Excursion:

Proposition $f_n : X \subset \mathbb{R}^n \to \mathbb{R}$

$f_n \to f$ uniformily

$\forall \varepsilon > 0 \ \exists n_\varepsilon$ st $|f_n(x) - f(x)| < \varepsilon \ \forall n > n_\varepsilon \ \forall x \in X$

Assume f_n, f are integrable

$$\lim_{n \to \infty} \int_X f_n = \int_X f \qquad y \in \partial B_1(0)$$

Example: $g \in C_c^0(\mathbb{R}^n)$ $\quad f_\varepsilon(x) = g(x + \varepsilon y)$

Prove that $\lim_{\varepsilon \to 0} \int f_\varepsilon(x) = \int g$ ✓

Show $f_\varepsilon \to g$ uniformily

$\forall \eta > 0 \ \exists \delta > 0$ st $|g(x + \varepsilon y) - g(x)| < \eta$ if $\varepsilon < \delta$.

Similar $g \in C^0(\mathbb{R}^n)$

$g|_{\underbrace{\partial B_1(0)}_{\text{cpt}}}$ $\qquad f_\varepsilon(x) = g(x + \varepsilon y)$

$x \in \partial B_1(0)$

$\Rightarrow g|_{\partial B_1(0)}$ uniformily cont.

same argument as before.

<u>Another example:</u> $g \in C^1(\mathbb{R}^n)$ $\quad t > 0$

$$\frac{\partial}{\partial t}\Big|_0 \int_{\partial B_1} g(x + ty)\, dx = \int_{\partial B} \frac{\partial}{\partial t}\Big|_0 g(x + ty)\, dx$$

$$h_t(x) = \frac{g(x + ty) - g(x)}{t} \to \langle \nabla g \cdot y \rangle = \frac{d}{dt}\Big|_0 g(x + ty)$$

$$h_t \longrightarrow \frac{d}{dt}\Big|_0 g(x+ty) \quad \text{uniformly}$$

$$h_t(x) = \frac{g(x+ty) - g(x)}{t} = \frac{d}{ds} g(x + \xi y)$$

$$\xi_t \in (0,t)$$

$g \in C^1(\mathbb{R}^n) \Rightarrow \nabla g$ is unif. cont. $\overline{B_3(0)}$

$$\Rightarrow \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \left| \frac{d}{ds}\Big|_\xi g(x+sy) - \frac{d}{ds}\Big|_0 g(x+sy) \right| < \varepsilon$$

if $t < \delta \quad \forall x$

$$\Rightarrow \lim_{t \to 0} \int_{\partial B_1} h_t = \int_{\partial B_1} \frac{d}{ds}\Big|_0 g(x+sy)\, dy \quad \square$$

Wave equation with source term

$$\left.\begin{array}{l} u_{tt} - c^2 \Delta u = f(x,t) \\ u(x,0) = 0, \quad u_t(x,0) = 0 \end{array}\right\} \quad (**)$$

For ϕ, ψ initial conditions

then $u(x,t) = \underbrace{t\,\Psi(x,t)}_{(\mathcal{T}(t)\psi)(x)} + \underbrace{\frac{d}{dt}\, t\,\bar{\Phi}(x,t)}_{\frac{d}{dt}\mathcal{T}(t)\phi(x)}$ is the solution of wave eq.

Duhamel's principle suggests

$$v(x,t) = \int_0^t \mathcal{T}(t-s) f(x,s)\, ds \quad \text{solves } (**).$$

$$v(x,t) = \int_0^t \underbrace{(t-s)\,\bar{g}(x, t-s, s)}_{\Sigma(t-s)\, g(x,s)} \, ds$$

surface measure

$$= \int_0^t \cancel{(t-s)} \frac{1}{4\pi c^2 (t-s)^{\cancel{2}}} \int_{\partial B_{c(t-s)}(x)} g(y,s)\, dS(y)\, ds$$

