MAT351 PARTIAL DIFFERENTIAL EQUATIONS
– LECTURE NOTES –

Contents

Lecture 01

1. Basic Notations and Definitions

1.1. **Partial Derivatives.** Consider a function u of several variables:

$$u = u(x,y,z) \text{ or more generally } u = u(x_1, x_2, \dots, x_n)$$

for $(x,y,z) \in U \subset \mathbb{R}^3$ or $(x_1,\dots,x_n) \in U \subset \mathbb{R}^n$. We also write $\mathbf{x} = \overrightarrow{x} = (x_1,\dots,x_n)$.

U is a domain $\Leftrightarrow$ U connected, $U^\circ \neq \emptyset$ and ∂U smooth.

x, y, z (or $x_1, \dots, x_n$) are called **independent variables**.

Notation. Let u be sufficiently smooth (e.g. $u \in C^1(U)$). We denote the partial derivatives with

$$\lim_{h\to 0} \frac{u(\mathbf{x}+he_i) - u(\mathbf{x})}{h} = \frac{\partial u}{\partial x_i}(\mathbf{x}) = u_{x_i}(\mathbf{x}) \quad i = 1,\dots,n,$$

where $e_i = (0,\dots,0,\underbrace{1}_{i},0,\dots,0)$. For partial derivatives of order $k \in \mathbb{N}$ we write

$$\frac{\partial^k u}{\partial x_{i_1} \dots \partial x_{i_k}}(\mathbf{x}) = u_{x_{i_1},\dots,x_{i_k}}(\mathbf{x}) \quad i_1,\dots,i_k \in \{1,\dots,n\}.$$

For the collection of all partial derivatives of order $k \in \mathbb{N}$ we write

$$\left\{ u_{x_{i_1},\dots,x_{i_k}} : i_1, \cdots i_k \in \{1,\cdots,n\} \right\} =: D^k u.$$

1.2. Differential Operators.

- **Gradient:** The vector $(u_{x_1}, \dots, u_{x_n}) =: \nabla u$ is called the gradient of u.
- **Directional Derivative:** Given a vector $v = (v_1, \dots, v_n)$

$$\nabla u \cdot v = \sum_{i=1}^{n} u_{x_i} v_i = \frac{\partial u}{\partial v} \quad \text{derivative of } u \text{ in direction } v.$$

  In particular $\nabla u \cdot (0, \dots, 0, \underbrace{1}_{i}, 0, \dots, 0) = u_{x_i}$

- **Differential of a vectorvalued map, Divergence:**
  For $V : U \subset \mathbb{R}^n \to \mathbb{R}^n$, $V(\mathbf{x}) = (V^1(\mathbf{x}), \dots, V^n(\mathbf{x}))$ one defines

$$DV = \begin{pmatrix} V^1_{x_1} & \dots & V^1_{x_n} \\ \dots & \dots & \dots \\ V^n_{x_1} & \dots & V^n_{x_n} \end{pmatrix} \quad \text{and} \quad \operatorname{tr} DV = \sum_{i=1}^{n} V^i_{x_i} =: \nabla \cdot V =: \operatorname{Div} V$$

- **Hessian and Laplace operator:** $u(\mathbf{x}) = u(x_1, \dots, x_n)$ smooth, $\mathbf{x} \in U$. Then $\nabla u : U \subset \mathbb{R}^n \to \mathbb{R}^n$ and

$$D\nabla u = \begin{pmatrix} u_{x_1,x_1} & \dots & u_{x_1,x_n} \\ \dots & \dots & \dots \\ u_{x_n,x_1} & \dots & u_{x_n,x_n} \end{pmatrix} \quad \text{and} \quad \operatorname{tr} D^2 u = \sum_{i=1}^{n} u_{x_i,x_i} =: \Delta u.$$

1.3. What is a Partial Differential Equation (PDE).

Definition 1.1. A PDE is an equation which relates an unknown function u, its partial derivatives and its independent variables.

