Last time:

Let $\Omega \subset \mathbb{R}^n$ open, bounded. $\partial\Omega$ smooth

Theorem: if Ω connected $\longrightarrow$ $\frac{\partial G(\cdot, y)}{\partial N} = K_y > 0$ on $\partial\Omega$

Theorem: (Boundary Maximum Principle)

$u \in C^2(\overline{\Omega})$ $\Delta u = 0$ on $\overline{\Omega}$

If $x \in \partial\Omega$ st $u(x) = \max_{\overline{\Omega}} u = M$

then $\frac{\partial u}{\partial N}(x) > 0$ or $u \equiv M$

where N unit normal vector in $x \in \partial\Omega$.

Proof: Exercise.

Theorem (local estimate)

$u \in C^2(\Omega)$, Ω open s.t. $\Delta u = 0$. $x_0 \in \Omega$ $\overline{B_r(x_0)} \subset \Omega$

Then $$|\nabla u|(x_0) \leq \frac{c(n)}{r^{n+1}} \cdot \int_{B_r(x_0)} |u(x)|\, dx,$$

Recall: ① MVP
$$u(x_0) = \frac{1}{vol(B_r(x_0))} \int_{B_r(x_0)} u(x)\, dx$$

$$\Longrightarrow |u(x_0)| \leq \frac{c(n)}{r^n} \int_{B_r(x_0)} |u(x)|\, dx$$

② Similar:
$$|\nabla^k u|(x_0) \leq \frac{c(n)}{r^{n+k}} \cdot \int_{B_r(x_0)} |u(x)|\, dx \qquad k \in \mathbb{N}.$$

<u>Proof:</u> $(\Delta u)_{x_i} = 0 = \Delta(u_{x_i}) \Rightarrow u_{x_i}$ harmoni

MVP $$u_{x_i}(x_0) = \frac{1}{vol(B_{\frac{r}{2}}(x_0))}\int_{B_{\frac{r}{2}}(x_0)} u_{x_i}(x)\,dx = (*)$$

Note: $u_{x_i} = div(V)$ $\quad V = \begin{pmatrix} 0 \\ \vdots \\ u \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$

$$(*) = \frac{1}{vol(B_{r/2}(x_0))}\cdot\int_{\partial B_{r/2}} V\cdot N\,ds = \frac{1}{\dots}\int_{\partial B_{r/2}} \underbrace{u\cdot N_i}_{|\,|\le |u|\,\underbrace{|N|}_{\le 1}}\,ds$$

$$\Rightarrow |u_{x_i}(x_0)| \le \frac{1}{vol(B_{r/2}(x_0))}\cdot vol(\partial B_{r/2}(x_0))\,\sup_{\partial B_{r/2}(x_0)}|u|$$

$B_{r/2}(x)\subset B_r(x_0)$

∂B_r

$$\sup_{x\in\partial B_{r/2}(x_0)}|u(x)| \le \frac{1}{vol(B_{r/2}(x))}\cdot\int_{B_{r/2}(x)}|u(y)|\,dy$$

$$\le \frac{1}{\dots}\int_{B_r(x_0)}|u(y)|\,dy \quad \forall x\in\partial B_{r/2}(x_0)$$

$$|u_{x_i}(x_0)| \leq \underbrace{\frac{\mathrm{vol}(\partial B_{r/2}(x_0))}{\mathrm{vol}(B_{r/2}(x_0))}}_{\frac{2n}{r}} \cdot \frac{1}{\underbrace{\mathrm{vol}(B_{r/2}(x))}_{\mathrm{vol}(B_1)\left(\frac{r}{2}\right)^n}} \int_{B_r(x_0)} |u(y)|\,dy$$

$$= n \cdot \mathrm{vol}(B_1) \cdot \left(\frac{2}{r}\right)^{n+1} \cdot \int_{B_r(x_0)} |u(y)|\,dy$$

$$|\nabla u|(x_0) = \sqrt{\sum_{i=1} (u_{x_i})^2}$$ $\blacksquare$

<u>Corollary</u>: (Liouville theorem)

