Last time: $B_a = B_a(0) \subset \mathbb{R}^n$ $\quad x \in \overline{B_a}$, $y \in B_a$

green fct: $G(x,y) = \dfrac{1}{n(n-2)\,\mathrm{vol}(B_1)}\left[\dfrac{-1}{|x-y|^{n-2}} + \dfrac{a}{|y|}\dfrac{1}{|x-y^*|^{n-2}}\right]$

where $y^* = \dfrac{a^2}{|y|^2}\, y$ $\qquad B_1 \subset \mathbb{R}^n$

Poisson formula

$u \in C^2(\overline{B_a})$ s.t. $\Delta u = 0$ on $\overline{B_a}$, $y \in B_a$

$$\Longrightarrow u(y) = \frac{a^2 - |y|^2}{n\,\mathrm{vol}(B_1)\cdot a}\int_{\partial B_a} \frac{u(x)}{|x-y|^n}\, ds(x)$$

Corollary: Let $R = a$ $B_R(x_0)$ $x_0 \neq x \in \overline{B_R(x_0)}$ with $|x_0 - x| = r$. Let $u \in C^2(\overline{B_R(x_0)})$ $u \geq 0$ $\Delta u = 0$

then
$$\frac{1 - \frac{r}{R}}{(1 + \frac{r}{R})^{n-1}} u(x_0) \leq u(x) \leq \frac{1 + \frac{r}{R}}{(1 - \frac{r}{R})^{n-1}} u(x_0)$$

In particular $r \in (0, \frac{R}{2})$:
$$\frac{1}{2^n} = \frac{1}{2} \frac{1}{2^{n-1}} u(x_0) \leq u(x) \leq 2^n u(x_0)$$

Proof of Corollary
$$u(x) = \frac{R^2 - r^2}{n \operatorname{vol}(B_1) \cdot R} \int_{\partial B_R(x_0)} \frac{u(y)}{|x-y|^n} ds$$

$$\underbrace{|y - x_0|}_{R} - \underbrace{|x - x_0|}_{r} \leq |x - y| \leq \underbrace{|x - x_0|}_{r} + \underbrace{|x_0 - y|}_{R}$$

$$\frac{R-r}{(R+r)^{m-1}} = \frac{(R+r)(R-r)}{(R+r)^m} \leq \frac{R^2-r^2}{|x-y|^m} \leq \frac{(R+r)(R-r)}{(R-r)^m} = \frac{R+r}{(R-r)^{m-1}}$$

$\underline{\underline{u \geq 0}}$ + Mean value property $\Rightarrow$ Statement $\square$

Corollary (Harnack's inequality 2nd version)

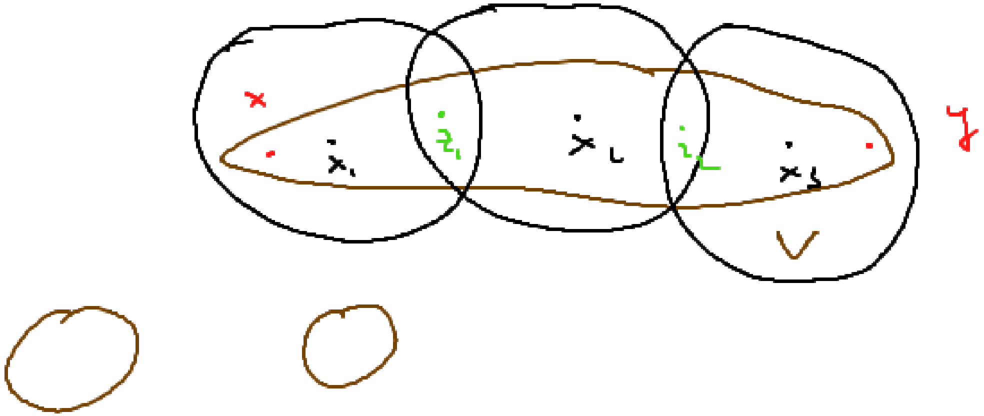
$\Omega \subset \mathbb{R}^m$ open, connected, $\Delta u = 0$ on Ω, $u \in C^2(\Omega)$

$V \subset \Omega$ open. $\overline{V} \subset \Omega$ s.t. $\exists R > 0$ $x_1, \dots, x_N \in V$

s.t. $\overline{V} \subset \bigcup_{i=1}^{N} B_{\frac{R}{2}}(x_i)$ $\quad \overline{B_R(x_i)} \subset \Omega$

$\Rightarrow$ $x, y \in V$ $x \neq y$ $\Rightarrow$ $u(y) \leq 2^{2\cdot m\cdot N} \cdot u(x)$