A general PDE on a domain $U \subset \mathbb{R}^n$ can be written as

$$F(\mathbf{x}, u, D^1 u, \dots, D^k u) = F(\mathbf{x}, u(\mathbf{x}), D^1 u(\mathbf{x}), \dots, D^k u(\mathbf{x})) = g(\mathbf{x}), \quad \mathbf{x} \in U \tag{1}$$

for functions

$$g(\mathbf{x}) \text{ and } F(\mathbf{x}, \theta, \theta^1, \dots, \theta^k)$$

where $(x_1, \dots, x_n) = \mathbf{x} \in U$, $\theta \in \mathbb{R}$ and $\theta^i = (\theta^i_1, \dots, \theta^i_{n^i}) \in \mathbb{R}^{n^i}$ and $i = 0, \dots, k$.

u and $D^1 u, \dots, D^k u$ are also called dependent variables.

When we study a PDE often the domain U is not specified yet in the beginning.

Definition 1.2. The order of a PDE is the highest order of a partial derivative that appears in the equation.

The most general form of a first order PDE for 2 independent variables is

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = g(x, y).$$

1.4. Linear PDEs.

Definition 1.3. A PDE of the form

$$F(\mathbf{x}, u, D^1 u, \dots, D^k u) = g(\mathbf{x}) \tag{2}$$

is called **linear** if the function

$$(\theta, \theta^1, \dots, \theta^k) \in \mathbb{R} \times \mathbb{R}^n \times \dots \times \mathbb{R}^{n^k} \mapsto F(\mathbf{x}, \theta, \theta^1, \dots, \theta^k) \in \mathbb{R}$$

is linear.

A linear PDE of order 2 in n indpendent variables can always be written in the form

$$\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x})u_{x_i,x_j} + \sum_{k=1}^{n} b_k(\mathbf{x})u_{x_k} + c(\mathbf{x})u = g(\mathbf{x})$$

with coefficients $(a_{i,j}(\mathbf{x}))_{i,j=1,\dots,n}, (b_k(\mathbf{x}))_{k=1,\dots,n}, c(\mathbf{x})$ that are functions in $\mathbf{x}$.

Example 1.4 (Poisson equation).

$$\Delta u = \sum_{i=1}^{n} u_{x_i,x_j} = g(\mathbf{x}) \text{ where } a_{i,j} = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}.$$

1.5. **Nonlinear PDEs.**

Definition 1.5. A PDE of the form

$$F(\mathbf{x}, u, D^1 u, \dots, D^k u) = g(\mathbf{x}) \tag{3}$$

is called

- **semi linear** if we can write
$$F(\mathbf{x}, \theta, \theta^1, \dots, \theta^k) = L(\mathbf{x}, \theta^k) + G(\mathbf{x}, \theta, \theta^1, \dots, \theta^{k-1})$$
and the function $\theta^k \in \mathbb{R}^{n^k} \mapsto L(\mathbf{x}, \theta^k)$ is linear.
- **quasi linear** if we can write
$$F(\mathbf{x}, \theta, \theta^1, \dots, \theta^k) = L(\mathbf{x}, \theta, \theta^1, \dots, \theta^{k-1}, \theta^k) + G(\mathbf{x}, \theta, \theta^1, \dots, \theta^{k-1})$$
and the function $\theta^k \in \mathbb{R}^{n^k} \mapsto L(\mathbf{x}, \theta, \theta^1, \dots, \theta^{k-1}, \theta^k)$ is linear.
- **fully nonlinear** if the PDE is not $\Big[$ linear, semilinear or quasilinear $\Big]$.

The following implications are clear: linear $\Longrightarrow$ semi-linear $\Longrightarrow$ quasi-linear $\Longrightarrow$ fully non-linear.