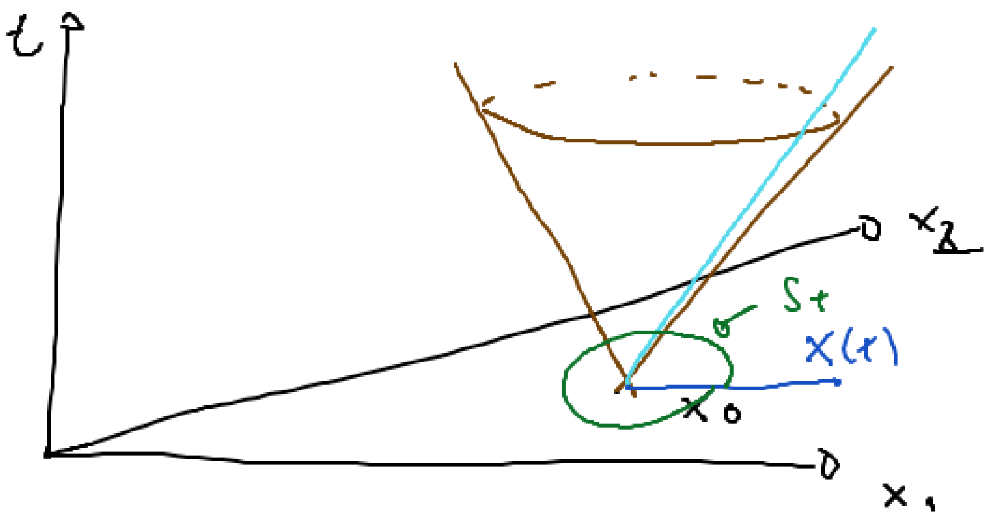
$$c(t-s) = |x-y|$$

$$= \int_0^t \frac{1}{4\pi c} \int_{\partial B_{c(t-s)}(x)} \frac{g\left(y, t - \frac{|y-x|}{c}\right)}{|y-x|}\, dS(y)\, ds$$

$$= \cancel{\int} \frac{1}{4\pi c} \int_{B_{ct}(x)} g(y, t - |y-x|/c)/|y-x| \, \underline{dy}$$

compare formula with the Poisson formula for solutions of the Laplace equation.

Relativistic geometry



$$X(t) = \underbrace{x_0}_{\in \mathbb{R}^3} + \underbrace{v_0}_{\in \mathbb{R}^3} t$$

$(X(t), t) \in \mathbb{R}^4$ light ray

v_0 $|v_0|^2 = c$

speed of the wave

Characteristic surfaces

Let $S \subset \mathbb{R}^4$ be 3D surfaces. $S_t = S \cap \underbrace{\{(x,t) : x \in \mathbb{R}^3\}}_{t = \text{const}}$

2D surface

S is called characteristic surface if it is a union of light rays of which each projection is orthogonal to S_t in $\mathbb{R}^3$ $\forall t \in \mathbb{R}$: $X'(t) \perp S_t$.

Theorem

$\{f = k\} = S$ is a characteristic surf $\forall k \in \mathbb{R}$ $\Longleftrightarrow$ g satisfies $|\nabla g| = \frac{1}{c}$ (eikonal equation)

$f : \mathbb{R}^4 \to \mathbb{R}$ smooth

More precisely $f(x,t) = t - g(x)$

for $g : \mathbb{R}^3 \to \mathbb{R}$ smooth

Proof: Assume $S = \{f = k\}$ characteristic surf $\forall k$

$x_0 \in \mathbb{R}^3 \Longrightarrow$ $\exists\ (x(t), t)$ light in S s.t. $x(0) = x_0$ and $\forall t : \frac{d}{dt} x(t) = v_0 \perp S_t$

$f(x_0, 0) = g(x_0) = k_0 \Longrightarrow f(x(t)) = t - g(x(t)) = k_0 \ \forall t$

we consider $S = \{f = k_0\}$

$0 = \frac{d}{dt} f \circ x(t) = 1 - \langle \nabla g(x(t)), v

$$\Longrightarrow \quad 1 = \langle \nabla g(x_0), v_0 \rangle$$

$$S_0 = \{ x \in \mathbb{R}^3 : f(x, 0) = k_0 = g(x) \}$$

$$\Longrightarrow \quad \nabla g(x) \perp S_0 \quad \forall x \in S_0$$

$$x'(0) = v_0 \perp S_0$$

$$\Longrightarrow \quad \nabla g(x_0) \parallel v_0$$

$$\Longrightarrow \quad \langle \nabla g(x_0), v_0 \rangle = |\nabla g(x_0)| \cdot \overbrace{|v_0|}^{= c}$$

