Bessel diff. eqn.: $u''(z) + \frac{1}{z} u'(z) + \left(1 + \frac{s^2}{z^2}\right) u(z) = 0$

$s = m \in \mathbb{N}$, $u(0) < \infty$ $\qquad z > 0$

then $J_m(z) = \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(m+j)!} \left(\frac{z}{2}\right)^{2j+m}$

the coefficients of the powerseries are given by a recursion formula where $a_0 = \frac{1}{2^m m!}$.

Let $s \in (0, \infty)$. $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad x > 0$.

- $\Gamma(m) = (m-1)! \quad \forall m \in \mathbb{N}$
- $\Gamma(s+1) = s \cdot \Gamma(s) = s(s-1) \cdot \ldots \cdot (s-m)\Gamma(s-m)$

Setting $a_0 = \frac{1}{2^s \Gamma(s+1)}$: $J_s(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j+s+1)} \cdot \left(\frac{z}{2}\right)^{2j+s}$

Bessel fct of order s.

Γ can also be defined on $(-1, 0)$.

- J_s and J_{-s} are lin. independent sol. of Bessel eqn.
  but $J_{-s}(z) \to \infty$ $z \to 0$
- Assymptotic behavior: $J_s(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi \cdot s}{2} - \frac{\pi}{4}\right) + O(z^{-3/2})$
- Recursion formula: $J_{s \pm 1}(z) = \frac{s}{z} J_s(z) \mp J_s'(z)$

  $$\Leftarrow \quad J_{s+1} = z^s \left(\frac{d}{dz} z^{-s} J_s(z)\right)$$

for instance $s = \frac{1}{2} + (m-1)$ $\quad J_{s+1}(z) = (-1)^m z^{m+\frac{1}{2}} \left(z^{-1} \frac{d}{dz}\right)^m$

- $s = \frac{1}{2}$: $\quad V(z) = z^{\frac{1}{2}} u(z)$ where $u(z) = J_{\frac{1}{2}}(z) \quad z^{-\frac{1}{2}} J_{\frac{1}{2}}(z)$

  $$\Rightarrow \quad V'' + V = 0 \qquad V(0) = 0$$

  $$\Rightarrow \quad V(z) = c \cdot \sin z$$

$$\Rightarrow \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{1}{z}} \cdot \sin z$$

(Similar $J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos(z)$

$J_{m+\frac{1}{2}}(z) = \ldots$ ✓.

Eigenfcts of $-\Delta$ on $B_a(0) \subset \mathbb{R}^3$ with Dir. BC

$$\tilde{v}_{emj}(r,\phi,\theta) = \frac{1}{\sqrt{r}} \cdot J_{\frac{1}{2}+e}(\sqrt{\lambda_{ej}} \cdot r)\, P_e^{|m|}(\cos\theta) \cdot e^{im\phi}$$

for EV λ_{ej} where λ_{ej} $j \in \mathbb{N}$ roots of $\lambda \mapsto J_{\frac{1}{2}+e}(\sqrt{\lambda}a)$.

$m \in \mathbb{Z},\ e \in \mathbb{N}_0 \quad \ell \geq |m|$

- $P_e^{|m|}(s)$ are the associated Legendre fct. given by $$P_e^m(s) = \frac{(-1)^m}{2^e e!}\left((1-s^2)\right)^{m/2} \frac{d^{m+e}}{ds^{m+e}}(s^2-1)^e$$ $e, m \in \mathbb{N} \quad \ell \geq m$

- $\frac{d^e}{ds^e}(s^2-1)^e = P^e$ Legendre Polynomial

  It satisfies $((1-s^2)P')' + \ell(e+1)P = 0$ Legendre eqn.

  $$((1-s^2)P')' + \left(e(e+1) + \frac{m^2}{1-s^2}\right)P = 0$$

  associat. Legendre eqn.

Orthogonality

$$(\tilde{V}_{lmj}, \tilde{V}_{kmt}) = \int_0^a \int_0^{2\pi} \int_0^{\pi} \tilde{v}_{lmj}(r,\varphi,\theta)\, \tilde{v}_{kmt}(\dots) \sin\theta \; r^2\, dr\, d\varphi\, d\theta$$

$$= \underbrace{\int_0^a J_{\frac{1}{2}+l}(\sqrt{\lambda} r)\, J_{\frac{1}{2}+k}(\sqrt{\lambda} r) \cdot r\, dr}_{= 0 \text{ if } (l,j) \neq (k,t)} \cdot \underbrace{\int_0^{2\pi} e^{i(m-n)\varphi}\, d\varphi}_{= 0 \text{ if } m \neq n}$$

$$\cdot \underbrace{\int_0^{\pi} P_l^{|m|}(\cos\theta)\, P_k^{|m|}(\cos(\theta)) \sin\theta\, d\theta}_{\int_{-1}^{1} \dots\, ds = 0 \text{ if } (m,l) \neq (m,k)}$$

Spherical harmonics

$$Y_l^m(\phi, \theta) = P_l^m(\theta)\, e^{im\phi}$$

* form a complete orthogonal system of eigenfct for Laplace operator on $\partial B_1(0)$

$$\frac{1}{\sin^2\theta} Y_{\phi\phi} + \frac{1}{\sin\theta}\left(\sin\theta\, Y_\theta\right)_\theta$$

Application

$$\Delta u = 0 \quad \text{on } B_1(0)$$

$$u = g \quad \text{on } \partial B_1(0)$$

$$\Rightarrow \quad g(\phi, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m\, Y_l^m(\phi, \theta)$$

Assume $R(r) \cdot Y(\phi, \theta)$

then $R'' + \frac{2}{r} R' + \frac{l(l+1)}{r^2} R = 0$

Sol. is $R(r) = c \cdot r^{\alpha}$ Euler type

$\alpha = l$

$\Rightarrow \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m r^l \cdot Y_l^m(\phi, \theta) = u(r, \phi, \theta)$

solves the previous Laplace equat.