Consider a quasi linear PDE $F(\mathbf{x}, u, D^1 u) = g(\mathbf{x})$. Hence F has the form

$$F(\mathbf{x}, \theta, \theta^1) = \sum_{i=1}^{n} a_i(\mathbf{x}, \theta)\theta^1 + G(\mathbf{x}, \theta).$$

The coefficients $(a_i)_{i=1,\dots,n}$ are functions in $\mathbf{x}$ and θ. The PDE takes the form

$$\sum_{i=1}^{n} a_i(\mathbf{x}, u)u_{x_i} + G(\mathbf{x}, u) = g(\mathbf{x}).$$

Example 1.6 (Inviscid (or Non-viscous) Burger's equations).

$$u_t + (u^2)_x = 0 \implies u_t + uu_x = 0$$

is a quasi-linear PDE of order 1 in 2 independent variables: $t = x_1$ and $x = x_2$. Here we have $a_1(\mathbf{x}, u) = 1$, $a_2(\mathbf{x}, u) = u$ and $G = g \equiv 0$.

Consider a PDE of order 2 $F(\mathbf{x}, u, D^1 u, D^2 u) = g(\mathbf{x})$. If the PDE is quasi-linear, it can be writen in the general form

$$\sum_{i,j=1}^{n} a_{i,j}(\mathbf{x}, u, D^1 u)u_{x_i,x_j} + G(\mathbf{x}, u, D^1 u) = g(\mathbf{x}).$$

$(a_{i,j})_{i,j=1,\dots,n}$, G are functions in $\mathbf{x}$, θ and θ^1.

1.6. **Solutions.**

Definition 1.7. Consider a PDE of order k:

$$F(\mathbf{x}, u, D^1 u, \dots, D^k u) = g(\mathbf{x}) \tag{4}$$

A classical solution of (4) on a domain $\Omega \subset \mathbb{R}^n$ where n is the number of independent variables, is a sufficiently smooth function $u(\mathbf{x})$ that satisfies (4).

If $k \in \mathbb{N}$ is the order of the PDE, then, by sufficiently smooth, we man that $u \in C^k(\Omega)$.

Example 1.8. The function $u(x,t) = \frac{x}{t}$ solves

$$u_t + u \cdot u_x = 0 \quad \text{on} \quad \mathbb{R} \times (0, \infty) \subset \mathbb{R}^2. \tag{5}$$

1.7. **Homogeneous/Inhomogeneous Linear PDEs.**

Definition 1.9. Consider a linear PDE of order k:

$$L(\mathbf{x}, u, D^1 u, \dots, D^k u) = g(\mathbf{x}) \tag{6}$$

If $g(\mathbf{x}) \equiv 0$, the PDE is called homogeneous. Otherwise, the PDE is called inhomogeneous.

Remark 1.10.

- If u and v solve the homogeneous linear PDE

$$L(\mathbf{x}, u, D^1 u, \dots, D^k u) = 0 \quad \text{on a domain } \Omega \subset \mathbb{R}^n \tag{7}$$

  then also $\alpha u + \beta v$ solves the same homogeneous linear PDE on the domain Ω for $\alpha, \beta \in \mathbb{R}$. *(Superposition Principle)*
- If u solves the homogeneous linear PDE (7) and w solves the inhomogeneous linear pde (6) then $v + w$ also solves the same inhomogeneous linear PDE.
- We can see the map

$$u \in\mapsto \mathcal{L}u \text{ where } (\mathcal{L}u)(\mathbf{x}) = L(\mathbf{x}, u, D^1 u, \dots, D^k u)$$

  as a linear (differential) operator.

  Hence, it makes sense to specify appropriate function vector spaces V and W such that $u \in V$ and $\mathcal{L}u \in W$.

  For instance: For a PDE of order 2, we can choose $V = C^2(\Omega)$ and $W = C^0(\Omega)$.

  For instance, for a linear PDE of order one for independent variable x and y, we could set $V = C^1(\mathbb{R}^2)$ and $W \in C^0(\mathbb{R}^2)$.

Lecture 02

1.8. **Important Theorems.**

- $V = (V^1, \dots, V^n) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is in $C^k(\Omega, \mathbb{R}^n)$ if $V^i \in C^k(\Omega)$, $i = 1, \dots, n$.
- $U, V \subset \mathbb{R}^n$ open. $\Phi : U \to V$ is a C^k-diffeomorphism if Φ is one-to-one and onto and $\Phi \in C^k(U, \mathbb{R}^n)$ and $\Phi^{-1} \in C^k(V, \mathbb{R}^n)$.

