# MAT351 Partial Differential Equations
# Lecture 2

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## Last Lecture

- A PDE
$$F(\mathbf{x}, u, D^1 u, \ldots, D^k u) = g(x) \text{ for } \mathbf{x} \in \Omega$$
is linear if the map
$$(\theta, \theta^1, \ldots, \theta^k) \mapsto F(\mathbf{x}, \theta, \theta^1, \ldots, \theta^k)$$
is linear for all $\mathbf{x} \in \Omega$.
- The PDE is called **homogeneous** if $g \equiv 0$, otherwise the PDE is called **nonhomogeneous**.
- There is a linear map
$$\mathcal{L} : C^k(\Omega) \to C^0(\Omega),\ \mathcal{L}u = F(\mathbf{x}, u, D^1 u, \ldots, D^k u).$$
We call $\mathcal{L} : C^k(\Omega) \to C^0(\Omega)$ **Linear Differential Operator**.

### Example (Linear PDE of order 2)

$$\mathcal{L}u := \sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}) u_{x_i,x_j} + \sum_{k=1}^{n} b_k(\mathbf{x}) u_{x_k} + c(\mathbf{x})u.$$

If $a_{i,j}(\mathbf{x}) = \delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$ (delta function), $b_k \equiv 0$ and $c \equiv 0$, then

$$\mathcal{L}u = \sum_{i=1}^{n} u_{x_i,x_i} = \Delta u$$ is called the **Laplace operator**.

# Important Theorems

- $V = (V^1, \dots, V^n) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is in $C^k(\Omega, \mathbb{R}^n)$ if $V^i \in C^k(\Omega)$, $i = 1, \dots, n$.
- $U, V \subset \mathbb{R}^n$ open. $\Phi : U \to V$ is a $C^k$-diffeomorphism if $\Phi$ is one-to-one and onto and $\Phi \in C^k(U, \mathbb{R}^n)$ and $\Phi^{-1} \in C^k(V, \mathbb{R}^n)$.

## Theorem (Transformation formula)

*Let $U, V \subset \mathbb{R}^n$ be open and let $\Phi : U \to V$ be a $C^1$-diffeomorphism. Then, a function $f : V \to \mathbb{R}$ is integrable if and only if $(f \circ \Phi)|\det D\Phi| : U \to \mathbb{R}$ is integrable. Moreover, it holds*

$$\int_V f(\mathbf{x}) d\mathbf{x} = \int_U f \circ \Phi(\mathbf{y}) |\det D\Phi(\mathbf{y})| d\mathbf{y}.$$

## Theorem (Divergence (also Gauss) Theorem)

*Let $\Omega \subset \mathbb{R}^n$ be closed and bounded with smooth boundary $\partial\Omega$. Let $N : \partial\Omega \to \mathbb{R}^n$ be the outer unit normal vector field of $S$. Let $V \in C^1(\Omega, \mathbb{R}^n)$. Then*

$$\int_{\partial\Omega} N \cdot V dS = \int_\Omega \nabla \cdot V d\mathbf{x}.$$

The Divergence Theorem generalizes the **Fundamental Theorem of Calculus**:

$$f(b) - f(a) = \int_a^b f'(x) dx, \quad f \in C^1([a, b]).$$

# Simple Transport Equation

Let

$$V : \mathbb{R}^n \to \mathbb{R}^n, \quad V \in C^1(\mathbb{R}^n, \mathbb{R}^n), \quad \nabla \cdot V = 0 \text{ and } V(\mathbf{x}) \neq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

We imagine a medium in $\mathbb{R}^n$ that moves with a speed that is equal to $V(\mathbf{x})$ at any point $\mathbf{x} \in \mathbb{R}^n$. Solve $\frac{d}{dt}\gamma_x(t) = V \circ \gamma_x(t)$, $\gamma_x(0) = x$. The flow of $V$ is the map

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \ \Phi_t(x) = \gamma_x(t).$$

$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$-diffeomorphism.

Let $u(x,t)$ be the density of a substance $\Psi$ that is released "into" the flow of $V$.

**What is the evolution law for $u$?**

Let $\Omega \subset \mathbb{R}^n$ be any open domain and bounded. **We assume:**

$$\int_\Omega u(\mathbf{x}, t) d\mathbf{x} = \int_{\phi_h(\Omega)} u(\mathbf{y}, t+h) d\mathbf{y}$$

$\Phi_t|_\Omega : \Omega \to \Phi_t(\Omega)$ is a $C^1$-diffeomorphism. The transformation formula yields

$$\int_\Omega u(\mathbf{x}, t) d\mathbf{x} = \int_{\Phi_h(\Omega)} u(\mathbf{y}, t+h) d\mathbf{y} = \int_\Omega u(\Phi_h(\mathbf{x}), t+h) |\det D\Phi_h(\mathbf{x})| d\mathbf{x}.$$

