Deriving the wave equation (for sound waves)

We start with equations of motion of a compressible fluid / gas
$\rightsquigarrow$ compressible Euler equations:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla (f \circ \rho) \qquad (1)$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \cdot u) = 0 \qquad (2)$$

where
- $u(x,y,z,t) \in \mathbb{R}^3$ the velocity of a particle in (x,y,z) and at time t
- $\rho(x,y,z,t) \in \mathbb{R}_{\geq 0}$ density of particles in (x,y,z) and at t

$f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the internal pressure that is fct of ρ. One assumes f increasing wrt $\rho \rightsquigarrow f' \geq 0$.

$f(\rho) = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}$ for air

γ is the adiabatic index (~ 1.4)

p_0 sea level atmospheric pressure

ρ_0 density w.r.t a reference temp

<u>Assumptions</u>: $|\rho - \rho_0|$ and the absolute value of the derivatives of $\rho - \rho_0$ are small $\sim \varepsilon$

$|u|$ and $|\nabla u|$ are small $\sim \varepsilon$

Then in the following we neglect all terms of order $\sim \varepsilon^2$

$$(2): \quad 0 = \rho_t + \operatorname{div}(\rho \cdot u)$$

$$= \rho_t + \underbrace{\operatorname{div}((\rho - \rho_0)\cdot u)}_{\langle \nabla(\rho-\rho_0), u\rangle \; + \; \underbrace{(\rho-\rho_0)\operatorname{div} u}_{\sim \varepsilon^2}} + \rho_0 \underbrace{\operatorname{div} u}_{\sim \varepsilon}$$

$$\rightsquigarrow \quad 0 = \rho_t + \rho_0 \operatorname{div} u \sim \varepsilon^2 \quad (2)'$$

$$(1): \quad u \cdot \nabla u \sim \varepsilon^2$$

$$\nabla(g \circ \rho) = \underbrace{g'(\rho)}_{g'(\rho_0) + g''(\rho_0)\underbrace{(\rho-\rho_0)}_{\sim\varepsilon} + \underbrace{o(|\rho-\rho_0|)}_{\sim\varepsilon^2}} \overbrace{\nabla(\rho-\rho_0)}^{\sim\varepsilon}$$

$$\frac{1}{\rho} = \frac{1}{\rho_0} - \frac{1}{\rho_0^2}\underbrace{(\rho-\rho_0)}_{\sim\varepsilon^2} + \underbrace{o(|\rho-\rho_0|)}_{\sim\varepsilon^2}$$

$$\frac{\nabla(g\circ\rho)}{\rho} \rightsquigarrow \frac{g'(\rho_0)}{\rho_0}\nabla(\rho-\rho_0)$$

We replace (2) with $u_t = -\frac{f'(\rho_0)}{\rho_0} \nabla(\rho - \rho_0)$

$$= -\frac{f'(\rho_0)}{\rho_0} \nabla \rho$$

$$\Longrightarrow \quad \rho_{tt} = -\rho_0 (\operatorname{div} u)_t$$

$$= -\rho_0 \operatorname{div} u_t$$

$$= \cancel{\rho_0} \cdot \frac{f'(\rho_0)}{\cancel{\rho_0}} \operatorname{div} \nabla \rho = \underbrace{f'(\rho_0)}_{= c^2} \Delta \rho$$

Kirchhoff's formula (3 D)

Theorem

Let $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ be a solution of $u_{tt} = c^2 \Delta u$ with initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$

Then $$u(\bar{x},\bar{t}) = \frac{1}{4\pi c^2 \bar{t}} \int_{\partial B_{c\bar{t}}(\bar{x})} \psi(x)\, ds(x) + \frac{\partial}{\partial \bar{t}} \left[\frac{1}{4\pi c^2 t} \int_{\partial B_{c\bar{t}}(x)} \phi(x)\, ds(x) \right]$$

Remark

Hence $u(\bar{x},\bar{t})$ does only depend on $\partial B_{c\bar{t}}(\bar{x})$

$\leadsto$ Huygens's principle

Lemma (Euler-Poisson-Darboux equations)