Theorem 1.11 (Transformation formula)**.** *Let $U, V \subset \mathbb{R}^n$ be open and let $\Phi : U \to V$ be a C^1-diffeomorphism. Then, a function $f : V \to \mathbb{R}$ is integrable if and only if $(f \circ \Phi)|\det D\Phi| : U \to \mathbb{R}$ is integrable. Moreover, it holds*

$$\int_V f(\mathbf{x}) d\mathbf{x} = \int_U f \circ \Phi(\mathbf{y}) |\det D\Phi(\mathbf{y})| d\mathbf{y}.$$

Theorem 1.12 (Divergence (also Gauss) Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be closed and bounded with smooth boundary $\partial\Omega$. Let $N : \partial\Omega \to \mathbb{R}^n$ be the outer unit normal vector field of $\partial\Omega$. Let $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$\int_{\partial\Omega} N \cdot V dS = \int_\Omega \nabla \cdot V d\mathbf{x}.$$

The Divergence Theorem generalizes the **Fundamental Theorem of Calculus**:

$$f(b) - f(a) = \int_a^b f'(x) dx, \quad f \in C^1([a, b]).$$

2. Some important exmples of PDEs from physical context

2.1. **Simple Transport Equation.** Let

$$V : \mathbb{R}^n \to \mathbb{R}^n, \quad V \in C^1(\mathbb{R}^n, \mathbb{R}^n), \quad \nabla \cdot V = 0 \text{ and } V(\mathbf{x}) \neq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

We imagine a medium in $\mathbb{R}^n$ that moves with a speed that is equal to $V(\mathbf{x})$ at any point $\mathbf{x} \in \mathbb{R}^n$.

Solve $\frac{d}{dt}\gamma_{\mathbf{x}}(t) = V \circ \gamma_{\mathbf{x}}(t)$, $\gamma_{\mathbf{x}}(0) = \mathbf{x}$. The flow of V is the map

$$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n, \ \Phi_t(\mathbf{x}) = \gamma_{\mathbf{x}}(t).$$

$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism (this follows from smooth dependence of ODEs on initial values and will not be proven).

Remark 2.1. In this context the subscripts t and $\mathbf{x}$ in Φ_t and $\gamma_{\mathbf{x}}$ do not stand for the partial derivative of Φ or γ w.r.t. t and $\mathbf{x}$, but denote parameters.

Let $u(\mathbf{x}, t)$ be the density of a substance Ψ that is released “into” the flow of V.

What is the evolution law for u?

Let $\Omega \subset \mathbb{R}^n$ be any open domain and bounded. **We assume:**

$$\int_\Omega u(\mathbf{x}, t) d\mathbf{x} = \int_{\phi_h(\Omega)} u(\mathbf{y}, t + h) d\mathbf{y}$$

$\Phi_t|_\Omega : \Omega \to \Phi_t(\Omega)$ is a C^1-diffeomorphism. The transformation formula yields

$$\int_\Omega u(\mathbf{x}, t) d\mathbf{x} = \int_{\Phi_h(\Omega)} u(\mathbf{y}, t + h) d\mathbf{y} = \int_\Omega u(\Phi_h(\mathbf{x}), t + h) |\det D\Phi_h(\mathbf{x})| d\mathbf{x}.$$

Let us assume $\det D\Phi_h(\mathbf{x}) > 0$. Since Ω was arbitrary, we get

$$u(\mathbf{x}, t) = u(\phi_h(\mathbf{x}), t + h) \det D\Phi_h(\mathbf{x}).$$

Differentiate w.r.t. h at $h=0$ on both sides:

$$0 = \nabla_x u \cdot \underbrace{\frac{d}{dh}|_{h=0}\Phi_0(\mathbf{x})}_{V(\mathbf{x})} + u_t(\mathbf{x},t) + u(\underbrace{\Phi_0(\mathbf{x})}_{\gamma_x(0)=x}, t)\frac{d}{dh}|_{h=0}\det D\Phi_h(\mathbf{x}).$$

The matrix $D(\phi_h)$ is invertible, $D\Phi_0 = E_n$ and $(\mathbf{x},h)\mapsto \Phi_h(\mathbf{x}) = \gamma_{\mathbf{x}}(h)$ is a C^2 map.