$u \in C^2(\mathbb{R}^n)$ s.t. $\Delta u = 0$ and $|u| \leq M$ for $M > 0$

$|u(x)| \leq M \quad \forall x \in \mathbb{R}^n$

$\Longrightarrow \quad u \equiv \mathrm{const}$

<u>Proof</u> $\quad B_R(x) \subset \mathbb{R}^n \quad \forall x \in \mathbb{R}^n \; \forall R > 0$

$$\Longrightarrow \quad |\nabla u|(x) \leq C(n) \cdot M \cdot \frac{1}{R^{n+1}} \xrightarrow[R \to 0]{} 0$$ $\blacksquare$

Theorem: Given $h \in C^0(\partial B_a(0))$ then

$$u(x) = \begin{cases} \frac{a^2 - |x|^2}{n \operatorname{vol}(B_1(0)) \cdot a} \cdot \int_{\partial B_a} \frac{h(y)}{|x-y|^n} \, dy & x \in B_a \\ h(x) & x \in \partial B_a \end{cases}$$

is in $C^2(B_a(0)) \cap C^0(\overline{B_a(0)})$ and solves

$$\Delta u = 0 \quad \text{on } B_a(0)$$

$$u|_{\partial B_a} = h$$

Direct Method of Calculus of Variations

Ω as before + connected

Dirichlet Principle

$u \in C^2(\bar\Omega)$ solves $\begin{cases} \Delta u = 0 \text{ on } \Omega \\ u|_{\partial\Omega} = h \end{cases}$ $\iff$ u minimizes $E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx$ for $\underbrace{v \in C^1(\bar\Omega),\ v|_{\partial\Omega} = h}_{\mathcal{E}_h}$

where $h = w|_{\partial\Omega}$ $w \in C^1(\bar\Omega)$

Find Minimizer of E on $\mathcal{E}_h$

① $E(v) \geq 0 \implies I = \inf_{v \in \mathcal{E}_h} E(v) \geq 0 > -\infty$

② $\exists\, v_m \in \mathcal{E}_h$ s.t. $E(v_m) \longrightarrow I$ (Minimal sequence)

Does (v_m) converge? In which sense?

Define $\|\cdot\|_{1,2} = \sqrt{\int_\Omega v^2 dx + E(v)}$

$v_n \rightsquigarrow v_n - w = \tilde{v}_n \in \mathcal{E}_0 = \{ v \in C^1(\bar{\Omega}) . \; v|_{\partial\Omega} = 0 \}$

$\Longrightarrow (\mathcal{E}_0, \|\cdot\|_{1,2})$ Norm space

Claim $(\tilde{v}_n)$ Cauchy sequence

- $E(\tilde{v}_n - \tilde{v}_m) = E(v_n - v_m) = E(v_n) + E(v_m) - \int_\Omega \langle \nabla v_n, \nabla v_m \rangle$

$$= 2E(v_n) + 2E(v_m) - \underbrace{4E\left(\frac{v_n + v_m}{2}\right)}_{\geq 4I}$$

$$\leq 2E(v_n) + 2E(v_m) - 4I \xrightarrow{n,m \to \infty} 0$$

- Poincaré inequality

$v \in \mathcal{E}_0 : \int_\Omega |v|^2 dx \leq C(\Omega) \int_\Omega |\nabla v|^2 dx$

$\tilde{v}_n - \tilde{v}_m \in \mathcal{E}_0 \Longrightarrow \int |\nabla(\tilde{v}_n - \tilde{v}_m)|^2 dx \to 0 \Longrightarrow \|\tilde{v}_n - \tilde{v}_m\|_{1,2} \xrightarrow[n,m \to \infty]{} 0$

Consider: $\overline{\mathcal{E}_0}^{\|\cdot\|_{1,2}} = H_0^{1,2}(\Omega)$ the completion of $\mathcal{E}_0$ w.r.t. $\|\cdot\|_{1,2}$

$\tilde{v}_n$ Cauchy $\Rightarrow \exists\, \tilde{v} \in H_0^{1,2}(\Omega)$ s.t. $\|v_n - \tilde{v}\|_{1,2} \to 0$

Does $\tilde{v} + w = v$ solves Laplace equation?