Since $\Omega$ was arbitrary, we get

$$u(\mathbf{x}, t) = u(\phi_h(\mathbf{x}), t + h) \det D\Phi_t(\mathbf{x}).$$

Differentiate w.r.t. $h$ at $h = 0$ on both sides:

$$0 = \nabla_x u \cdot \underbrace{\frac{d}{dh}|_{h=0}\Phi_0(\mathbf{x})}_{V(\mathbf{x})} + u_t(\mathbf{x}, t) + u(\underbrace{\Phi_0(\mathbf{x})}_{\gamma_x(0)=x}, t)\frac{d}{dh}|_{h=0} \det D\Phi_h(\mathbf{x}).$$

The matrix $D(\phi_h)$ is invertible, $D\Phi_0 = E_n$ and $(\mathbf{x}, h) \mapsto \Phi_h(\mathbf{x}) = \gamma_{\mathbf{x}}(h)$ is a $C^2$ map. Hence $h \mapsto D(\Phi_h)(\mathbf{x}) =: A(h)$ is differentiable at $h = 0$.

$$\frac{d}{dh}|_{h=0} \det A(h) = \det A(0) \operatorname{trace}[A^{-1}(0)\frac{d}{dh}|_{h=0}A(h)] = \operatorname{trace}[\frac{d}{dh}|_{h=0}A(h)]$$

On the other hand we can compute that

$$\frac{d}{dh}|_{h=0}D\Phi_h(\mathbf{x}) = D\frac{d}{dh}|_{h=0}\Phi_h(\mathbf{x}) = DV(\mathbf{x})$$

It follows

$$\frac{d}{dh}|_{h=0} \det D\Phi_h = \operatorname{trace} DV = \nabla \cdot V = 0.$$

So the PDE that governs $u(\mathbf{x}, t)$ is

$$u_t + V \cdot \nabla u = 0.$$

# Simple Transport, revisited

Let us think the previous model from a different perspective.

We set

$$\int_{\Omega} u(\mathbf{x}, t) d\mathbf{x} = M(\Omega, t).$$

Then

$$\frac{d}{dt} M(\Omega, t) = \frac{d}{dt} \int_{\Omega} u(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} u_t(\mathbf{x}, t) d\mathbf{x}$$

How can we understand the change of $M(\Omega, t)$ in time $t$?

Let $\mathbf{F}(\mathbf{x}, t) = \mathbf{F}_t(\mathbf{x}) \in \mathbb{R}^n$ the (infinitesimal) rate and direction of change of $u$ in $\mathbf{x}$ at time $t$:

$$\mathbf{F}(\mathbf{x}, t) = u(\mathbf{x}, t) V(\mathbf{x}).$$

The **total flux** of $u$ through $\partial\Omega$ is then

$$\int_{\partial\Omega} N(\mathbf{x}) \cdot F(\mathbf{x}, t) dS(\mathbf{x}) = \int_{\partial\Omega} N(\mathbf{x}) \cdot u(\mathbf{x}, t) V(\mathbf{x}) dS(\mathbf{x}),$$

the net value of how much of the substance $\Psi$ has flown in and out of $\Omega$.

But clearly

$$\frac{d}{dt} M(\Omega, t) = \int_{\partial\Omega} N(\mathbf{x}) \cdot u(\mathbf{x}, t) V(\mathbf{x}) dS(\mathbf{x})$$

Applying the divergence theorem yields

$$\int_\Omega u_t(\mathbf{x}, t) d\mathbf{x} = \int_\Omega \nabla \cdot [u(\mathbf{x}, t) V(\mathbf{x})] \, d\mathbf{x} = (1)$$

By the chain rule this becomes

$$(1) = \int_\Omega \left[ \nabla u(\mathbf{x}, t) \cdot V(\mathbf{x}) + u(\mathbf{x}, t) \underbrace{\nabla \cdot V(\mathbf{x})}_{=0} \right] d\mathbf{x} = \int \nabla u(\mathbf{x}, t) \cdot V(\mathbf{x}) d\mathbf{x}.$$

Since $\Omega$ was arbitrary, it follows

$$u_t + V(\mathbf{x}) \cdot \nabla u = 0.$$

# Diffusion equation

Imagine a liquid in $3D$ or higher.

Let $u(\mathbf{x}, t)$ be a concentrations function of substance $\Psi$ released into the liquid.

The substance $\Psi$ moves from regions with higher concentration to regions with lower concentrations. We call this process **Diffusion**.