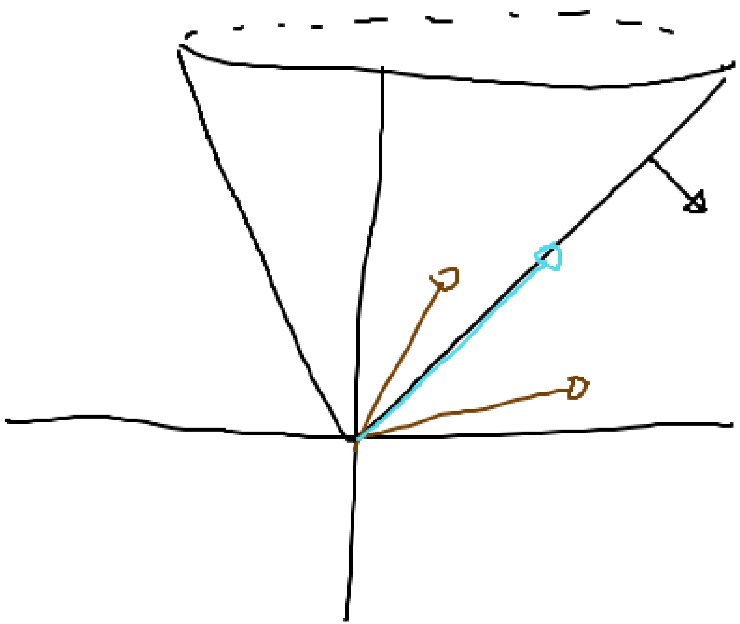
$$\Longrightarrow \quad \frac{1}{c} = |\nabla g|(x_0)$$

Since x_0 was arbitrary we get the result. $\boxed{\text{QED}}$

Def: if $(v,t) \in \mathbb{R}^4$ satisfies $t > c|v|$, then (v,t) is called timelike

if (v,t) satisfies $t < c|v|$, then ... spacelike

if (v,t) " $t = c|v|$, then ... null



$\mathbb{R}^4$ equipped with the Gelin form $\langle (v,t), (w,s) \rangle_1 = c \sum_{i=1}^{3} v_i w_i - ts$ is called Minkowski space time.

$|(v,t)|_1^2 = \langle (v,t), (v,t) \rangle_1 < 0$

timelike

and similar for spacelike and null

Lemma. $S = \{(x,t) : \underbrace{t - g(x)}_{= f(x,t)} = k\}$ is charact. iff $\forall N \in \mathbb{R}^4$ normal vector one has N is null.

Proof: N normal $\iff N = \alpha \nabla f = \alpha \begin{pmatrix} -\nabla g \\ 1 \end{pmatrix}$

N null $\iff c|\nabla g| = 1$ on S $\iff \frac{1}{c} = |\nabla g|$ on S

Example

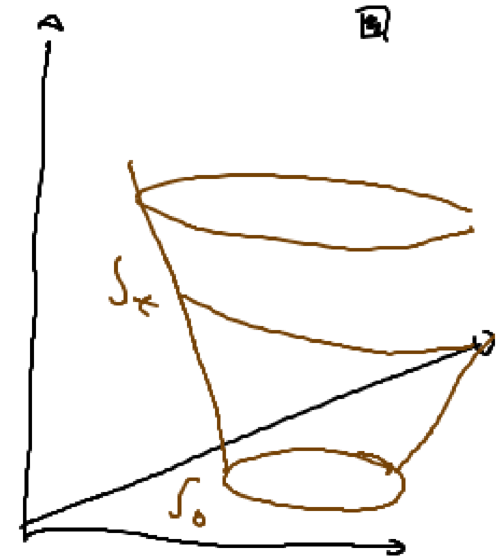
$$g(x) = \sum_{i=1}^{3} \frac{a_i}{c} \cdot x_i \quad \text{with } a_1^2 + a_2^2 + a_3^2 = 1$$