Def: (Sobolev fct)

$f : \Omega \to \mathbb{R}$ L^2-integrable $\left(\int_\Omega |f|^2 dx < \infty \right)$

has a weak partial derivative $f_{x_i} : \Omega \to \mathbb{R}$ that is L^2-integrable if

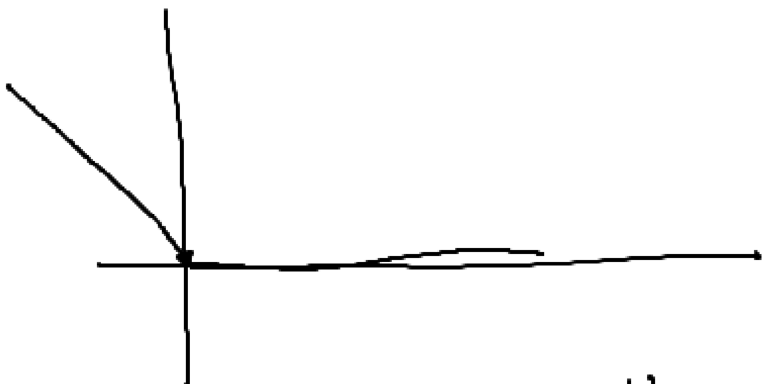
$$\int_\Omega f \cdot \varphi_{x_i} = - \int_\Omega f_{x_i} \varphi \, dx \quad \forall \varphi \in C_c^1(\Omega) \subset \mathcal{E}_0$$

$f \in W^{1,2}(\Omega)$ if all weak derivatives f_{x_i} ex.

$\rightsquigarrow$ weak gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$

Example:

$$f(x) = \begin{cases} -x & x \in [-1, 0) \\ 0 & x \in (0, 1] \end{cases} \Rightarrow \begin{array}{l} f \notin C^1((-1,1)) \\ f \in W^{1,2}((-1,1)) \end{array}$$



Theorem: $H^{1,2}_0(\Omega) \subset W^{1,2}(\Omega)$ exist $\forall v \to \frac{1}{2}\int_\Omega |\nabla v| dx$

$v \in H^{1,2}_0(\Omega) \to$ weak gradient

$$\|v\|_{1,2} = \sqrt{\int_\Omega v^2 dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx}$$

Moreover if $v_n \to v$ in $W^{1,2}(\Omega)$: $\liminf_{n\to\infty} E(v_n) \geq E(v)$

$\tilde{v}_m = v_m - w \qquad v_m \to v = \tilde{v} + w$

Minimal sequence

$$\underbrace{\liminf E(v_m)}_{I} \geq E(v) \geq I \Longrightarrow E(v) = I$$

Dirichlet principle in weak context

$$0 = \frac{d}{dt}\Big|_0 E(v + t\varphi) = \frac{d}{dt}\Big|_0 E(v) + t^2 E(\varphi) + t\int \langle \nabla\varphi, \nabla v\rangle$$

$$\varphi \in C^1_c(\Omega) \qquad = \int_\Omega \langle \nabla\varphi, \nabla v \rangle \, dx$$

Def: $v \in W^{1,2}(\Omega)$ is a weak solution of

$\Delta v = 0$ on $\bar{\Omega}$

$v|_{\partial\Omega} = h$ on $\partial\Omega$ if $\int \langle \nabla\varphi, \nabla v \rangle \, dx = 0 \quad \forall \varphi \in C^1_c(\Omega)$

$v - w \in H^{1,2}_0(\Omega)$

$h = w|_{\partial\Omega}$

Question: Is a weak solution also a classical solution

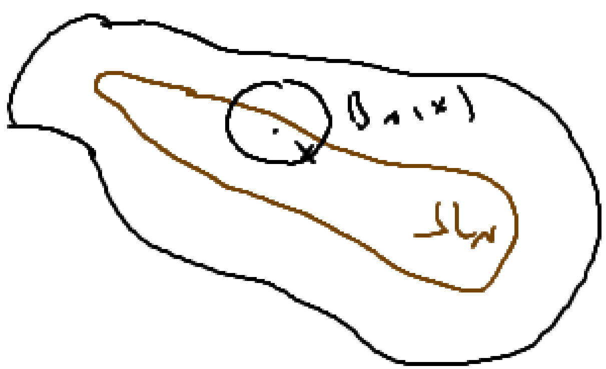
Theorem: (Weyl Lemma)

If u is a weak solution to $\Delta u = 0$ then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ on Ω classically.

