Let $\Omega \subset \mathbb{R}^n$ open, bounded, $\partial\Omega$ smooth

Ω can be disconnected

Dirichlet BC Problem:
$$-\Delta u = \lambda u \quad \text{on } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

EVs: $0 < \lambda_1 \leq \lambda_2 \leq \dots$

orthogonal

Eigenfcts: $v_1, v_2, \dots, v_{n-1}, \dots$ in $\underline{\mathcal{E}_0 \cap C^2(\bar\Omega)}$. $\underbrace{\|v_n\|_2^2}_{\int_\Omega v_n^2 dx} = 1$

We showed if $u \in \mathcal{E}_0 \cap C^2(\bar\Omega)$

s.t. $R(u) = \min R(w) \quad w \in \mathcal{E}_0 \setminus \{0\}$

$(w, v_i) = 0 \quad \forall i = 1, \dots, n-1$

then $\lambda_n = R(u)$ and u is an eigenfct for $\lambda = \lambda_n$

Now let $w_1, \dots, w_{n-1}$ that are square integrable

$\int_\Omega w_i^2 dx < \infty$, ~~[illegible]~~ Assume w_i is piecewise continuous

$j = 1, \dots, l \quad \exists\, \Omega_j \subset \Omega \quad \dot\bigcup \Omega_j, \quad \bigcup \bar\Omega_j = \bar\Omega \quad w_i|_{\Omega_j}$ cont.

$\leadsto \lambda_m^* = \inf \frac{R(\omega)}{\|\omega\|_2^2}$ $\quad \omega \in \mathcal{E}_0 \setminus \{0\}$

$0 = (\omega, \omega_i) \quad \forall i = 1, \dots, m-1$

<u>Lemma:</u> $\lambda_m^* \leqslant \lambda_m$

<u>Proof:</u> $\exists\, c_k \quad k = 1, \dots, m \quad$ s.t. $\omega = \sum_{k=1}^{m} c_k \underset{\text{Eigenfcts}}{v_k}$

s.t. at least one c_k ist $\neq 0$ and $(\omega, \omega_i) = 0 \quad \forall i = 1, \dots, m-1$

$$\longrightarrow \lambda_m^* \leqslant R(\omega) = \frac{E(\omega)}{\|\omega\|_2^2} = \frac{\sum_{k,l=1}^{m} c_k c_l \int_\Omega \langle \nabla v_k, \nabla v_l \rangle dx}{\sum_{k,l=1}^{m} c_k c_l \int_\Omega v_k v_l \, dx}$$

$$= \frac{\sum c_k c_l \int \Delta v_k v_l \, dx}{\sum c_k^2 \int v_k^2 \, dx} = \frac{\sum c_k^2 \lambda_k}{\sum c_k} \leqslant \frac{\sum c_k^2 \cdot \lambda_m}{\sum c_k^2}$$

$= \lambda_m \quad \blacksquare$

Theorem (Max Min Principle)

$$\lambda_n = \max_{\substack{w_1, \dots, w_{n-1} \\ \text{square integrable}}} \left(\inf_{\substack{w \in \mathcal{E}_0 \ (w, w_i) = 0 \\ \forall i = 1, \dots, n-1}} R(w) \right)$$

Remark

Analog theorem for Neumann EVs $\tilde{\lambda}_n$ also holds.

but replace $\mathcal{E}_0$ with $\mathcal{E} = C^1(\bar{\Omega})$.

Corollary: $\tilde{\lambda}_n \leq \lambda_n \quad \forall n \in \mathbb{N}$

Proof: $\lambda_n^* = \inf R(w) \quad w \in \mathcal{E}_0 \setminus \{0\} \quad 0 = (w, w_i)_{i=1,\dots,n-1}$

$\tilde{\lambda}_n^* = \inf R(w) \quad w \in \mathcal{E} \setminus \{0\} \quad \dots$

Since $\mathcal{E}_0 \subset \mathcal{E}$, $\Longrightarrow$ corollary $\blacksquare$

Completeness:

Theorem: Let $(v_n)_{n \in \mathbb{N}}$ eigenfcts for EVs λ_n

Assume they orthogonal.

If f is square integrable

then $$f = \sum_{n=1}^{\infty} c_n v_n \qquad c_n = \frac{(f, v_n)}{(v_n, v_n)}$$

L^2-sense

Proof: for $f \in C^2(\bar{\Omega})$ see notes.

Rm: Important $\lambda_n \longrightarrow \infty$ if $n \to \infty$

<u>Theorem</u> (<u>Weyl</u>)

① Let $n = 2$ (dimension) $\Omega \subset \mathbb{R}^2$.

