General eigenvalue problem

Let $\Omega \subset \mathbb{R}^n$ ($n = 3$, or $n \in \mathbb{N}$) a domain (bounded, open, connected, smooth $\partial\Omega$)

EV Problem:

$$\boxed{\begin{aligned} -\Delta u &= \lambda u \quad \text{on } \Omega \qquad (*) \\ u &= 0 \quad \text{on } \partial\Omega \\ &\text{(homogeneous Dir. BC)} \end{aligned}}$$

We saw $\lambda > 0$.

We assume $\exists\ 0 < \lambda_1 \leq \lambda_2 \leq \dots$

How can we find the EVs $(\lambda_i)_{i \in \mathbb{N}}$?

Recall $E(u) = \int_\Omega |\nabla u|^2 \, dx \qquad u \in C^1(\bar{\Omega})$

Minimizers of $E(u)$ are harmonic

if $u|_{\partial\Omega} = 0$ and harmonic $\Rightarrow u = 0 \Rightarrow \lambda = 0$ is not EV

New Minimization problem (+ additional constraints)

Find Minimizer of $E(u)$ on $\mathcal{E}_0 = \{u \in C^1(\bar{\Omega}) : u|\partial\Omega = 0\}$

(MP) with $\int_\Omega u^2 \, dx = 1$

$\Leftrightarrow$ Find minimizer of

$$\underbrace{R(u) = \frac{E(u)}{\|u\|_2^2}}_{\text{Rayleigh quotient}} \quad u \in \mathcal{E}_0 \setminus \{0\}$$

<u>Theorem</u> (Minimum principle of 1st EV)

Assume $u \in C^2(\bar{\Omega}) \cap \mathcal{E}_0 \setminus \{0\}$ solves (MP). Then $\frac{E(u)}{\|u\|_2^2} = R(u) =: \lambda_1$ is the smallest EV and u solves $(*)$ for $\lambda = \lambda_1$.

Proof: Let $\varphi \in C^1(\Omega)$ with compact in Ω

$\Longrightarrow \quad w(\varepsilon) = u + \varepsilon\varphi \in \mathcal{E}_0 \quad \forall \varepsilon \in (-\delta, \delta)$

$$\Longrightarrow \quad f(\varepsilon) = E(w(\varepsilon)) = \int_\Omega |\nabla u|^2 dx + 2\varepsilon \int_\Omega \langle \nabla u, \nabla \varphi \rangle dx + \varepsilon^2 \int_\Omega |\nabla \varphi|^2 dx$$

$$f'(0) = 2 \cdot \int_\Omega \langle \nabla u, \nabla \varphi \rangle dx$$

$$g(\varepsilon) = \|w(\varepsilon)\|_2^2 = \int_\Omega u^2 dx + 2\varepsilon \int_\Omega u \cdot \varphi \, dx + \varepsilon^2 \int_\Omega \varphi^2 dx$$

$$0 = \frac{d}{d\varepsilon}\Big|_0 \underbrace{\frac{f(\varepsilon)}{g(\varepsilon)}}_{R(w(\varepsilon))} = \frac{f'(0) \cdot g(0) - g'(0) \cdot f(0)}{g(0)^2}$$

$$0 = \int_\Omega \langle \nabla u, \nabla \varphi \rangle dx \cdot \|u\|_2^2 - \int_\Omega u \cdot \varphi \, dx \cdot E(u)$$

$$\Longrightarrow \quad \underbrace{R(u)}_{\lambda_1} \cdot \int_\Omega u \cdot \varphi \, dx = \int_\Omega \langle \nabla u, \nabla \varphi \rangle dx \overset{\text{Gaum}}{=} -\int_\Omega \Delta u \cdot \varphi \, dx$$

$$\Longrightarrow \quad -\Delta u = \underset{R(u)}{\underset{\shortparallel}{\lambda_1}} u \quad \text{in } \Omega$$

If λ_2 is any EV with eigenfct $v \in C^2(\Omega)$

$\|v\|_2^2$

$$\Longrightarrow \quad -\int_\Omega \Delta v \cdot v \, dx = \int_\Omega \langle \nabla v, \nabla v \rangle \, dx = \lambda_2 \overbrace{\int_\Omega v^2 \, dx}$$

$E(v)$ $\quad \|v\|_2^2$

$$\Longrightarrow \quad \lambda_2 = R(v) \geq R(u) = \lambda_1 . \quad \boxtimes$$

Remark: $u, v \in C^1(\Omega)$. $(u, v) = \int_\Omega u \cdot v \, dx$

$$\rightsquigarrow (u \cdot u) = \|u\|_2^2$$

Example: $\Omega = (0, \pi)$. $-\frac{d^2}{dt^2} u = \lambda u$

Solutions are $u(t) = \sin(t) \Longrightarrow \underline{1} = \lambda_1 = \dfrac{\int_0^\pi u'(t)^2 \, dx}{\int_0^\pi u^2(t) \, dx}$

