Last time:

Theorem (Weyl)

$\Omega \subset \mathbb{R}^2$ open, bounded, $\partial\Omega$ smooth

Let $(\lambda_n)_{n\in\mathbb{N}}$ be Dirichlet EVs.

Then $\lim\limits_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{\text{Area}(\Omega)}$. In particular $\lambda_n \to \infty$

Example: for a cube $Q = (0,a)\times(0,b) = \Omega$

Dirichlet EVs are $\lambda_{nm} = \frac{n^2}{(\frac{a}{\pi})^2} + \frac{m^2}{(\frac{b}{\pi})^2}$ $\quad n,m \in \mathbb{N}$

$\pi^2 \underline{a\cdot\sqrt{\lambda}\cdot b\sqrt{\lambda}}$

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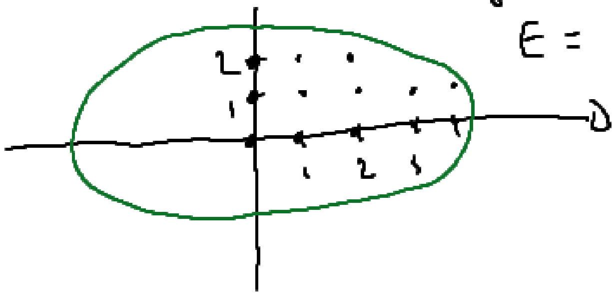
$N(\lambda)$ counting fct



$E = \{(x,y) : \frac{x^2}{(\frac{a}{\pi})^2} + \frac{y^2}{(\frac{b}{\pi})^2} < \lambda\}$

$\Rightarrow N(\lambda) \sim \frac{\text{Area}(E)}{4} \pm C\sqrt{\lambda}$

$\lim N(\lambda)/\lambda = \frac{\pi^{-1} a\cdot b}{4}$

Remark $\tilde{N}(\lambda)$ counting for Neumann EVs

$$Q = (0,a) \times (0,b) \rightsquigarrow \tilde{\lambda}_{n,m} = \frac{n^2}{\left(\frac{a}{\pi}\right)^2} + \frac{m^2}{\left(\frac{b}{\pi}\right)^2}$$

$n, m \in \mathbb{N}_0$

$$\left[N(\lambda)/\lambda \sim \frac{\frac{4\pi \cdot a(E)}{4}}{\lambda} = \frac{\frac{1}{4}\pi \cdot \left(\frac{a\sqrt{\lambda}}{\pi}\right) \cdot \left(\frac{b\sqrt{\lambda}}{\pi}\right)}{\lambda} = \frac{a \cdot b}{4\pi} \right]$$

Lemma: $\Omega_1 \subset \Omega_2$. Then

(1) $\lambda_n^{\Omega_2} \leq \lambda_n^{\Omega_1}$

(2) $\tilde{\lambda}_n^{\Omega_2} \geq \tilde{\lambda}_n^{\Omega_1}$ where $\tilde{\lambda}_n^{\Omega}$ are the Neumann EVs.

Proof: (1) Let $w_1, \dots, w_{m-1}$ square integrable on Ω_2

$$\lambda_m^{*\Omega_2} = \inf R(w) \quad w \in \mathcal{E}_0 \quad (w, w_i) = 0 \quad \forall i = 1, \dots, m-1$$

Now if $w \in \mathcal{E}_0(\Omega_1) = \{ C^1(\bar{\Omega}_1) \text{ s.t. } u|_{\partial\Omega_1} = 0 \}$
then $w \in \mathcal{E}_0(\Omega_2)$ & $\tilde{w} = \begin{cases} w & \text{on } \Omega_1 \\ 0 & \text{on } \Omega_2 \setminus \Omega_1 \end{cases}$

to make this rigorous $\quad\quad \notin C^1(\Omega_2)$

take $w \in C^1_c(\Omega_1) \Rightarrow \tilde{w} \in C^1(\Omega_2)$.

$\tilde{w}_i = w_i|_{\Omega_1} \Rightarrow \tilde{w}_i$ square integrable on Ω_1

$$\int_{\Omega_1} \tilde{w}_i^2 \leq \int_{\Omega_2} w_i^2$$

$\Rightarrow \quad \lambda_m^{*\Omega_2} \leq \inf R(w) \quad w \in C^1_c(\Omega_1) \quad (w, w_i) = (w, \tilde{w}_i) = 0$
$= \lambda_m^{*\Omega_1} \Rightarrow \lambda_m^{\Omega_2} \leq \lambda_m^{\Omega_1}$.