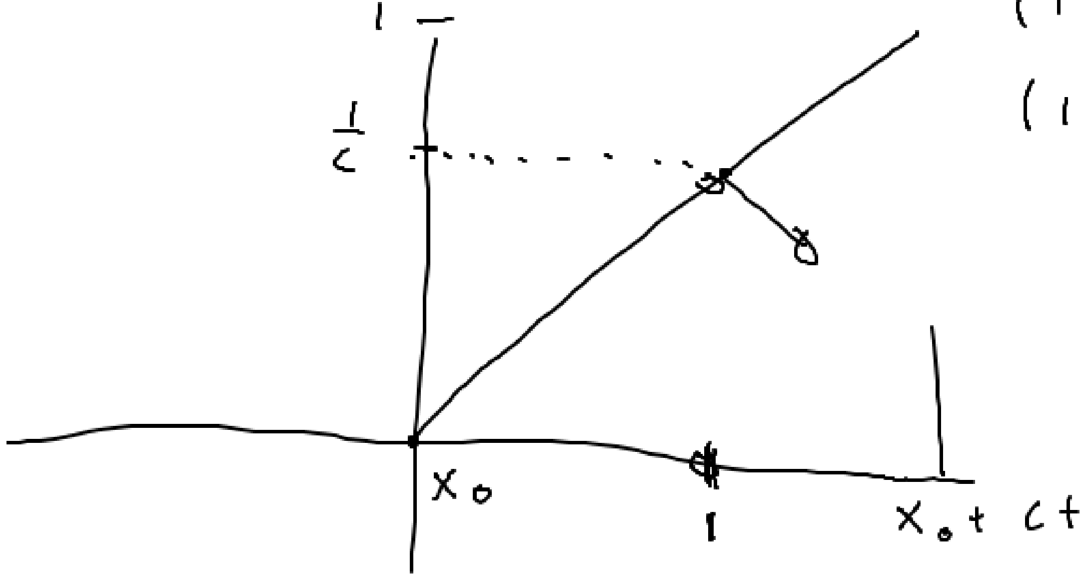
$$\Rightarrow \nabla g(x) = \begin{pmatrix} \frac{a_1}{c} \\ \frac{a_2}{c} \\ \frac{a_3}{c} \end{pmatrix}$$

$$\Rightarrow |\nabla g| = \frac{1}{c}$$

$t = 0 : S_0 = \{g = \frac{b}{c}\}$

$S_t = \{g = \frac{b}{c} + t\}$





$(1, \frac{1}{c})$

$(1, -c)$

$t =$

A surface $F = \{ (x,t) \in \mathbb{R}^4 \quad f(x,t) = k_0 \}$

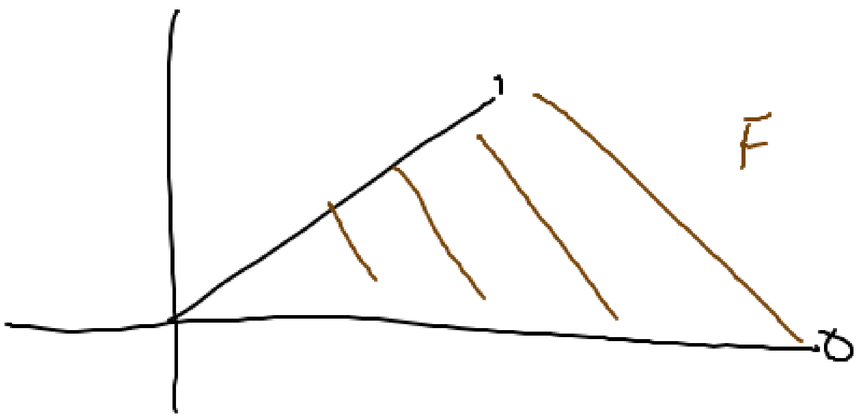
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t - g(x)$

is spacelike if $|\nabla g| < \frac{1}{c}$ ∀ on F so normal vectors are timelike

For instance if $g = \text{const}$

$\leadsto f(x,t) = t$

$\Rightarrow |\nabla g| = 0 < \frac{1}{c}$ ✓



Theorem

Let $S \subset \mathbb{R}^4$ spacelike $\in C^2(\mathbb{R}^4)$

Then $\exists!$ solution u of $c^2 \Delta u = u_{tt}$

with $u = \Phi$ and $\frac{\partial u}{\partial N} = \Psi$ on S

for $\Phi, \Psi \in C^2(S)$. N normal vector on S

For instance if $S = \{ g(x) = t \}$

then $N = \frac{1}{\alpha} \begin{pmatrix} -\nabla g \\ 1 \end{pmatrix} \rightsquigarrow \frac{\partial u}{\partial N} = \frac{1}{\alpha} \langle \nabla u, \begin{pmatrix} -\nabla g \\ 1 \end{pmatrix} \rangle$

$$= \frac{-1}{\alpha} \langle \nabla u, \nabla g \rangle_{\mathbb{R}^3} + \frac{1}{\alpha} u_t$$

Example: $u_{tt} = c^2 u_{xx}$ on $\mathbb{R} \times \mathbb{R}$

$$S = \{ (x, t) \mid \gamma(x) = t \} \quad \text{for } \gamma \in C^2(\mathbb{R}, \mathbb{R})$$