Theorem (Minimum principle for m'th EV)
Assume $0 < \lambda_1 \leq \dots \leq \lambda_{m-1}$ are the first $m-1$ EVs with eigenfcts $v_1, \dots, v_{m-1} \in C^2(\bar\Omega) \cap \mathcal{E}_0$
Then the m'th EV is given by

$$(MP)_m \quad \lambda_m = \min R(w) \text{ where } w \neq 0,\ w \in \mathcal{E}_0$$
$$0 = (w, v_1) = \dots = (w, v_{m-1}).$$

If $u \in C^2(\bar\Omega)$ solves $(MP)_m$ then u is an eigenfct for EV λ_m.

Proof: Let $u \in C^2(\bar\Omega)$ be a minimizer of $(MP)_m$
Let $\varphi \in C^1(\Omega)$ with compact support in Ω

Assume $v_1, \dots, v_{m-1}$ orthogonal

Define $\tilde{\varphi} = \varphi - \sum_{k=1}^{m-1} c_k v_k \qquad c_k = \frac{(\varphi, v_k)}{(v_k, v_k)}$

$\underline{\in C^1(\bar{\Omega})} \qquad \tilde{\varphi}|_{\partial\Omega} = 0$

Define $\tilde{w}(\varepsilon) = u + \varepsilon \cdot \tilde{\varphi}$

$\rightsquigarrow \quad (\tilde{w}(\varepsilon), v_k) = \underbrace{(u, v_k)}_{0} + \varepsilon \underbrace{(\tilde{\varphi}, v_k)}_{0} \quad \forall k = 1, \dots, m-1$

Define $f(\varepsilon) = E(\tilde{w}(\varepsilon))$, $g(\varepsilon) = \|\tilde{w}(\varepsilon)\|_2^2$

$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} R(w(\varepsilon)) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{f(\varepsilon)}{g(\varepsilon)}$

$\Rightarrow \quad \underset{\displaystyle = -\int_\Omega \Delta u \cdot \tilde{\varphi}\, dx}{\int_\Omega \langle \nabla u, \nabla \tilde{\varphi} \rangle \, dx} = \underset{\displaystyle = R(u)}{\lambda^*} \int_\Omega u \cdot \tilde{\varphi} \, dx$

$$\Rightarrow \quad -\int_\Omega (\Delta u - \lambda^* u)\,\varphi \, dx = -\sum_{k=1}^{m-1} c_k \int_\Omega (\Delta u - \lambda^* u)\, v_k \, dx$$

$$= -\sum_{k=1}^{m-1} c_k \int_\Omega (\Delta v_k - \lambda^* v_k)\, u \, dx$$

$$= -\sum_{k=1}^{m-1} c_k \underbrace{\int_\Omega (\lambda_k - \lambda^*)\, v_k u \, dx}_{0}$$

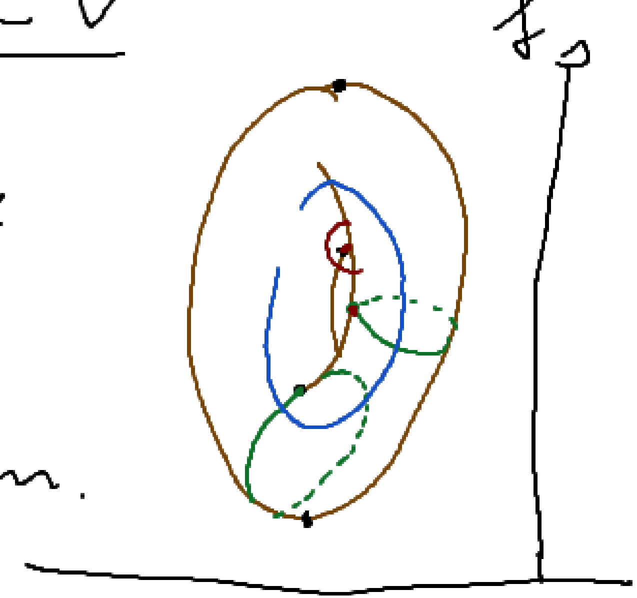
$$\Rightarrow \quad -\int_\Omega (\Delta u - \lambda^* u)\,\varphi \, dx = 0$$

$$\Rightarrow \quad -\Delta u = \lambda^* u \quad \text{for } \lambda^* = R(u) \text{ on } \Omega$$

Since u ~~...~~ and $\{v_1, \dots, v_{m-1}\}$ are lin. indep. ☺

$\Rightarrow \quad \lambda^* = \lambda_m$ is m'th EV.