(ii) Let $w_1, \dots, w_{m-1}$ square integrable on Ω_1 $(\subset \Omega_2)$

$\Longrightarrow$ $\tilde{\lambda}^{*\Omega_1}_m = \inf R(w)$ $\quad w \in C^1(\bar{\Omega}_1)$ $\quad (w, w_i) = 0$
$\forall i = 1, \dots, m-1$

If $w \in C^1(\bar{\Omega}_2)$ $\quad w|_{\Omega_1} = \tilde{w} \in C^1(\bar{\Omega}_1)$

Define $\tilde{w}_i = \begin{cases} w_i & \text{on } \Omega_1 \\ 0 & \text{on } \Omega_2 \setminus \Omega_1 \end{cases}$ square integr.

$\Longrightarrow$ $\tilde{\lambda}^{*\Omega_1}_m \leq \tilde{\lambda}^{*\Omega_2}_m = \inf R(w)$ $\quad w \in C^1(\bar{\Omega}_2)$
$0 = (w, \tilde{w}_i) = (w, w_i)$
$\forall i = 1, \dots, m-1$

Maximum w.r.t. $w_1, \dots, w_{m-1}$

yield $\tilde{\lambda}^{\Omega_1}_m \leq \tilde{\lambda}^{\Omega_2}_m$.

∎

<u>Remark</u>: $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ $\quad \Omega_1 \cap \Omega_2 = \emptyset$

If $\{v_n^{\Omega_1}\}$, $\{v_n^{\Omega_2}\}$ are eigenfcts. w.r.t. Ω_1 and Ω_2

then $\tilde{v}_n^{\Omega_1} = \begin{cases} v_n^{\Omega_1} & \text{on } \Omega_1 \\ 0 & \text{on } \Omega_2 \end{cases}$ $\quad \tilde{v}_n^{\Omega_2} = \begin{cases} v_n^{\Omega_2} & \text{on } \Omega_2 \\ 0 & \text{on } \Omega_1 \end{cases}$

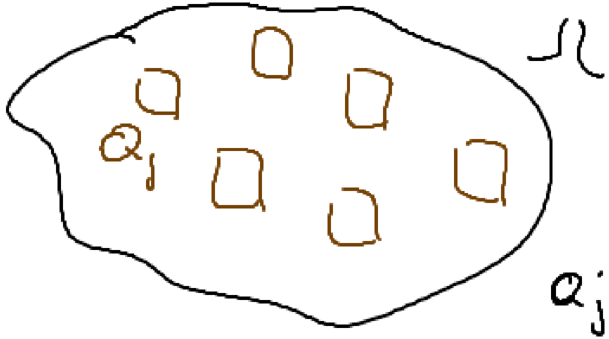
are the eigenfcts of $\Omega_1 \cup \Omega_2$

(w.r.t. Dirichlet BC or also Neumann BC)

$\leadsto$ EVs $= \{\lambda_n^{\Omega_1}\}_{n \in \mathbb{N}} \cup \{\lambda_n^{\Omega_1}\}_{n \in \mathbb{N}}$

Proof:

$$\underline{\Omega} = \bigcup_{j=1}^{\ell} Q_j \subset \Omega$$



Q_j disjoint

$Q_j \cong (0,a_j)\times(0,c_j)$

Then $\lambda_m^{\underline{\Omega}} \geq \lambda_m^{\Omega} \quad \forall m \in \mathbb{N}$

Let N^{Ω} and $N^{\underline{\Omega}}$ counting fct.

$$\Longrightarrow \quad N^{\Omega}(\lambda) \geq N^{\underline{\Omega}}(\lambda)$$

Remark $\Longrightarrow \quad N^{\underline{\Omega}}(\lambda) = \sum_{j=1}^{\ell} N^j(\lambda)$

Example

$$\frac{N^j(\lambda)}{\lambda} \to \frac{\text{Area}(Q_j)}{4\pi}$$

$$\Longrightarrow \quad \liminf_{\lambda\to\infty} \frac{N^{\Omega}(\lambda)}{\lambda} \geq \sum_{j=1}^{\ell} \frac{\text{Area}(Q_j)}{4\pi} = \frac{\text{Area}(\underline{\Omega})}{4\pi}.$$

