Let $\Omega \subset \mathbb{R}^n$, $L = -\Delta$ on Ω, Ω bounded, $\partial\Omega$ is smooth

Theorem: L together with homog Dir. BC. Then all eigenvalues are positive.

(Neumann or Robin BC $\Rightarrow$ $\forall$ EV are nonneg.)

Proof: 1st. Green id., u eigenfct. for EV λ

$$\Rightarrow -\int_\Omega \Delta u \cdot u \, dx = \int_\Omega |\nabla u|^2 dx - \underbrace{\int_{\partial\Omega} (\ldots)}_{=0}$$

$$\Rightarrow \lambda = \int_\Omega |\nabla u|^2 dx \Big/ \int_\Omega u^2 dx \geq 0$$

if "=" $\Rightarrow |\nabla u|^2 = 0 \Rightarrow \nabla u = 0 \Rightarrow u = c = \text{const}$

Dir. BC $\Rightarrow u = 0$ $\lightning$

$\square$

Example: What eigenfcts of $Q = [0, \pi]^m = \Omega$
with Dir. BC.

$$-\Delta v = -\sum_{i=1}^{m} \underbrace{u_{x_i x_i}}_{\Delta^{1D} u(\dots x_i \dots)}$$

Eigenfct of 1D problem are $\sin(kx) = v_k$ $\lambda_k = k^2$
$k \in N$

$$\Rightarrow \quad v_{\vec{k}}(x_1, \dots, x_m) = \prod_{i=1}^{m} v_{k_i}(x) \qquad \vec{k} = (k_1, \dots, k_m) \in N^m$$

is a complete of eigenfct for $-\Delta$ on Q
with EV $\lambda_{\vec{k}} = \sum_{i=1}^{m} \lambda_{k_i}$.

Vibrations of a (2D) disk.

$$(*) \quad \begin{cases} u_{tt} = \Delta u & \text{on } \Omega = U_a(0) \subset \mathbb{R}^2 \\ u(\cdot, t) = 0 & \text{on } \partial\Omega \\ u(x,0) = \Phi(x), \ u_t(x,0) = f(x) \end{cases}$$

$\Longrightarrow$ EV equation: $$\left.\begin{aligned} -\Delta v &= \lambda v && \text{on } \Omega \\ v(x) &= 0 && \text{on } \partial\Omega \end{aligned}\right\} (**)$$

Note $\lambda > 0$.

Polar coord. $\tilde{v}(r, \theta) = v(r\cos\theta, r\sin\theta)$

$$(**) \Longrightarrow -\left(\tilde{v}_{rr} + \frac{1}{r}\tilde{v}_r + \frac{1}{r^2}\tilde{v}_{\theta\theta}\right) = \lambda v$$

$$v(a, \theta) = 0 \quad \forall \theta, \quad \tilde{v}(0, \theta) < \infty$$

$$v(r, \theta) \quad 2\pi \text{ periodic in } \theta$$

Separate Var. $\tilde{v}(r, \theta) = R(r) \cdot \Theta(\theta)$

$$\Longleftarrow \quad \frac{\left(R'' + \frac{1}{r}R' + \lambda R\right) r^2}{R} = -\frac{\Theta''}{\Theta} = \gamma \in \mathbb{R}$$

$V(r,\cdot)$ 2π-periodic $\Longrightarrow$ $\Theta(\theta) = A_m \cos(m\theta) + B_m \sin(m\theta)$

and $\gamma = m^2$ $m \in \mathbb{N}_0$.

$$\Longrightarrow \quad R'' + \frac{1}{r} R' + \left(\lambda - \frac{m^2}{r^2}\right) R = 0$$

$$R(0) < \infty, \quad R(a) = 0$$

New variable: $\rho = \sqrt{\lambda} \cdot r \quad \leadsto \quad \tilde{R}(\rho) = R\left(\rho/\sqrt{\lambda}\right)$

$$\Longrightarrow \quad \tilde{R}'' + \frac{1}{\rho} \tilde{R}' + \left(1 - \frac{m^2}{\rho^2}\right) \tilde{R} = 0$$

Bessel different. equation.

<u>Remark</u>: Coeff. are singular in $\rho = 0$.

But the point $\rho = 0$ is a so-called regular sing. point.