Hence $h \mapsto D(\Phi_h)(\mathbf{x}) =: A(h)$ is differentiable at $h=0$.

$$\frac{d}{dh}|_{h=0}\det A(h) = \det A(0)\operatorname{tr}[A^{-1}(0)\frac{d}{dh}|_{h=0}A(h)] = \operatorname{tr}[\frac{d}{dh}|_{h=0}A(h)]$$

On the other hand we can compute that

$$\frac{d}{dh}|_{h=0}D\Phi_h(\mathbf{x}) = D\frac{d}{dh}|_{h=0}\Phi_h(\mathbf{x}) = DV(\mathbf{x})$$

It follows

$$\frac{d}{dh}|_{h=0}\det D\Phi_h = \operatorname{tr} DV = \nabla\cdot V = 0.$$

So the PDE that governs $u(\mathbf{x},t)$ is

$$u_t + V\cdot\nabla u = 0.$$

Simple Transport, revisited. Let us think the previous model from a different perspective.

We set

$$\int_\Omega u(\mathbf{x},t)d\mathbf{x} = M(\Omega,t).$$

Then

$$\frac{d}{dt}M(\Omega,t) = \frac{d}{dt}\int_\Omega u(\mathbf{x},t)d\mathbf{x} = \int_\Omega u_t(\mathbf{x},t)d\mathbf{x}$$

How can we understand the change of $M(\Omega,t)$ in time t?

Let $\mathbf{F}(\mathbf{x},t) = \mathbf{F}_t(\mathbf{x}) \in \mathbb{R}^n$ defined as

$$\mathbf{F}(\mathbf{x},t) = u(\mathbf{x},t)V(\mathbf{x}).$$

The **total flux** of u through $\partial\Omega$ is then

$$\int_{\partial\Omega} N(\mathbf{x})\cdot F(\mathbf{x},t)dS(\mathbf{x}) = \int_{\partial\Omega} N(\mathbf{x})\cdot u(\mathbf{x},t)V(\mathbf{x})dS(\mathbf{x}),$$

the net value of how much of the substance Ψ flows in and out of Ω at time t.

But clearly

$$\frac{d}{dt}M(\Omega,t) = -\int_{\partial\Omega} N(\mathbf{x})\cdot u(\mathbf{x},t)V(\mathbf{x})dS(\mathbf{x})$$

Applying the divergence theorem yields

$$\int_\Omega u_t(\mathbf{x},t)d\mathbf{x} = -\int_\Omega \nabla\cdot[u(\mathbf{x},t)V(\mathbf{x})]\,d\mathbf{x} = (1)$$

By the chain rule this becomes

$$(1) = -\int_\Omega \Big[\nabla u(\mathbf{x},t)\cdot V(\mathbf{x}) + u(\mathbf{x},t)\underbrace{\nabla\cdot V(\mathbf{x})}_{=0}\Big]d\mathbf{x} = -\int \nabla u(\mathbf{x},t)\cdot V(\mathbf{x})d\mathbf{x}.$$

Since Ω was arbitrary, it follows

$$u_t + V(\mathbf{x})\cdot\nabla u = 0.$$

2.2. **Diffusion equation.** Imagine a liquid in $3D$ or higher.

Let $u(\mathbf{x}, t)$ be a concentrations function of substance Ψ released into the liquid.

The substance Ψ moves from regions with higher concentration to regions with lower concentrations. We call this process **Diffusion**.