The rate and direction of change of $\Psi$ in $x$ and $t$ is proportional to the gradient of $u$ w.r.t. $\mathbf{x} \in \mathbb{R}^n$ at time $t$.This is known as **Fick's law**:

$$\mathbf{F}(\mathbf{x}, t) = \lambda \nabla_{\mathbf{x}} u(\mathbf{x}, t) = \lambda \begin{pmatrix} u_{x_1}(\mathbf{x}, t) \\ \dots \\ u_{x_n}(\mathbf{x}, t) \end{pmatrix}.$$

Let $\Omega$ be a compact domain with smooth boundary. Let

$$M(\Omega, t) = \int_\Omega u(\mathbf{x}, t) d\mathbf{x} \ \text{ and } \ \frac{d}{dt} M(\Omega, t) = \int_\Omega u_t(\mathbf{x}, t) dx$$

$\frac{d}{dt} M(\Omega, t)$ is equal to the total flux through the boundary $\partial\Omega$. Hence

$$\int_\Omega u_t(\mathbf{x}, t) d\mathbf{x} = \int_{\partial\Omega} N(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}, t) dS(\mathbf{x}) = \lambda \int_{\partial\Omega} N \cdot \nabla u dS(\mathbf{x}).$$

Hence, by the divergence theorem

$$\int_\Omega u_t(\mathbf{x}, t) d\mathbf{x} = \lambda \int_\Omega \nabla \cdot \nabla u(\mathbf{x}, t) d\mathbf{x} = \lambda \int_\Omega \Delta u(\mathbf{x}, t) d\mathbf{x}.$$

Since $\Omega \subset \mathbb{R}^n$ was arbitrary, it follows $u_t = \lambda \Delta u$.

# Nonlinear Scalar Conservation Laws

Imagine a "flowing" substance $\Psi$.

What if the infinitesimal flux $\mathbf{F}(\mathbf{x}, t)$ of $\Psi$ in $\mathbf{x}$ at time $t$ depends on the concentration function $u$ in $\mathbf{x}$ at time $t$?

We assume there exists $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ such that $\mathbf{F}(\mathbf{x}, t) = \mathbf{f} \circ u(\mathbf{x}, t)$.

Then

$$\frac{d}{dt} \int_{\Omega} u(\mathbf{x}, t) = \int_{\partial\Omega} N(\mathbf{x}) \cdot \mathbf{f} \circ u(\mathbf{x}, t) dS(\mathbf{x})$$

As before by the divergence theorem and differentiating under the integral we obtain

$$\int_{\Omega} u_t(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} \nabla \cdot \circ u d\mathbf{x} = \int_{\Omega} \mathbf{f}'(u)(\mathbf{x}) \cdot \nabla u(\mathbf{x}) d\mathbf{x}$$

where

$$\mathbf{f}'(r) = \begin{pmatrix} \mathbf{f}_1'(r) \\ \dots \\ \mathbf{f}_n'(r) \end{pmatrix}.$$

## Example

Consider the $1D$ case (for instance traffic in a street). Let $\mathbf{f}(r) = \frac{1}{2} r^2$.

Then $\mathbf{f}'(r) = r$. The corresponding PDE

$$u_t + u u_x = 0$$

is the inviscid Burger's equation.

## Theorem (Fundamental Theorem of Calculus of Variations)

*Consider* $f \in C^0(\mathbb{R}^n)$*. If*

$$\int f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = 0 \quad \forall \varphi \in C_c^0(\mathbb{R}^n),\ \varphi \geq 0 \quad \Longrightarrow \quad f \equiv 0.$$

*Proof.* Assume the contrary. We will derive a contradiction.

If $f \neq 0$, then there exists at least $\mathbf{x}_0 \in \mathbb{R}^n$ such that $f(\mathbf{x}_0) \neq 0$.

We can assume $f(\mathbf{x}_0) > 0$ (otherwise replace $f$ with $-f$, this does not change $\int f\varphi d\mathbf{x} = 0$).

In particular, there exist $\epsilon > 0$ such that $f(\mathbf{x}_0) - \epsilon > 0$.

Since $f$ is continuous, there exists $\delta = \delta(\epsilon) > 0$ such that

$$f^{-1}(B_\epsilon(f(\mathbf{x}_0))) = f^{-1}(\{r \in \mathbb{R} : |f(\mathbf{x}_0) - r| < \epsilon\}) \subset \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0|_2 < \delta\} =: B_\delta(\mathbf{x}_0)$$

where $|\mathbf{x}|_2 = \sqrt{\sum_{i=1}^n (x_i)^2}$. In particular, if $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ then $f(\mathbf{x}) \in B_\epsilon(f(\mathbf{x}_0))$. So $f(\mathbf{x}) > \epsilon > 0$.

We can choose $\varphi \in C_c^0(\mathbb{R}^n)$, $\varphi(\mathbf{x}) \equiv 0$ on $\mathbb{R}^n \backslash B_\delta(\mathbf{x}_0)$ and $\varphi(\mathbf{x}) = 1$ for $\mathbf{x} \in B_{\frac{\delta}{2}}(x_0)$. For instance

$$\varphi(\mathbf{x}) = \begin{cases} \min\{1 - \frac{1}{\delta}|\mathbf{x}|_2, 1\} & \text{for } \mathbf{x} \in B_\delta(\mathbf{x}_0) \\ 0 & \text{for } \mathbf{x} \in \mathbb{R}^n \backslash B_\delta(\mathbf{x}_0). \end{cases}$$

Then, it follows

$$0 = \int f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \int_{B_\delta(\mathbf{x})} f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} \geq \epsilon \int \phi(\mathbf{x})d\mathbf{x} = \epsilon > 0. \quad \square$$