Then $\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{\text{Area}(\Omega)}$

② In 3D: $\lim_{n \to \infty} \frac{\lambda_n^{3/2}}{n} = \frac{6\pi^2}{\text{vol}(\Omega)}$.

<u>Examples</u>

1D case: $\Omega = (0, L)$ Eigenfcts $v_n(x) = \sin\left(n\pi \frac{x}{L}\right)$

$\rightsquigarrow$ EVs: $\lambda_n = \frac{n^2\pi^2}{L^2}$. $\frac{\sqrt{\lambda_n}}{n} = \frac{\pi}{L}$

2D: $\Omega = (0,a) \times (0,b)$

Eigenfcts $v_n(x) \cdot w_m(y) = V_{nm}(x,y)$

EV's $\lambda_{nm} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$

$\rightsquigarrow$ $\hat{\lambda}_n \nearrow$ $n \to \infty$ Reordering of λ_{nm}

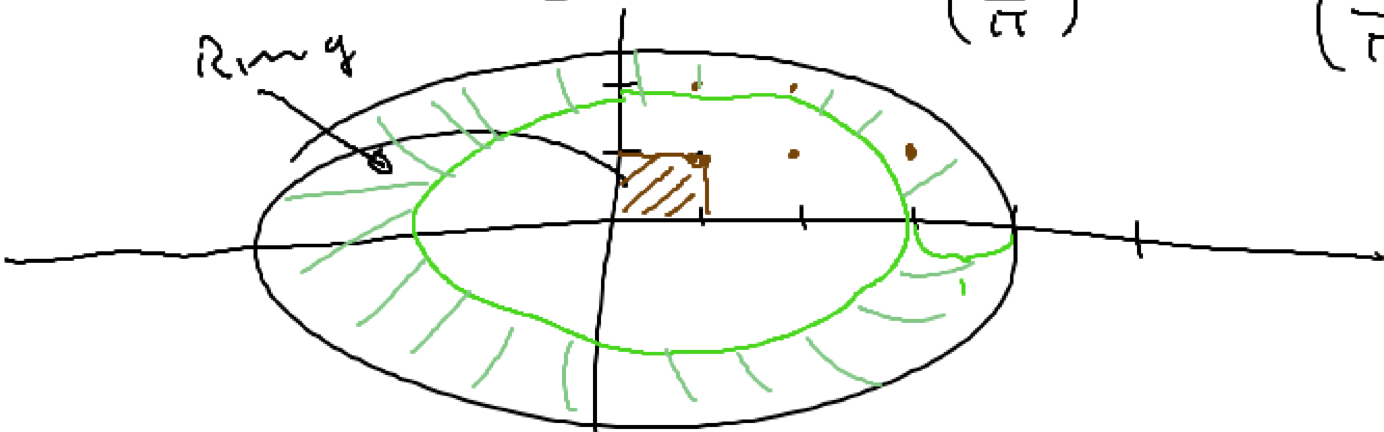
Define a counting fct.

$N(\lambda)$ = number of EVs $\hat{\lambda}_n \leq \lambda$

So $N(\hat{\lambda}_n) = n$

$\Longrightarrow$ $\lambda \geq \lambda_{nm} = \frac{n^2}{\left(\frac{a}{\pi}\right)^2} + \frac{m^2}{\left(\frac{b}{\pi}\right)^2}$

Ellipse $\left\{(x,y): \frac{x^2}{\left(\frac{a}{\pi}\right)^2} + \frac{y^2}{\left(\frac{b}{\pi}\right)^2} < \lambda\right\} = E$



$$\Longrightarrow \quad N(\lambda) \le \frac{\text{Area}(E)}{4} = \frac{\pi}{4}\,\frac{a\cdot\sqrt{\lambda}}{\pi}\cdot\frac{b\cdot\sqrt{\lambda}}{\pi} = \frac{a\cdot b\;\lambda}{4\pi} = \frac{\text{Area}(\Omega)\cdot\lambda}{4\pi}$$

$$\frac{a\cdot b\cdot\lambda}{4\pi} - \underbrace{C\sqrt{\lambda}}_{\text{"Area(Ring)"}} \le N(\lambda) \Longrightarrow \frac{a\cdot b}{4\pi} - \frac{C}{\sqrt{\lambda}} \le \frac{N(\lambda)}{\lambda} \le \frac{a\cdot b}{4\pi}$$

$$\frac{N(\lambda_k)}{\lambda_k} = \frac{k}{\lambda_k}.$$