The rate and direction of change of Ψ in x and t is proportional to the gradient of u w.r.t. $\mathbf{x} \in \mathbb{R}^n$ at time t. This is known as **Fick's law**:

$$\mathbf{F}(\mathbf{x}, t) = \lambda \nabla_{\mathbf{x}} u(\mathbf{x}, t) = \lambda \begin{pmatrix} u_{x_1}(\mathbf{x}, t) \\ \dots \\ u_{x_n}(\mathbf{x}, t) \end{pmatrix}, \quad \lambda > 0.$$

Let Ω be a compact domain with smooth boundary. Let

$$M(\Omega, t) = \int_\Omega u(\mathbf{x}, t) d\mathbf{x} \text{ and } \frac{d}{dt} M(\Omega, t) = \int_\Omega u_t(\mathbf{x}, t) dx$$

$\frac{d}{dt} M(\Omega, t)$ is equal to the total flux through the boundary $\partial\Omega$. Hence

$$\int_\Omega u_t(\mathbf{x}, t) d\mathbf{x} = -\int_{\partial\Omega} N(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}, t) dS(\mathbf{x}) = -\lambda \int_{\partial\Omega} N \cdot \nabla u dS(\mathbf{x}).$$

Hence, by the divergence theorem

$$\int_\Omega u_t(\mathbf{x}, t) d\mathbf{x} = -\lambda \int_\Omega \nabla \cdot \nabla u(\mathbf{x}, t) d\mathbf{x} = -\lambda \int_\Omega \Delta u(\mathbf{x}, t) d\mathbf{x}.$$

Since $\Omega \subset \mathbb{R}^n$ was arbitrary, it follows $u_t + \lambda\Delta = 0$.

2.3. **Nonlinear Scalar Conservation Laws.** Imagine a "flowing" substance Ψ.

What if the infinitesimal flux $\mathbf{F}(\mathbf{x}, t)$ of Ψ in $\mathbf{x}$ at time t depends on the concentration function u in $\mathbf{x}$ at time t?

We assume there exists $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ such that $\mathbf{F}(\mathbf{x}, t) = \mathbf{f} \circ u(\mathbf{x}, t)$.

Then

$$\frac{d}{dt} \int_\Omega u(\mathbf{x}, t) = -\int_{\partial\Omega} N(\mathbf{x}) \cdot \mathbf{f} \circ u(\mathbf{x}, t) dS(\mathbf{x})$$

As before by the divergence theorem and differentiating under the integral we obtain

$$\int_\Omega u_t(\mathbf{x}, t) d\mathbf{x} = -\int_\Omega \nabla \cdot \mathbf{f} \circ u d\mathbf{x} = -\int_\Omega \mathbf{f}'(u)(\mathbf{x}) \cdot \nabla u(\mathbf{x}) d\mathbf{x}$$

where

$$\mathbf{f}'(r) = \left(\mathbf{f}_1'(r), \dots, \mathbf{f}_n'(r)\right).$$

Example 2.2. Consider the $1D$ case (for instance traffic in a street). Let $\mathbf{f}(r) = \frac{1}{2}r^2$.

Then $\mathbf{f}'(r) = r$. The corresponding PDE

$$u_t + u u_x = 0$$

is the inviscid Burger's equation.

2.4. **Fundamental Theorem of Calculus of Variations.**

Theorem 2.3 (Fundamental Theorem of Calculus of Variations)**.** *Consider* $f \in C^0(\mathbb{R}^n)$*. If*

$$\int f(\mathbf{x})\varphi(\mathbf{x}) d\mathbf{x} = 0 \quad \forall \varphi \in C_c^0(\mathbb{R}^n), \ \varphi \geq 0 \quad \Longrightarrow \quad f \equiv 0.$$

Proof. Assume the contrary. We will derive a contradiction.

If $f \neq 0$, then there exists at least $\mathbf{x}_0 \in \mathbb{R}^n$ such that $f(\mathbf{x}_0) \neq 0$.

We can assume $f(\mathbf{x}_0) > 0$ (otherwise replace f with $-f$, this does not change $\int f\varphi d\mathbf{x} = 0$).

In particular, there exist $\epsilon > 0$ such that $f(\mathbf{x}_0) - \epsilon > 0$.