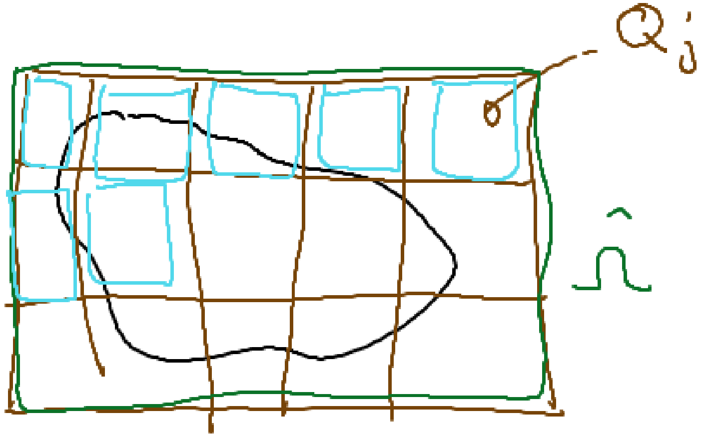
On the other hand:

Q_j disjoint $\quad \Omega \subset \bigcup_{j=1}^{\ell} \overline{Q}_j = \overline{\bigcup_{j=1}^{\ell} Q_j} = \overline{Q}$, where $\overline{Q}_j = [0,a_j]\times[0,c_j]$

Set $\hat{\Omega} = (\overline{Q})^{\circ}$

$\hat{\Omega}$ is different from $Q = \bigcup_{j=1}^{\ell} Q_j$

Still $Q \subset \hat{\Omega}$



Lemma $\quad \tilde{\lambda}_m^{\hat{\Omega}} \geq \tilde{\lambda}_m^{Q}$

$\Longrightarrow \quad \tilde{N}^{\hat{\Omega}}(\lambda) \leq \tilde{N}^{Q}(\lambda)$

Remark: $\quad \tilde{N}^{Q}(\lambda) = \sum_{j=1}^{\ell} \tilde{N}^{Q_j}(\lambda)$ $\quad \sim \quad \dfrac{\tilde{N}^{Q_j}(\lambda)}{\lambda} \longrightarrow \dfrac{\text{Area}(Q_j)}{4\pi}$

$\Longrightarrow \quad \limsup_{\lambda\to\infty} \dfrac{\tilde{N}(\lambda)}{\lambda} \leq \dfrac{\text{Area}(\hat{\Omega})}{4\pi}$

$$\Longrightarrow \quad \frac{\mathrm{Area}(\Omega)}{4\pi} \leq \liminf \frac{N^{\Omega}(\lambda)}{\lambda}$$

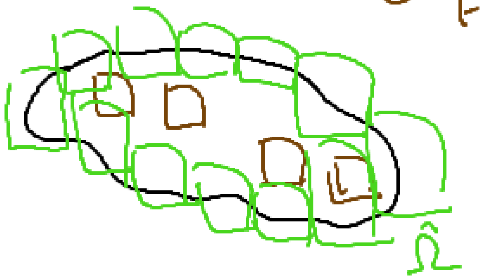
$$\frac{\mathrm{Area}(\hat{\Omega})}{4\pi} \geq \limsup_{\lambda \to \infty} \frac{N^{\hat{\Omega}}(\lambda)}{\lambda}$$

We have $\hat{\Omega} \supset \Omega$

$$\Longrightarrow \quad \tilde{\lambda}_m^{\hat{\Omega}} \leq \lambda_m^{\hat{\Omega}} \leq \lambda_m^{\Omega} \Longrightarrow \tilde{N}^{\hat{\Omega}}(\lambda) \geq N^{\Omega}(\lambda)$$

$$\Longrightarrow \quad \frac{\mathrm{Area}(\Omega)}{4\pi} \leq \liminf \frac{N^{\Omega}(\lambda)}{\lambda} \leq \limsup \frac{\tilde{N}^{\hat{\Omega}}(\lambda)}{\lambda}$$

$$\sim \frac{\mathrm{Area}(\Omega)}{4\pi} \qquad \leq \frac{\mathrm{Area}(\hat{\Omega})}{4\pi}$$



$$N^{\Omega}(\lambda_k) = k \qquad \sim \frac{\mathrm{Area}(\Omega)}{4\pi}$$

$\square$

Fourier transform

$f : \mathbb{R} \to \mathbb{R}$ cont. 2ℓ periodic

Fourier series of f: $\sum_{n \in \mathbb{Z}} c_n e^{i n \pi \frac{x}{\ell}} = \mathcal{F}(f)$