$\tilde{R}(\rho) \sim c \cdot \rho^\alpha$ for $\rho \to 0$ for some $\alpha \geq 0$

Hence. Assume $\tilde{R}(\rho) = \omega(\rho) \cdot \rho^{\alpha}$ for $\omega(\rho) = \sum_{k=0}^{\infty} a_k \rho^k$

$$= \sum_{k=0}^{\infty} a_k \rho^{\alpha+k}$$

$\xrightarrow{\text{ODE}}$ $0 = \sum_{k=0}^{\infty} a_k (\alpha+k)(\alpha+k-1) \rho^{\alpha+k-2} + \sum_{k=0}^{\infty} a_k (\alpha+k) \rho^{\alpha+k-1}$

$$\sum_{k=0}^{\infty} \left(a_k \rho^{k+\alpha} - m^2 a_k \rho^{k+\alpha-2} \right)$$

$\Rightarrow$ $k=0$: $(\alpha(\alpha-1) + \alpha - m^2) a_0 = 0$

$$\Rightarrow \alpha^2 = m^2 \Rightarrow \alpha = \pm m$$

Assume $\alpha = m$

$k=1$: $(\alpha(\alpha+1) + (\alpha+1) - m^2) a_1 = 0$

$$\Rightarrow a_1 = 0$$

$k \geq 2$ $((\alpha+k)(\alpha+k-1) + (\alpha+k) - m^2) a_k = -a_{k-2}$

$\Rightarrow$ $a_k = 0$ if $k = 2j+1$ $a_k = -\dfrac{a_{2j-2}}{(m+2j)^2 - m^2} =$

$$= -\frac{a_{2j-2}}{(m+2j)^2 - m^2} = -\frac{a_{2j-2}}{2j\cdot(2m+2j)} = -\frac{a_{2j-2}}{2\cdot 2\; j\,(m+j)}$$

if $a_0 = 1 \Longrightarrow a_{2j} = (-1)^j \cdot \frac{1}{j!\,2^{2j}(m+1)\cdot\ldots\cdot(m+j)}$

if $a_0 = \frac{1}{2^m m!}$ $\quad a_{2j} = (-1)^j \frac{1}{j\,2^{2j+m}(m+j)!}$

$$\Longrightarrow \sum_{j=0}^{\infty} (-1)^j \left(\tfrac{1}{2}\rho\right)^{2j+m} \frac{1}{j!\,(m+j)!} =: J_m(\rho)$$

Bessel fct of order m.

<u>Remark</u>: $\quad \rho \to 0: \; J_m(\rho) \sim c\cdot\rho^m$

$\quad \rho \to \infty: \; J_m(\rho) \sim \sqrt{\frac{2}{\pi\rho}}\cdot\cos\left(\rho - \frac{\pi}{4} - \frac{m\pi}{2}\right) + O\left(\frac{1}{\rho^{3/2}}\right)$

$R(r) = c\cdot J_m(\sqrt{\lambda}\cdot r)$.

Homog. Dir BC require $R(a) = c\, J_m(\sqrt{\lambda}\, a) = 0$

$$\lambda \longmapsto J_m(\sqrt{\lambda}\cdot a) \text{ has roots } 0 < \lambda_{m1} < \lambda_{m2} \ldots$$

<u>Theorem</u>: eigenfcts of $-\Delta$ with homog. Dirichlet BC are

$$\tilde{v}_{mn}(r,\theta) = J_m(\sqrt{\lambda_{mn}}\cdot r)\cdot \cos(m\theta)$$

$$\tilde{w}_{mn}(r,\theta) = J_m(\sqrt{\lambda_{mn}}\, r)\cdot \sin(m\cdot\theta)$$

$m, n \in \mathbb{N}_0$ The eigenvalues is λ_{mn}.

<u>Remark</u>

$\tilde{v}_{mn}$ are orthogonal

$$\rightsquigarrow \int_0^{2\pi}\int_0^a \tilde{v}_{mn}(r,\theta)\, \tilde{v}_{kl}(r,\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \cos(m\theta)\cdot\cos(k\theta)\,d\theta \int_0^a J_m(\sqrt{\lambda_{mn}}\,r)\cdot J_k(\sqrt{\lambda_{kl}}\,r)\,r\,dr$$

$$= 0 \text{ iff } (m,n) \neq (k,l)$$

Example

The solutions of (x) are $\tilde{u}(r,\theta,t) = \dots$

$\tilde{u}(r,\theta,0) = 0$, $\tilde{u}_t(r,\theta,0) = \tilde{\varphi}(r)$

Expand $\tilde{\varphi}(r) = \sum_{m=1}^{\infty} \underline{\beta_{0m} C_{0m}} J_0(\beta_{0m} \cdot r)$

$$\beta_{0m} = \sqrt{\lambda_{0m}}$$

$$\Longrightarrow \tilde{u}(r,\theta,t) = \sum_{m=1}^{\infty} \underline{\underline{C_{0m}}} J_0(\beta_{0m} \cdot r) \cdot \sin(\beta_{0m} t)$$

$$\beta_0 C_{0m} = \int_0^a \tilde{\varphi}(r) J_0(\beta_{0m} r) \, r \, dr \Big/ \int_0^a J_0(\beta_{0m} r)^2 \, r \, dr$$

<u>Now</u>:

$$\left.\begin{aligned} -\Delta v &= \lambda v \quad \text{on } \Omega = B_a(0) \subset \mathbb{R}^3 \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \text{(xxx)}$$

Spherical coord. $\tilde{v}(r,\phi,\theta) = v(r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta)$