Since f is continuous, there exists $\delta = \delta(\epsilon) > 0$ such that

$$f^{-1}(B_\epsilon(f(\mathbf{x}_0))) = f^{-1}(\{r \in \mathbb{R} : |f(\mathbf{x}_0) - r| < \epsilon\}) \subset \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0|_2 < \delta\} =: B_\delta(\mathbf{x}_0)$$

where $|\mathbf{x}|_2 = \sqrt{\sum_{i=1}^n (x_i)^2}$. In particular, if $\mathbf{x} \in B_\delta(\mathbf{x}_0)$ then $f(\mathbf{x}) \in B_\epsilon(f(\mathbf{x}_0))$. So $f(\mathbf{x}) > \epsilon > 0$.

We can choose $\varphi \in C_c^0(\mathbb{R}^n)$, $\varphi(\mathbf{x}) \equiv 0$ on $\mathbb{R}^n \backslash B_\delta(\mathbf{x}_0)$ and $\varphi(\mathbf{x}) = 1$ for $\mathbf{x} \in B_{\frac{\delta}{2}}(x_0)$. For instance

$$\varphi(\mathbf{x}) = \begin{cases} \min\{1 - \frac{1}{\delta}|\mathbf{x}|_2, 1\} & \text{for } \mathbf{x} \in B_\delta(\mathbf{x}_0) \\ 0 & \text{for } \mathbf{x} \in \mathbb{R}^n \backslash B_\delta(\mathbf{x}_0). \end{cases}$$

Then, it follows

$$0 = \int f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \int_{B_\delta(\mathbf{x})} f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} \geq \epsilon \int \phi(\mathbf{x})d\mathbf{x} = \epsilon > 0.$$

$\square$

Lecture 03.

3. First order PDEs

3.1. **Introduction to the Method of Characteristics.** We found the general solution of $au_x + bu_y = 0$. Solutions are constant on lines parallel to (a, b).

Now, we consider

$$u_x + yu_y = 0 \Longleftrightarrow (1, y) \cdot \nabla u = 0 \text{ in } \mathbb{R}^2.$$

Instead of straight lines, now we looking for curves $(x, y(x))$ such that

$$\frac{d}{dx}(x, y(x)) = (1, y) \Leftrightarrow \frac{dy}{dx} = y.$$

Hence $y(x) = Ce^x$, and a solution u satisfies

$$\frac{d}{dx}u(x, y(x)) = \nabla u \cdot (1, y) = 0$$

and

$$u(x, y(x)) = u(0, y(0)) = u(0, C)$$

is independent of x.

Let $g \in C^1(\mathbb{R})$. For every tuple (x, y) there exists a unique $C(x, y)$ such that $(x, y) = (x, C(x, y)e^x)$.

Then $u(x, y) := u(0, C(x, y)) = g(ye^{-x})$ satisfies

$$u_x + yu_y = g'(ye^{-x})(-ye^{-x}) + yg'(ye^{-x})e^{-x} = 0.$$

Therefore $u(x, y) = g(ye^{-x})$ solves the PDE with auxiliary condition $g(y) = u(0, y)$.

3.2. **Method of characteristics for linear equations.** We consider a general linear PDE of order 1 with 2 independent variables x, y:

$$a(x, y)u_x + b(x, y)u_y = c_1(x, y)u + c_2(x, y) \text{ in } \mathbb{R}^2 \tag{8}$$

We assume there is an auxiliary condition given as follows.

$$u(x, y) = g(x, y) \text{ for } (x, y) \in \Gamma \text{ and } g : \Gamma \to \mathbb{R}$$

where Γ is a suitable 1 dimensional subset in $\mathbb{R}^2$, for instance the image of a curve $\gamma : \mathbb{R} \to \mathbb{R}^2$: $\mathrm{Im}(\gamma) = \Gamma$.

We can rewrite (8) as follows

$$V(x, y) \cdot \nabla u = c_1(x, y)u + c_2(x, y)$$

for a vectorfield $(x, y) \mapsto V(x, y) = (a(x, y), b(x, y)) \in \mathbb{R}^2$.

