$$\left.\begin{aligned} & u_{tt} = c^2 \Delta u \quad \text{on } \mathbb{R}^3 \times [0, \infty) \\ & \lim_{(x,t)\to(x_0,0)} u(x,t) = \phi(x) \\ & \lim_{(x,t)\to(x_0,0)} u_t(x,t) = \psi(x) \quad \text{on } \mathbb{R}^3 \end{aligned}\right\} (*)$$

__Kirchhoff's formula__

$u \in C^2(\mathbb{R}^3 \times [0, \infty))$ solves $(*)$

$$\Longrightarrow \quad u(x,t) = t\,\bar{\Phi}(x,t) + \frac{\partial}{\partial t}\left(t\,\bar{\Psi}(x,t)\right)$$

where $\bar{\Psi}(x,t) = \fint_{\partial B_{ct}(x)} \psi(y)\,dS(y) = \frac{1}{4\pi c^2 t^2} \int_{\partial B_{ct}(x)} \psi(y)\,dS(y)$

$\bar{\Phi}(x,t) = \ldots$

Remark (Alternative form)

$$\frac{\partial}{\partial t}\left(t\,\bar{\Phi}(x,t)\right) = \frac{\partial}{\partial t}\left(\cancel{t}\cdot\frac{1}{4\pi c^2 t^{\cancel{2}}}\int_{\partial B_{ct}(x)} \phi(y)\,ds(y)\right)$$

$$= \frac{\partial}{\partial t}\left(t\,\frac{1}{4\pi \cancel{c^2t^2}}\int_{\partial B_1(0)} \phi(x+cty)\cdot\cancel{c^2t^2}\,ds(y)\right)$$

$$= \bar{\Phi}(x,t) + t\cdot\frac{1}{4\pi}\int_{\partial B_1(0)} \langle\nabla\phi(x+cty),\, c\cdot y\rangle\,ds(y)$$

$$= \bar{\Phi}(x,t) + \frac{\cancel{tc}}{4\pi c^2t^2}\int_{\partial B_{ct}(x)} \left\langle\nabla\phi(y),\, \frac{y-x}{\cancel{ct}}\right\rangle ds(y)$$

$$= \fint_{\partial B_{ct}(x)} \left(\phi(y) + \langle\nabla\phi(y),\, y-x\rangle\right) ds(y)$$

$$u(x,t) = \fint_{\partial B_{ct}(x)} \left(t\psi(y) + \phi(y) + \langle\nabla\phi(y),\, y-x\rangle\right) ds(y)$$

Similar for $n = 2$:

$$u(x,t) = \frac{1}{2} \int_{B_{ct}(x)} \frac{t\psi(y) + \phi(y) + t\langle \nabla\phi(y), y-x\rangle}{\sqrt{t^2 - |y-x|^2}} \, ds(y)$$

<u>Theorem</u>: (Solution for wave equa. in 3D)

$n = 3$. $\phi, \psi \in C^2(\mathbb{R}^3)$. Define u by Kirchhoff's formula

Then $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ and solves $(*)$.

<u>Proof</u>: Assume $\phi = 0$

$$\Longrightarrow \quad u(x,t) = \underline{t \cdot \Psi(x,t)} = t \cdot \frac{1}{4\pi c^2 t^2} \int_{\partial B_{ct}(x)} \psi(y) \, ds(y)$$

$$= t \cdot \frac{1}{4\pi} \int_{\partial B_1(0)} \psi(x + cty) \, ds(y)$$

$$\Longrightarrow u \in C^2(\mathbb{R}^3 \times [0,\infty))$$

$$\left[\lim_{t \to 0} \int f(x,t) \, dx = \int \lim_{t \to 0} f(x,t) \, dx \right.$$

$f_t(x) \to f_0(x)$

- $u_t(x,t)$