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(y) e^{-i n \pi y/\ell} \, dy .$$

This is a map $f \in C^0(\mathbb{R})$ 2ℓ period $\longmapsto (c_n)_{n \in \mathbb{Z}}$

$$\sum_{n \in \mathbb{Z}} |c_n|^2 < \infty$$

Question: Can we drop 2ℓ-periodicity somehow?

$$\tilde{f}(x) = \frac{1}{2} \sum_{m \in \mathbb{Z}} \underbrace{\int_{-\ell}^{\ell} f(y)\, e^{-i\pi y \frac{m}{\ell}}\, dy\; e^{i\pi x \frac{m}{\ell}}}_{g\left(\frac{m}{\ell}\right)} \cdot \frac{1}{\ell}$$

$$\xrightarrow{\text{"}\ell \to \infty\text{"}} \text{"} \quad \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)\, e^{-i\pi y \cdot \xi}\, dy\; e^{i\pi x \cdot \xi}\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(y)\, e^{-i y \cdot \xi}\, dy}_{\hat{f}(\xi)}\; e^{i \cancel{\pi} x \cdot \xi}\, d\xi$$

Def: $\hat{f}$ is called Fourier transform.

where $f : \mathbb{R} \to \mathbb{C}$ is integrable: $\int_{-\infty}^{\infty} |f|\, dx < \infty$

We want to justify a formula of the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\xi x}\, d\xi = f(x) \quad (?)$$

However $\hat{f}$ might not be integrable.

<u>Def</u>: A function $f \in C^\infty(\mathbb{R}, \mathbb{C})$ is in the Schwartz space $\mathcal{S}(\mathbb{R})$ if f is rapidly decreasing:

$$\sup_{x\in\mathbb{R}} |x|^k \cdot \left|\frac{d^\ell}{dx^\ell} f(x)\right| \leq C < \infty \quad \forall k,\ell \in \mathbb{N}$$

<u>Example</u>: $e^{-x^2/2} \in \mathcal{S}(\mathbb{R})$

<u>Remark</u>: if $f \in \mathcal{S}(\mathbb{R}) \Longrightarrow \frac{d^\ell}{dx^\ell} f \in \mathcal{S}(\mathbb{R}) \ \forall \ell \in \mathcal{S}(\mathbb{R})$ and $x \cdot f(x) \in \mathcal{S}(\mathbb{R})$

Proposition: $f \in S(\mathbb{R})$

(1) $f \longmapsto \hat{f}$ is linear map

(2) $\frac{d}{dx} f \longmapsto i\xi \cdot \hat{f}(\xi)$

(3) $x \cdot f(x) \longmapsto i \frac{d}{d\xi} \hat{f}(\xi)$

(4) $f(x-a) \longmapsto e^{-ia\xi} \hat{f}(\xi)$

(5) $e^{iax} f(x) \longmapsto \hat{f}(\xi - a)$

(6) $f(ax) \longmapsto \frac{1}{|a|} \cdot \hat{f}(\xi/a)$.

<u>Proof:</u> (3)

$$(*) = \frac{i}{h}\left(\hat{f}(\xi + h) - \hat{f}(\xi)\right) - (x \cdot f(x))^{\wedge}(\xi)$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x} \left[i\, \frac{e^{-ixh} - 1}{h} - x \right] dx$$

$$\rightsquigarrow \underbrace{\int_{\{|x| \geq N\}} \ldots\ldots dx}_{a} + \underbrace{\int_{-N}^{N} \ldots\ldots dx}_{b}$$

$$b) = \left| \int_{-N}^{N} f(x)\, e^{-i\xi x} [\quad]\, dx \right| \leq \int_{-N}^{N} |f(x)|\, \underbrace{|[\quad]|}_{\substack{\leq C \cdot \varepsilon \\ \text{for } h \text{ small}}} dx$$

$$a) = |\quad| \leq C \cdot \varepsilon \text{ for } N \text{ big.}$$

$|(*)| \leq C \cdot \varepsilon$ for any $\varepsilon > 0$ $\square$

<u>Corollary</u>: $f \in S(\mathbb{R}) \Longrightarrow \hat{f} \in S(\mathbb{R})$