Let $u \in C^2(\mathbb{R}^m \times [0,\infty))$ be a solution of $u_{tt} = c^2 \Delta u$ on $\mathbb{R}^m$ with initial cond. ϕ, ψ

Define $$\bar{u}(r,t) = \fint_{\partial B_r(0)} u(x,t)\, ds(x)$$

Then $\bar{u} \in C^2((0,\infty) \times [0,\infty))$ and $\bar{u}$ solves

$$\bar{u}_{tt} = c^2 \left(\bar{u}_{rr} - \frac{m-1}{r} u_r \right)$$
$$\bar{u}(r,0) = \bar{\Phi}(r), \quad \bar{u}_t(r,0) = \bar{\Psi}(r)$$

where $$\bar{\Phi}(r) = \fint_{\partial B_r(0)} \phi(x)\, ds(x), \quad \bar{\Psi}(r) = \int_{\partial B_r(0)} \psi(x)\, ds(x).$$

Remark

$\frac{d^2}{dr^2} - \frac{m-1}{r} \frac{d}{dr}$ is the radial part of Δ in ~~polar~~ spherical coordinates.

Proof (Lemma a)

$$\bar{u}(r,t) = \fint_{\partial B_r(0)} u(x,t)\, ds(x) = \underbrace{\frac{r^{m-1}}{vol(\partial B_r(0))}}_{C(m)} \int_{\partial B_1(0)} u(rx,t)\, ds(x)$$

$$u(rx) \leq \max_{B_{2r_0}(0)} u \qquad r \in (0, 2r_0) \quad \forall x \in \partial B_1(0)$$

$$\frac{d}{dr} u(rx) = \langle \nabla u(rx), x \rangle \leq \max_{B_{2r_0}(0)} |\nabla u| \qquad r \in (0, 2r_0)$$

$$\Longrightarrow \frac{d}{dr} \bar{u}(r,t) = C(m) \int_{\partial B_1(0)} \frac{\partial u}{\partial r}(rx)\, ds(x) \qquad x \in \partial B_1(0)$$

Similar for $\frac{d^2}{dr^2}$ and for derivatives w.r.t t.

$$\Longrightarrow u \in C^2((0,\infty) \times [0,\infty))$$

$$\frac{d}{dr}\bar{u}(r,t) = c(m)\int_{\partial B_1(0)} \langle \nabla u(rx), x\rangle \, ds(x)$$

$$= \frac{1}{vol(\partial B_r(0))} \cdot \int_{\partial B_r(0)} \langle \nabla u(y), \underbrace{\frac{y}{r}}_{\nu} \rangle \, ds(y)$$

$$= \underbrace{\frac{1}{vol(\partial B_r(0))}}_{m\alpha(m)\, r^{m-1}} \cdot \int_{B_r(0)} \underbrace{\Delta u(x)}_{u_{tt}\cdot\frac{1}{c^2}} \, dx$$

$$\underbrace{\frac{d}{dr}\left(r^{m-1}\frac{d}{dr}\bar{u}(r,t)\right)} = \frac{1}{m\alpha(m)}\frac{d}{dr}\int_{B_r(0)} \frac{1}{c^2} u_{tt}\, dx$$

$$(m-1)r^{m-2}\bar{u}_r + r^{m-1}\bar{u}_{rr}$$

$$= \frac{1}{m\alpha(m)}\int_{\partial B_r(0)} \frac{1}{c^2} u_{tt}\, ds(x)$$

$$= r^{m-1}\frac{1}{c^2}\frac{d^2}{dt^2} \fint_{\partial B_r(0)} u \, ds(x) = \frac{r^{m-1}}{c^2}\bar{u}_{tt}$$

Proof (Theorem)

$n=3$: $\quad \bar{u}_{tt} = c^2\left(\bar{u}_{rr} - \frac{2}{r}\bar{u}_r\right)$

(1) $\rightsquigarrow \quad v(r,t) = r\,u(r,t)$

$\Longrightarrow \quad v_r = \bar{u} + r \cdot \bar{u}_r$

$v_{rr} = \bar{u}_r + \bar{u}_r + r\,\bar{u}_{rr} = 2\bar{u}_r + r\bar{u}_{rr}$

$= \frac{1}{c^2}\bar{u}_{tt} = \frac{1}{c^2} v_{tt}$

(2) $\bar{u}(r,t) \longrightarrow u(0,t) \quad r \to 0$ because u is cont.