Min Max Principle for the n'th EV



Let $v_1, \dots, v_n \in \mathcal{E}_0 \cap C^2(\bar{\Omega})$ eigenfcts orthogonal for first n EVs

Normalize: $\|v_i\|_2 = 1 \quad i = 1, \dots, n$

$V = \operatorname{span}\{v_1, \dots, v_n\}$ finit dim.

If $w \in V$: $w = \sum_{k=1}^{n} c_k v_k$

then $$E(w) = \sum_{l,k=1}^{n} c_k c_k \int_{\Omega} \langle \nabla v_k, \nabla v_l \rangle \, dx$$

$$= \sum_{k=1}^{n} c_k^2 \int_{\Omega} |\nabla v_k|^2 \, dx = \sum_{k=1}^{n} c_k^2 \lambda_k \underbrace{\int_{\Omega} v_k^2 \, dx}_{=1}$$

On V we have E corresponds to $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

$$E(w) = (c_1, \dots, c_n)\, A\, (c_1, \dots, c_n)^T \quad \text{for } w = \sum_{k=1}^{n} c_k v_k$$

$$\Longrightarrow \quad \lambda_m = \max_{c \in \mathbb{R}^m} \frac{c A c^T}{|c|_2^2} = \max_{w \in V} \frac{E(w)}{\|w\|_2^2}$$

Linear algebra.

Now consider $w_1, \dots, w_m \in E_0 \setminus \{0\}$ arbitrary but lin. indep.

$$\Longrightarrow \quad \exists\, w = \sum_{k=1}^{m} x_k w_k \quad \text{s.t. } (w, v_i) = 0 \quad \forall i = 1, \dots, m-1$$

To see that solve: $(w, v_i) = \sum_{k=1}^{m} x_k \underbrace{(w_k, v_i)}_{(d_{ki}) = D} \quad i = 1, \dots, m-1$

$$\underset{m \times (m-1)}{D} \cdot x = 0 \qquad \Longrightarrow \quad \exists\, (x_1, \dots, x_m) \neq 0 \quad \text{s.t. } (w, v_i) = 0$$

$R(w) \geq \lambda_m$ for this $w \in W = \text{span}(w_1, \dots, w_m)$

$$\max_{w \in W} R(w) \geq \lambda_m .$$

<u>Theorem</u>: (Min Max Principle for EVs)

The mth EV of $-\Delta$ on Ω with hom. Dir. BC is given by

$$\lambda_m = \min_{\substack{w_1, \dots, w_m \in \mathcal{E}_0 \setminus \{0\} \\ \text{lin. independent}}} \left(\max_{\substack{w \in W \\ = \text{span}(w_1, \dots, w_m)}} R(w) \right)$$

Proof: See above.

Ritz-Rayleigh Approximation.

Let $\tilde{v}_1, \ldots, \tilde{v}_m \in \mathcal{E}_0$ be lin independent in $\mathcal{E}_0$

$$\rightsquigarrow \quad a_{ij} = \int_\Omega \langle \nabla \tilde{v}_i, \nabla \tilde{v}_j \rangle \, dx \quad \rightsquigarrow A$$

$$b_{ij} = \int_\Omega \tilde{v}_i \tilde{v}_j \, dx \quad \rightsquigarrow B$$

Let $\underset{\neq 0}{w^l} = \sum_{k=1}^{l} c_k^l v_k$ s.t $R(w^l) = \lambda_l^* = \dfrac{c^l \cdot A (c^l)^T}{c^l B (c^l)^T}$

$$c^l = (c_1^l, \ldots, c_k^l, 0, \ldots, 0) \in \mathbb{R}^m$$

$\Longrightarrow \quad c^l (A - \lambda_l^* B)(c^l)^T = 0$

$\Longrightarrow \quad \det(A - \lambda_l^* B) = 0 \qquad \exists\, w^l$ s.t $R(w^l) \geq \lambda_l$

$\Longrightarrow \quad P(\lambda) = \det(A - \lambda B) = 0$ Solution „approximate" the first m EVs.

Neumann BC: $-\Delta u = \tilde{\lambda} u$ on Ω

$\frac{\partial u}{\partial N} = 0$ on $\partial\Omega$

where N is normal vector along $\partial\Omega$

EVs $0 = \lambda_1 \leq \lambda_2 \leq \dots$

Same Minimum principle applies.

(MP): $\min\limits_{w \in \mathcal{E} = C^1(\bar{\Omega}) \setminus \{0\}} \frac{E(w)}{\|w\|_2^2}$ with $\int_\Omega w \cdot c \, dx = 0$

$\iff \int_\Omega w \, dx = 0$

Remark: Since there is no condition on the trial fct., the Neumann BC is also called free