Proof: $\varphi \in C_c^\infty(B_1(0))$, $1 = \int_{\mathbb{R}^n} \varphi \, dx$ $\quad \varphi(x) = \varphi(-x)$

$\rightsquigarrow \varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \cdot \varphi\left(\frac{x}{\varepsilon}\right) \rightsquigarrow \varphi_\varepsilon \in C_c^\infty(B_\varepsilon(0))$

$\rightsquigarrow 1 = \int \varphi_\varepsilon(x) \, dx$



Ω

$\Omega_r = \{ x \in \Omega : |x - y| > r \;\; \forall y \in \partial\Omega \}$

$\Longrightarrow B_r(x) \subset \Omega \quad \forall x \in \Omega_r$

$$u_\varepsilon(x) = \int_{B_\varepsilon(x)} u(y) \cdot \varphi_\varepsilon(y-x)\, dy \qquad x \in \Omega_r$$

1. claim: $u_\varepsilon \in C^\infty(\Omega_r)$ and $\Delta u_\varepsilon = 0$ on Ω_ε

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$$

$$\frac{\partial u_\varepsilon}{\partial x_i}(x) = \lim_{h \to 0} \frac{u_\varepsilon(x + h e_i) - u_\varepsilon(x)}{h}$$

$$= \lim_{h \to 0} \int_{B_\varepsilon(x)} \underbrace{u(y) \frac{\varphi_\varepsilon(y - x - h e_i) - \varphi(y-x)}{h}}\, dx$$

$$\longrightarrow u(y) \frac{\partial \varphi_\varepsilon}{\partial x_i}(y-x)$$

$$\Longrightarrow \frac{\partial u_\varepsilon(x)}{\partial x_i} = \int_\Omega u(y) \frac{\partial \varphi_\varepsilon}{\partial x_i}(y-x)\, dy$$ unif convergence wrt y

Higher derivatives similar

$$\Longrightarrow \Delta u_\varepsilon(x) = \int u(y) \Delta \varphi_\varepsilon(y-x)\, dy = -\int \langle \nabla u, \nabla \varphi_\varepsilon(x-y) \rangle$$
$$= 0 \qquad x \in \Omega_r$$

2. $|\nabla u_\varepsilon|(x) \leq \frac{c(m)}{r^{m+1}} \int_{B_r(x)} |u_\varepsilon(x)| dx \leq \frac{c(m)}{r^{m+1}} \int_\Omega |u_\varepsilon(x)| dx$

$$\int_\Omega |u_\varepsilon| dx \leq \int_{\Omega}\int_{\Omega} |u(x)| \varphi_\varepsilon |y-x| \, dx \, dy$$

$$= \int_\Omega |u(x)| \, dx \; \cdot$$

$\Longrightarrow \exists \, C > 0 : |\nabla u_\varepsilon|(x) \leq C \quad x \in \Omega_r$

$\Longrightarrow u_\varepsilon$ C-Lipschitz uniformly

Arzelà-Ascoli $\Longrightarrow u_\varepsilon \xrightarrow{\sigma} v \in C^0(\Omega_r)$

u_ε is harmonic

$x \in \Omega_r$ $\quad u_\varepsilon(x) = \frac{1}{\cdots}\int_{B_\delta(x)} u_\varepsilon(y)\, dx$

$\downarrow$ $\quad$ since $u_\varepsilon \to v$ uniformly

$v(x) \quad \int_{B_\delta(x)} v(y)\, dy$

$\Longrightarrow$ v cont. and satisfies MVP

$\Longrightarrow$ v harmonic on Ω_r

$$\left.\begin{array}{l} u_\varepsilon \to u \quad \text{in } L^1 \\ u_\varepsilon \to v \quad \text{in } L^1 \end{array}\right\} \Longrightarrow u = v$$

$\Longrightarrow$ u is harmonic on Ω_r $\square$