0)} \Phi(x-x_0)\, f(x)\, dx \right| \leq \int_{B_\varepsilon(x_0)} |\Phi(x-x_0)\, f(x)|\, dx$$

$$\leq \underbrace{\sup_{\Omega} |f|}_{=M} \int_{B_\varepsilon(0)} |\Phi(x)|\, dx = M \int_0^{\varepsilon}\int_0^{2\pi}\int_0^{\pi} \frac{1}{4\pi} \cdot \frac{1}{r} \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= M \cdot \frac{1}{2} \cdot \varepsilon^2 \to 0$$

Theorem (Symmetry of Green fct)

Ω as before G is Green fct.

$\Longrightarrow G(a,b) = G(b,a) \quad a, b \in \Omega$

Proof: $\varepsilon > 0 \quad B_\varepsilon(a), B_\varepsilon(b) \subset \Omega \quad \overline{B_\varepsilon(a)} \cap \overline{B_\varepsilon(b)} = \emptyset$

$\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon(a)} \cup \overline{B_\varepsilon(b)}$

$v(x) = G(x,b)$, $u(x) = G(x,a)$

Green's 2nd Identity

$$0 = \int_{\Omega_\varepsilon} (v \cdot \Delta u - u \Delta v)\, dx = \underbrace{+\int_{\partial\Omega} (\ldots)}_{A} + \underbrace{\int_{\partial B_\varepsilon(a)} (\ldots)}_{B} + \underbrace{\int_{\partial B_\varepsilon(b)} (\ldots)}_{C}$$

$$A = \int_{\partial\Omega} \left(v \cdot \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N}\right) ds = 0$$

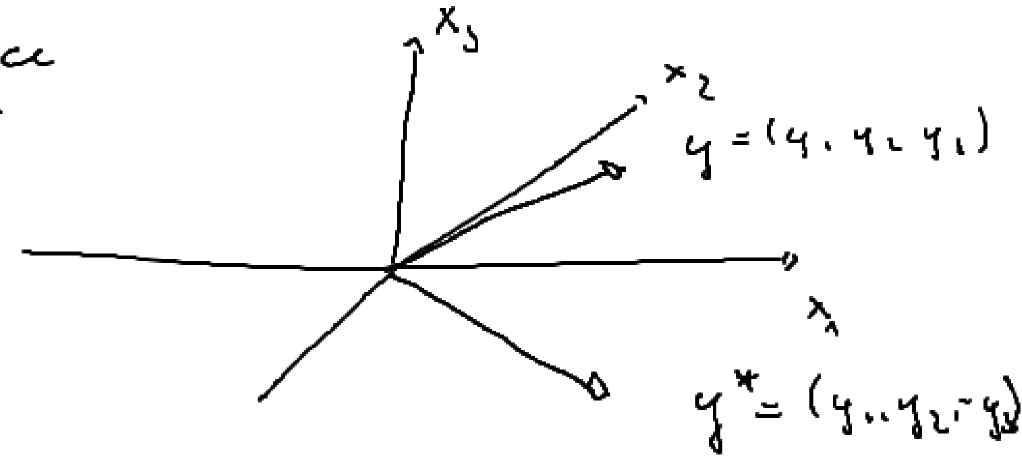
$$\int_{\partial B_\varepsilon(a)} \left(v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds \qquad u(x) = G(x, a) = H^a(x) + \phi(x-a)$$

$$= \underbrace{\int_{\partial B_\varepsilon(a)} \left(v \frac{\partial \phi(\cdot - a)}{\partial N} - \phi(\cdot - a) \frac{\partial v}{\partial N} \right) ds}_{\xrightarrow{\varepsilon \to 0} \ v(a) = G(a, b)} + \underbrace{\int_{\partial B_\varepsilon(a)} \left(v \frac{\partial H^a}{\partial N} + H^a \frac{\partial v}{\partial N} \right)}_{\underbrace{-\int_{B_\varepsilon(a)} v \underbrace{\Delta H^a}_{=0} - H^a \underbrace{\Delta v}_{=0}}_{=0}}$$

$$\int_{\partial B_\varepsilon(b)} \left(v \frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds = - \int_{\partial B_\varepsilon(b)} \left(u \frac{\partial v}{\partial N} - v \frac{\partial u}{\partial N} \right) \longrightarrow -u(b) = -G(b, a)$$

