# Some Remark on Lecture 29

Consider
$$\left.\begin{aligned} \Delta u &= 0 \quad \text{on } \Omega \\ u|_{\partial\Omega} &= h \quad \text{on } \partial\Omega \end{aligned}\right\} \quad (*)$$
where $h = w|_{\partial\Omega}$ for $w \in C^1(\bar{\Omega})$

$\Omega$ bounded open, $\partial\Omega$ smooth, $\Omega \subset \mathbb{R}^n$

Sobolev space: $f \in W^{1,2}(\Omega)$ if $\int_\Omega f^2\,dx < \infty$ and $\exists\, \hat{f}_{x_i}$:

s.t. $\int_\Omega (\hat{f}_{x_i})^2\,dx < \infty \quad i = 1, \dots, n$

and $\int_\Omega f\,\varphi_{x_i}\,dx = -\int_\Omega \hat{f}_{x_i}\,\varphi\,dx \quad \forall \varphi \in C_c^1(\Omega)$

$\rightsquigarrow$ Dirichlet Energy $E(f) = \frac{1}{2}\int_\Omega |\hat{\nabla} f|^2\,dx$

$\rightsquigarrow$ <u>Sobolev norm</u>: $f \in W^{1,2}(\Omega)$ $\|f\|_{1,2} = \sqrt{\int_\Omega f^2 dx + E(f)}$

$\rightsquigarrow$ $\overline{C_c^1(\Omega)}^{\|\cdot\|_{1,2}} =: H_0^{1,2}(\Omega)$

$$\begin{array}{c} C_c^1(\Omega) \\ \cap \\ W^{1,2}(\Omega) \end{array}$$

<u>Weak solution</u>:

$u \in W^{1,2}(\Omega)$ is a weak solution of $(*)$ if

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dx = 0 \quad \forall \varphi \in C_c^1(\Omega)$$

$$u - w \in H_0^{1,2}(\Omega)$$

<u>Dirichlet Principle</u>

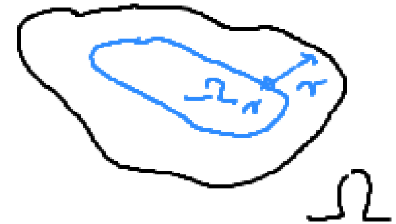
$u - w$ Minimizer of $E$ on $H_0^{1,2}(\Omega) \Longleftrightarrow$ $u$ is a weak solution of $(*)$

**Theorem:** $u$ weak sol. of $(*)$ $\Rightarrow$ $u \in C^\infty(\Omega)$ and $\Delta u = 0$

**Proof:** (Sketch)

① $u_\varepsilon(x) = \int_\Omega u(y)\,\varphi_\varepsilon(x-y)\,dy$ on $\Omega_r$, $\varepsilon \in (0,r)$

solves $\Delta u_\varepsilon = 0$ and $u_\varepsilon \in C^\infty(\Omega_r)$




② Local estimate

$$|\nabla u_\varepsilon| \le C(\Omega_r) =: C$$

$\Rightarrow$ $u_\varepsilon$ $C$-Lipschitz on $\overline{B_\delta(x)} \subset \Omega_r$, $x \in \Omega_r$

## Arzela - Ascoli theorem

Let $(u_m)_{m\in\mathbb{N}}$ $u_m \in C^0(\overline{B_\delta(x)})$ s.t (i) $|u_m(x)| \leq M(x)$

uniformly $\longrightarrow$ (ii) $\forall \varepsilon > 0 \ \exists \delta > 0$

equicontinous $|u_m(x) - u_m(y)| < \varepsilon$

$\forall x, y \in \overline{B_\delta(x)}$. $|x-y| < \delta$

$\forall m \in \mathbb{N}$

(i) + (ii) $\Longleftrightarrow$ $\exists (u_{m_i})_{i\in\mathbb{N}}$ subsequence

s.t $u_{m_i} \longrightarrow v$ uniformly

② $u_\varepsilon$ C-Lipschitz

$$|u_\varepsilon(y)| \leq |u_\varepsilon(x)| + |x-y| \leq |u_\varepsilon(x)| + \delta$$

$$u_\varepsilon(x) = \int_{B_\delta(x)} u_\varepsilon(y)\, dy = \int_{B_\delta(x)} \int u(z)\varphi(z-y)\, dy\, dz$$

(ii) follows from C-Lipschitz

$\Longrightarrow \quad \exists\, \varepsilon_i \to 0$ s.t. $u_{\varepsilon_i} \to v$ uniformly

$$u_{\varepsilon_i}(y) = \fint_{B_r(y)} u_\varepsilon(z)\,dz \qquad \forall y \in B_\delta(x)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$v(y) \qquad\qquad\qquad\qquad \fint_{B_r(y)} v(z)\,dz$$

$\Longrightarrow$ $v$ harmonic

Finally: $\displaystyle\int_{\Omega_r} |u_{\varepsilon_i} - u|\,dx \to 0$

$\Longrightarrow$ $u = v$ on any $B_\delta(x) \subset \Omega_r$