$r \in [0,a],\ \phi \in [0,2\pi]$

$\theta \in [0,\pi]$

$$\Longrightarrow \quad 0 = \tilde{v}_{rr} + \frac{2}{r}\tilde{v}_r + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin^2\theta}\tilde{v}_{\phi\phi} + \frac{1}{\sin\theta}(\sin\theta\, v_\theta)_\theta\right)}_{\text{Laplace op. on } \partial B_1(0)} + \lambda\tilde{v}$$

Sep. of Var.: $\tilde{v}(r,\phi,\theta) = R(r)\cdot Y(\phi,\theta)$

$$\Longrightarrow \quad R'' + \frac{2}{r}R' + \left(\lambda - \frac{\gamma}{r^2}\right)R = 0 \qquad \gamma \in \mathbb{R}$$

$$\frac{1}{\sin^2\theta}Y_{\phi\phi} + \frac{1}{\sin\theta}(\sin\theta\, Y_\theta)_\theta + \gamma Y = 0$$

Write: $\quad w(r) = \sqrt{r} \cdot R(r)$

$$\Longrightarrow \quad w'' + \frac{1}{r} w' + \left(\lambda - \frac{\gamma + \frac{1}{4}}{r^2} \right) w = 0$$

$\tilde{R}(\rho) = w\left(\rho/\sqrt{\lambda} \right)$ satisfies Bessel's eqn.

$$\Longrightarrow \quad w(r) = J_{\sqrt{\gamma + \frac{1}{4}}} \left(\sqrt{\lambda} \cdot r \right)$$

$$\Longrightarrow \quad R(r) = J_{\sqrt{\gamma + \frac{1}{4}}} \left(\sqrt{\lambda} \cdot r \right) \frac{1}{\sqrt{\lambda}} .$$

$R(0) < \infty \qquad R(a) = 0$

Seperation of Var: $Y(\Phi, \Theta) = q(\Phi) \cdot p(\Theta)$

$$\Longrightarrow \quad \frac{q''}{q} = -\frac{\sin\Theta(\sin\Theta\, p_\Theta)_\Theta}{p} - \gamma \sin^2\Theta = -\alpha$$

2π-periodic BC: $q(\Phi) = A\cos(m\Phi) + B\sin(m\Phi) \in \mathbb{R}$

$\alpha = m^2 \quad m \in \mathbb{N}_0$

$$\Longrightarrow \quad \frac{(\sin\Theta\, p_\Theta)_\Theta}{\sin\Theta} + \left(\gamma + \frac{m^2}{\sin^2\Theta}\right) \cdot p = 0$$

$p(0), p(\pi) < \infty$

$s = \cos\Theta$: $\quad \frac{d}{ds}\left((1-s)^2 \frac{d\tilde{p}}{ds}\right) + \left(\gamma - \left(\frac{m^2}{1-s^2}\right)\right) p = 0$

$\tilde{p}(s)$

$= p(\arccos(s))$ (associated) Legendre equation.

$\tilde{p}(-1), \tilde{p}(1) < \infty$

$\Longrightarrow \quad \gamma = \ell \cdot (\ell+1) \quad \ell \in \mathbb{N}, \ \ell \geq m$

with solutions $P_\ell^m(s) = \frac{(-1)^m}{2^\ell \cdot \ell!}(1-s^2) \cdot \frac{d^{\ell+m}}{ds^{\ell+m}}(s^2-1)^\ell$

associated Legendre Polynomials

<u>Hence:</u> $\tilde{V}(r,\phi,\theta) = R(r)\cdot q(\phi)\cdot p(\theta)$

$$= J_{\sqrt{\gamma+\frac{1}{4}}}(2\sqrt{\lambda}\, r)\frac{1}{\sqrt{\lambda}}\left(A\cos(m\phi)+B\sin(m\phi)\right) \cdot P_\ell^m(\cos\theta)$$

where $\ell \geq m$, $\gamma = \ell(\ell+1)$

$\rightsquigarrow \sqrt{\ell(\ell+1)+\frac{1}{4}} = \ell + \frac{1}{2}$

Replace sin and cos terms by $e^{im\phi}$, $m \in \mathbb{Z}$

And let $\lambda_{\ell j}$ be roots of $J_{\ell+\frac{1}{2}}(\sqrt{\lambda} \cdot a)$

$$\Longrightarrow \quad \tilde{V}_{\ell m j}(r, \phi, \theta) = \frac{J_{\ell+\frac{1}{2}}(\sqrt{\lambda_{\ell j}}\, a)}{\sqrt{\lambda_{\ell j}}} \cdot P_\ell^{|m|}(\cos\theta)\, e^{im\phi}$$

$$m \in \mathbb{Z}, \quad \ell \geq |m|, \quad j \in \mathbb{N}$$

EV is $\lambda_{\ell i}$ multiplicity $2\ell+1$

there are $2\ell+1$ m's

$m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell$. s.t. $\ell \geq |m|$