Fourier method

$u(x,0) = \phi(x)$
$u(x,t) = 0 \quad x \in \partial\Omega \quad t > 0$

Consider 3 equations

(1) $-iu_t = \underbrace{\frac{1}{2}k\Delta u + Vu}_{-Lu \text{ on } \mathbb{R}^m}$ (2) $u_t = \underbrace{k\Delta u}_{-Lu \text{ on } \Omega \subset \mathbb{R}^m}$ (3) $u_{tt} = \underbrace{c^2\Delta u}_{=Lu}$

together with boundary conditions (BC)

Assume the factor in front of Δ is 1.

Separation of variables: $u(x,t) = T(t)X(x)$

we have $T(t)$ satisfies (1) $-iu_t = \lambda u$ (2) $u_t = \lambda u$ 3) $u_{tt} = \lambda u$

and $X(x)$ s.t. $\boxed{\begin{array}{l} LX = \lambda X \\ \text{on } \mathbb{R}^m \text{ or } \Omega \subset \mathbb{R}^m + (BC) \end{array}}$ (xx)

" eigenvalue equation for L.

$X(x) = 0$ on $\partial\Omega$ $\forall x$

Goal: Find $(\lambda_k)_{k\in\mathbb{N}}$ and solutions v_k to (xx)

s.t. $\phi(x) = \sum_{k=0}^{\infty} A_k v_k$ in L^2-sense or

stronger. (for instance uniformly)

for as many ϕs as possible.

Let ϕ be an initial for (1) or (2)

then $u(x,t) = \sum_{k=0}^{\infty} T_{\lambda_k}(t) v_k(x)$ solves (1) or (2)

Remark: In case of (3) we need initial condition

ϕ and $\psi = \sum B_k v_k$

and $T_\lambda(t)$ is given $A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t$

$\Longrightarrow u(x,t) = \sum_{k=0}^{\infty} (A_k\cos(\sqrt{\lambda_k}t) + B_k\sin\sqrt{\lambda_k}t) v_k$

In general:

Existence of v_k depends on V, Ω and BC

Orthogonality of eigenfcts $(u, Av) = (Au, v)$, $(u, Lv) = (Lu, v)$.

Assume $g, f : \Omega \to \mathbb{C}$, inner product

$$(f, g) = \int_\Omega f \cdot \bar{g} \, dx$$

$$g = \mathrm{Re}(g) + i \, \mathrm{Im}(g)$$

$$\bar{g} = \mathrm{Re}(g) - i \, \mathrm{Im}(g)$$

$$\Rightarrow \|f\|^2 = (f, f) = \int_\Omega \mathrm{Re}(f)^2 + \mathrm{Im}(f)^2 \, dx$$

Remark

Equation (xx) makes sense also for $\mathbb{C}$ valued fcts

Green identity: Now Ω is bounded, $\partial\Omega$ smooth

$$\Rightarrow \int_\Omega Lu \cdot \bar{v} - \int_\Omega u \, \overline{Lv} = \int_\Omega (\Delta u \cdot \bar{v} - v \cdot \overline{\Delta u}) \, dx$$

~~Assume u, v are $\mathbb{R}$ valued~~ $$= \int_{\partial\Omega} \left(u \frac{\partial v}{\partial N} - v \frac{\partial u}{\partial N}\right) dx$$

$\Rightarrow$ RHS is 0 for homogeneous Dirichlet, Neumann and Robin conditions.

Corollary

① u, v ℝ-valued eigenfct. for real eigenvalues $\lambda_1 \neq \lambda_2$

$\Longrightarrow \quad 0 = (u, Lu) - (Lu, v) = (\lambda_1 - \lambda_2)(u, v)$

$\Longrightarrow \quad u, v$ orthogonal.

② $\lambda \in \mathbb{C}$ eigenvalue EV, u eigenfct.

$\Longrightarrow \quad 0 = (\bar{\lambda} - \lambda) \underbrace{(u, u)}_{> 0} \quad \Longrightarrow \quad \operatorname{Im}(\lambda) = 0$

$\Longrightarrow \quad \lambda$ is real.

In part. $\quad \Delta u = \Delta \operatorname{Re}(u) + i \Delta \operatorname{Im}(u)$

$\qquad\qquad = \lambda \cdot (\operatorname{Re}(u) + i \operatorname{Im}(u))$

$\Longleftrightarrow \quad \Delta \operatorname{Re}(u) = \lambda \operatorname{Re}(u)$

$\qquad\quad \Delta \operatorname{Im}(u) = \lambda \operatorname{Im}(u)$

Remark (EV with multiplicity)

If λ is EV and $\exists\, u_1, u_2$ Eigenfct.
and lin. independent.

Gram-Schmidt $\Rightarrow \exists\, \tilde{u}_2$ Eigenfct.
s.t. $(u_1, \tilde{u}_2) = 0$ and $u_1, \tilde{u}_2$ span the
same linear space as u_1 and u_2

Theorem: $\phi(x) = \sum_{n=0}^{\infty} A_n v_n$ in L^2-sense

$$\int_\Omega \left| \phi - \sum_{n=0}^{N} A_n v_n \right|^2 dx \longrightarrow 0$$

$$\Rightarrow \quad A_n = \frac{(\phi, v_n)}{(v_n, v_n)}.$$

Proof: $$(\phi, v_m) = \int_\Omega \left(\sum_{n=0}^{\infty} A_n v_n\right) v_m \, dx = \sum_{n=0}^{\infty} A_n \underbrace{\int v_n v_m \, dx}_{=0 \text{ if } n \neq m}$$

$\circledast$

Completeness

If $L = -\Delta$ with (D), (N) or (R) B C.

then $\Phi = \sum^{\infty} A_n v_n$ for a set eigenfct v_n on $\partial\Omega$

$\forall\, \Phi : \Omega \to \mathbb{R}$ and L^2-integrable.

(Ω as before).