$\Longrightarrow \quad v(r,t) \longrightarrow 0 \quad r \to 0$

Hence v solves

$v_{tt} = c^2 v_{rr}$ on $(0,\infty)\times[0,\infty)$

$\lim_{r\to 0} v(r,t) = 0$, $\quad v(r,0) = r\,\bar{\phi}(r) = \phi^*$, $\quad v_t(r,0) = \bar{\psi}(r) = \psi^*$

$\Longrightarrow$ Solution is given by

$$v(r,t) = \begin{cases} \frac{1}{2}\left[\Phi^*(ct+r) - \Phi^*(ct-r)\right] + \frac{1}{2c}\int_{ct-r}^{ct+r} \Psi^*(s)\,ds & 0 < r \leq c|t| \\ \frac{1}{2}\left[\Phi^*(r+ct) - \Phi^*(r-ct)\right] + \dots & r \geq c|t| \end{cases}$$

$$u(0,t) = \lim_{r\to 0} \bar{u}(r,t) = \lim_{r\to 0} \frac{1}{r} v(r,t)$$

$$= \lim_{r\to 0}\left(\frac{1}{2r}\left[\Phi^*(ct+r) - \Phi^*(ct-r)\right] + \frac{1}{2cr}\int_{ct-r}^{ct+r}\Psi^*(s)\,ds\right)$$

$$= (\Phi^*)'(ct) + \frac{1}{c}\Psi^*(ct)$$

$$= (r\,\bar{\Phi}(r))'\big|_{ct} + t\,\bar{\Psi}(ct)$$

$$= \frac{d}{dt}\left(t\,\overline{\Phi}(ct)\right) + t\,\overline{\Psi}(ct)$$

$$= \frac{1}{4\pi c^2 t} \int_{\partial B_r(0)} \psi(x)\, ds(x) + \dots \qquad \Box$$

$x = 0$ we are done

we get the formula for all x by translation with $-x$. $\Box$

Formula in 2D

Let $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ s.t. u solves $u_{tt} = c\Delta u$

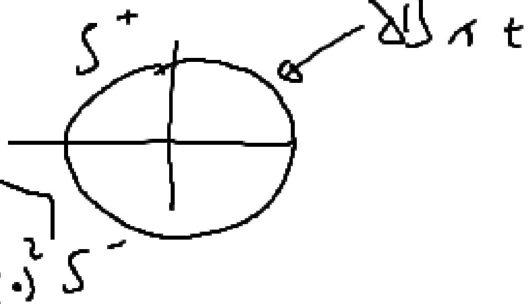
Define $v(x,y,z,t) = u(x,y,t)$

$\Longrightarrow$ v solves the wave equation

Kirchhoff's formula

$$\Longrightarrow v(\underbrace{x_0, y_0, 0}_{0}, t) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(0)} \psi(x,y,z,t)\, ds(x,y,z) + \dots$$

Note $\partial B_{ct}(0) = S^+ \cup S^-$



$$(x,y,z) \in S^{+/-} \Longleftrightarrow z = \sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}$$

$$ds(x) = \sqrt{1 + (z_x)^2 + (z_y)^2}\, dx\, dy$$

$$V(0,t) = \frac{1}{2\pi c^2 t} \int_{\{(x,y) : x^2+y^2 \leq c^2t^2\}} \psi(x,y) \sqrt{1+(z_x)^2+(z_y)^2} \, dx\, dy$$

$$+ \ldots$$

$$1 + (z_x)^2 + (z_y)^2 = 1 + \left(-\frac{x}{z}\right)^2 + \left(-\frac{y}{z}\right)^2$$

$$= \frac{c^2t^2}{z^2} = \frac{c^2t^2}{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}$$

$$V(0,t) = \frac{1}{2\pi c} \int_{\{x^2+y^2 \leq c^2t^2\}} \frac{\psi(x,y)}{\sqrt{c^2t^2-(x-x_0)^2-(y-y_0)^2}} \, dx\, dy$$

$$+ \ldots$$