$\Rightarrow$ $\sup_V u \leq 2^{2\cdot m \cdot N} \cdot \inf_V u$.



$$u(x) \leq 2^m u(x_1)$$

$$\leq 2^m 2^m \cdot u(z_1)$$

$$\leq \cdots \leq 2^{2\cdot 3 \cdot m} u(y)$$

Remarks on the Green fct

$\Omega \subset \mathbb{R}^m$ open, bd., smooth $\partial\Omega$

$\rightsquigarrow$ Green fct: $G(x,y) = \Phi_m(x-y) + H^y(x)$

$H^y \in C^2(\overline{\Omega})$ s.t. $\Delta H^y = 0$ on $\overline{\Omega}$

$H^y|_{\partial\Omega} = -\Phi_m(\cdot - y)$

① If Ω not connected, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$

If $y \in \Omega_1 \Longrightarrow G(x,y) = 0 \quad x \in \Omega_2$

This follows since $-\Phi_n(\cdot - y)$ solves

$$\Delta H^y = 0 \quad \text{on } \Omega_2$$
$$H^y|_{\partial\Omega} = -\Phi_n(\cdot - y).$$

(2) If Ω connected then $G(x,y) < 0$.

In general $G(x,y) \leq 0$

(3) $K_y(x) = \frac{\partial G(\cdot,y)}{\partial N}\Big|_x \quad x \in \partial\Omega$

$\Rightarrow K_y(x) \geq 0$ because $G(x,y) \leq 0 \quad \forall x,y \in \Omega$

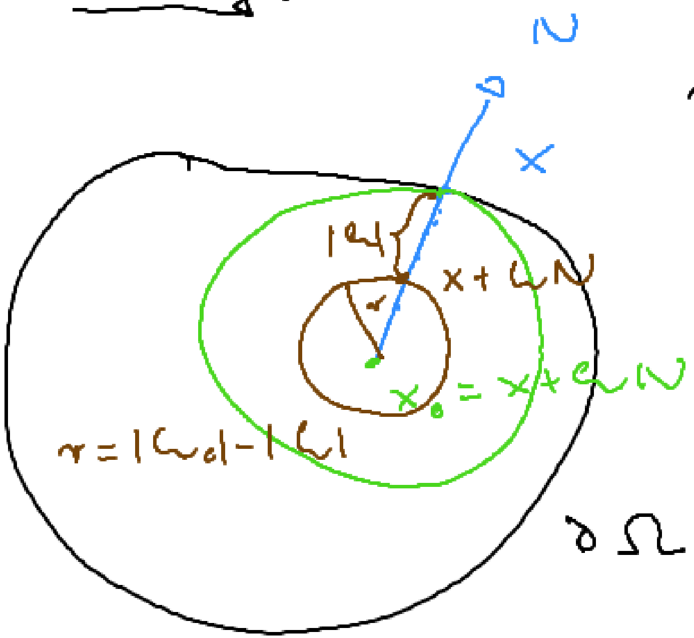
$G(x,y) = 0 \quad x \in \partial\Omega$

$K_y(x)$ Poisson kernel of Ω

Theorem: Ω is connected then $k_y(x) > 0$.

Proof:

$$\lim_{h<0} \frac{G(x+hN,y) - \overbrace{G(x,y)}^{=0}}{h}$$



We pick $h_0 < 0$ s.t.

$x_0 = x + h_0 N$: $B_{|h_0|}(x_0) \subset \Omega$

$\partial B_{|h_0|}(x_0) \cap \partial\Omega = \{x\}$

$h \in (h_0, 0)$

$$\frac{G(x+hN)}{h} = -\frac{G(x+hN)}{|h|} \geq \frac{1 - \frac{|h_0| - \cancel{|h|}}{|h_0|}}{\left(1 + \frac{|h_0|-|h|}{|h_0|}\right)^{n-1}} \cdot \frac{1}{\cancel{|h|}} \cdot (-G(x_0))$$

$$\geq \frac{1}{|h_0| 2^{n-1}} \cdot (-G(x_0)) > 0$$

$$\geq \frac{1}{|K_0|} \cdot \frac{1}{2^m} \cdot \Big(-\underbrace{g(x_0)}_{g(x_0,y)}\Big) > 0$$

$$\Longrightarrow \quad \frac{\partial G(\cdot, y)}{\partial N}\Big|_x = \lim_{h \searrow 0} \frac{g(x+hN)}{h} > c > 0$$

$$\Longrightarrow \quad K_y(x) > 0$$

$\square$