Fourier transform

$f: \mathbb{R} \to \mathbb{C}$ integrable ($\int_{-\infty}^{\infty} |f| \, dx < \infty$)

The Fourier transf. of f is $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ik \cdot x} dx$

Schwartz space

$$f \in S(\mathbb{R}) \longleftrightarrow \sup_{x \in \mathbb{R}} \left| x^t \frac{d^s}{dx^s} f(x) \right| < \infty$$

$$\rightsquigarrow f \in C^\infty(\mathbb{R}) \quad \forall s, t \in \mathbb{N}$$

Proposition: $f \in S(\mathbb{R})$

- $\left(\frac{d}{dx} f\right)^\wedge (k) = ik \cdot \hat{f}(k)$
- $(x f(x))^\wedge (k) = i \frac{d}{dk} \hat{f}(k)$

Corollary: $f \in S(\mathbb{R}) \Longrightarrow \hat{f} \in S(\mathbb{R})$

Proof:
$$\underbrace{(i)^{t+s}\, k^t \frac{d^s}{dk^s} \hat{f}(k)}_{(*)} = \left(\frac{d^t}{dx^t} x^s \cdot f(x)\right)^\wedge$$

$$\Longrightarrow |(*)| \leq \int_{-\infty}^{\infty} \left|\frac{d^t}{dx^t} x^s f(x)\right| \cdot |\underbrace{e^{-ikx}}_{=1}|\, dx$$

$$< \infty \quad \text{because } f \in S(\mathbb{R})$$

$$\Longrightarrow \hat{f} \in S(\mathbb{R}) \qquad \square$$

Examples:

① $f(x) = e^{-x^2/2} \Longrightarrow \sqrt{2\pi} \cdot e^{-k^2/2} = \hat{f}(k)$

Proof of ①: $\hat{f}(k) = \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ikx}\, dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ik)^2} dx \cdot e^{-k^2/2}$

$$= \sqrt{2\pi} \cdot e^{-k^2/2}$$

② $f(x) = e^{-a|x|} \Longrightarrow \hat{f}(k) = \dfrac{2a}{a^2+k^2}$ $a>0$

~~⇒~~ $f \notin S(\mathbb{R})$ because not diff. in $0 \in \mathbb{R}$

Theorem (Inversion formula)

$$f \in S(\mathbb{R}) \longrightarrow f(x) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\cdot e^{ikx}\,dk}_{(\hat{f})^{\vee}}$$

Tempered Distributions

Recall: $\tilde{F} : C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$ is called Distrib.

if (i) $\tilde{F}$ linear

(ii) $\tilde{F}(f_n) \longrightarrow \tilde{F}(f)$ if $\frac{d^t}{dx^t} f_n \xrightarrow{C^0} \frac{d^t}{dx^t} f$ $\forall t \in \mathbb{N}$.

Example: $\tilde{F}(\phi) = \delta_0(\phi) = \phi(0)$

Def: $\mathcal{F}$ is a tempered Distribution if in the previous Def. we replace $C_c^\infty(\mathbb{R})$ with $S(\mathbb{R})$.

Then the Fourier transf of a tempered Distr. is defined as $\hat{\mathcal{F}}(\phi) = \mathcal{F}(\hat{\phi})$.

Examples:

(i) $\mathcal{F}(\phi) = \int f \cdot \phi \, dx$ f integrable

$$\hat{\mathcal{F}}(\phi) = \mathcal{F}(\hat{\phi}) = \int f(x) \cdot \int \phi(y) e^{-ixy} \, dy \, dx$$

$$= \int \phi(y) \underbrace{\int f(x) e^{-ixy} \, dx}_{\hat{f}(y)} \, dy$$

$\Longrightarrow \hat{\mathcal{F}} \sim \hat{f}$

(ii) $F(\phi) = \delta_x(\phi) = \phi(x)$

$$\Longrightarrow \quad \hat{\delta}_x(\phi) = \delta_x(\hat{\phi}) = \hat{\phi}(x)$$

$$= \int_{-\infty}^{\infty} \phi(y)\, \underbrace{e^{-iyx}} \, dy$$

$$\Longrightarrow \quad \hat{\delta}_x \sim e^{-iyx} \quad \Longrightarrow \quad \hat{\delta}_0 \sim 1 \, .$$

(iii) $\tilde{F}(\phi) = \int_{-\infty}^{\infty} \phi \, dx \qquad \tilde{F} \sim 1$ not integrable

$$\tilde{F}_n(\phi) = \int_{-\infty}^{\infty} \phi(x) \cdot \underbrace{e^{-\frac{|x|}{n}}}_{f_n \to 1 \quad n \to \infty} dx \longrightarrow \tilde{F}(\phi)$$

$$\hat{f}_n(k) = \frac{2/n}{n^2 + k^2} \longrightarrow \delta_0 \quad n \to \infty \, .$$

Application of Fourier transf. to PDEs

① Fundamental sol. of the diff eqn.

$$S_t = S_{xx}$$

$$\text{"} \; S(x,0) = \delta_0 \; \text{"}$$

Fourier Transf: $\quad \hat{S}_t = (ik)^2 \hat{S}(k,t) = -k^2 \hat{S}(k,t)$

$$\hat{S}(x,0) = 1$$

$$\Longrightarrow \hat{S}(k,t) = e^{-k^2 t} \xrightarrow{\text{Prop.}} S(x,t) = \frac{1}{\sqrt{4\pi t}} \cdot e^{-x^2/4t}.$$

$$\uparrow$$

$$\mid \; f(x \cdot a) \rightsquigarrow \frac{1}{|a|} \hat{f}\left(\frac{k}{a}\right)$$

② Laplace equation on Half plane

$$H = \{(x,y) \in \mathbb{R}^2 : y > 0\}$$

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{on } H$$

$$\text{"} u(x,0) = \delta_0 \text{"} \quad \text{on } \partial H$$

$$\rightsquigarrow \quad -k^2 \hat{u}(k,y) + u_{yy} = 0 \quad \text{on } H$$

$$u(k,0) = 1$$

$$\Longrightarrow \quad \hat{u}(k,y) = e^{\pm yk} \quad \rightsquigarrow \quad \hat{u}(k,y) = e^{-y|k|}$$

$$(\hat{u})^{\vee}(x) = \frac{1}{2\pi} \int e^{ikx} \cdot e^{-y|k|} \, dk$$

$$= \frac{1}{2\pi} \int_0^\infty e^{k(ix-y)} \, dk + \int_{-\infty}^0 e^{k(ix+y)} \, dy$$

$$= \frac{1}{2\pi}\left(\frac{1}{ix-y} e^{k(ix-y)}\right)\Big|_0^{\infty} + \frac{1}{2\pi}\left(\frac{1}{ix+y} e^{k(ix+y)}\right)\Big|_{-\infty}^{0}$$

$$= \frac{y}{\pi(x^2+y^2)}.$$ Poisson kernel.

$$\left|\underbrace{e^{ikx}}_{1} \cdot e^{-yk}\right| \leq e^{-yk} \xrightarrow[k\to\infty]{} 0$$