$\boxtimes$

Green fct of the Half space



$$\mathbb{H}^3 = \{ x \in \mathbb{R}^3 : x_3 > 0 \}$$

$x = (x_1, x_2, x_3)$

$$G(x,y) = \Phi(x-y) + H^y(x)$$

where H^y solves

$$(**) \quad \begin{aligned} \Delta H^y &= 0 \text{ on } \mathbb{H} \\ H^y|_{\partial \mathbb{H}} &= -\Phi(\cdot - y)|_{\partial \mathbb{H}} \end{aligned}$$

Theorem:

$$H^y(x) = \frac{1}{4\pi |x-y^*|} \text{ solves } (**)$$

Hence

$$G(x,y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi|x-y^*|}$$

Proof: ① $H^y(x) = \frac{1}{4\pi}\frac{1}{|x-y^*|} \in C^\infty(\mathbb{H})$

② $\Delta H^y = 0$ on $\mathbb{H}$

③ $x \in \partial\mathbb{H}$, $x = (x_1, x_2, 0)$

$$H^y(x) = \frac{1}{4\pi}\cdot\frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\underbrace{(0-(-y_3))^2}_{\substack{y_3^2 \\ (0-y_3)^2}}}}$$

$$= \frac{1}{4\pi}\cdot\frac{1}{|x-y|} = -\Phi(x-y)$$

Hence $G(x,y) = -\frac{1}{4\pi}\frac{1}{|x-y|} - \frac{1}{4\pi}\frac{1}{|x-y^*|}$

Corollary Let $u \in C^2(\overline{\mathbb{H}})$ be a solution of

$\Delta u = 0$ on $\mathbb{H}$

$u|_{\partial\mathbb{H}} = h$ on $\partial\mathbb{H}$ $\Longrightarrow$ $u(x) = \frac{x_3}{\cancel{4}2\pi}\int_{\partial\mathbb{H}}\frac{h(y)}{|y-x|}\,ds(y)$

$\frac{1}{2\pi}\frac{x_3}{|x-y|}$ Poisson kernel of $\mathbb{H}$

Proof

$$u(x) = \int_{\partial \mathbb{H}} u(y) \cdot \frac{\partial G(y,x)}{\partial N} \, dS(y)$$

$$\frac{\partial}{\partial N} = -\frac{\partial}{\partial y^3} \rightsquigarrow \frac{\partial}{\partial y^3} G(y,x) = +\frac{1}{4\pi}\left(-\frac{y_3 - x_3}{|y-x|^3} + \frac{y_3 + x_3}{|y-x|^3}\right)$$

$$= \frac{1}{4\pi}\left(\frac{x_3 - \cancel{y_3} + \cancel{y_3} + x_3}{|y-x|}\right.$$

$$= \frac{1}{2\pi} \frac{x_3}{|y-x|} \quad \boxed{}$$

Remark $n \geq 2$. $\mathbb{H}^n$ n-dim. Half space.
Then the Green fct is given by

$$G(x,y) = -\frac{1}{n(n-2)} \frac{1}{\underbrace{\mathrm{vol}(B_1(0))}_{n\text{-dim.}}} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-y^*|^{n-2}}\right)$$

$y^* = (y_1, \ldots, y_{n-1}, -y_n)$

Theorem (Green fct for $B_a(0) \subset \mathbb{R}^3$)

$H^y(x) = \frac{a}{|y| \cdot} \frac{1}{4\pi} \frac{1}{|x - y^*|}$ solves

$\Delta H^y = 0$ on $B_a(0)$

$H^y|_{\partial B_a(0)} = -\Phi(\cdot - y)|_{\partial B_a(0)}$

y^*: $|y| \cdot |y^*| = a^2$ y and y^* colinear