Let us also consider the case for n independent variables. A general linear PDE of order 1 then takes the form

$$\sum_{i=1}^{n} a_i(\mathbf{x})u_{x_i} = c_1(\mathbf{x})u + c_2(\mathbf{x}) \text{ in } \mathbb{R}^n \tag{9}$$

with auxiliary condition

$$u(\mathbf{x}) = g(\mathbf{x}) \text{ for } \mathbf{x} \in \Gamma \text{ and } g : \Gamma \to \mathbb{R}$$

where Γ is a suitable $n-1$ dimensional subset in $\mathbb{R}^n$, for instance a $n-1$ dimensional submanifold.

We can again write (9) as follows

$$V(\mathbf{x}) \cdot \nabla u = c_1(\mathbf{x})u + c_2(\mathbf{x}).$$

for a vectorfield $V(\mathbf{x}) = (a_1(\mathbf{x}), \ldots, a_n(\mathbf{x}))$.

We want to find the flow curves of $V(\mathbf{x})$. That is, we have to solve the following ODE:

$$\frac{d}{dt}\gamma_{\mathbf{x}}(t) = \dot{\gamma}_{\mathbf{x}}(t) = V(\mathbf{x}), \quad \gamma_{\mathbf{x}}(0) = \mathbf{x}.$$

We also want to choose the initial point $\mathbf{x}$ such that $\mathbf{x} \in \Gamma$.

Recall that

$$\gamma_{\mathbf{x}}(t) = \begin{pmatrix} \gamma^1_{\mathbf{x}}(t) \\ \dots \\ \gamma^n_{\mathbf{x}}(t) \end{pmatrix}, \quad \dot{\gamma}_{\mathbf{x}}(t) = \frac{d}{dt}\gamma_{\mathbf{x}}(t) = \begin{pmatrix} \frac{d}{dt}\gamma^1_{\mathbf{x}}(t) \\ \dots \\ \frac{d}{dt}\gamma^n_{\mathbf{x}}(t) \end{pmatrix}$$

Now, we investigate how u evolves along a flow curve $\gamma_{\mathbf{x}}$:

$$\frac{d}{dt}u \circ \gamma_{\mathbf{x}}(t) = \nabla u(\gamma_{\mathbf{x}}(t)) \cdot \dot{\gamma}_x(t) = \nabla u(\gamma_{\mathbf{x}}(t)) \cdot V(\gamma_{\mathbf{x}}(t))$$

On the other hand

$$\nabla u(\gamma_{\mathbf{x}}(t)) \cdot V(\gamma_{\mathbf{x}}(t)) = c_1(\gamma_{\mathbf{x}}(t))u(\gamma_{\mathbf{x}}(t)) + c_2(\gamma_{\mathbf{x}}(t)).$$

This gives us an ODE for the composition $u \circ \gamma_{\mathbf{x}}(t) =: z_{\mathbf{x}}(t)$:

$$\frac{d}{dt}z_{\mathbf{x}}(t) = \dot{z}_{\mathbf{x}}(t) = c_1(\gamma_{\mathbf{x}}(t))z_{\mathbf{x}}(t) + c_2(\gamma_x(t)), \ z_{\mathbf{x}}(0) = u_0(\gamma_{\mathbf{x}}(0)).$$

3.3. **Characteristics equations.**

Definition 3.1. Consider linear PDE of order 1 with n independent variables in the form

$$V(\mathbf{x})\nabla u = c_1(\mathbf{x})u + c_2(\mathbf{x}) \ \text{ in } \mathbb{R}^n \ \text{ and } u(\mathbf{x}) = u_0(\mathbf{x}) \ \text{ on } \Gamma$$

where Γ is an $n-1$ dimensional subset.

The corresponding *characteristics equations* are

$$\begin{cases} \dot{\gamma}_{\mathbf{x}}(t) = V \circ \gamma_{\mathbf{x}}(t), & \gamma_{\mathbf{x}}(0) = \mathbf{x}, \\ \dot{z}_{\mathbf{x}}(t) = c_1(\gamma_{\mathbf{x}}(t))z_{\mathbf{x}}(t) + c_2(\gamma_{\mathbf{x}}(t)), & z_{\mathbf{x}}(0) = u_0(\mathbf{x}). \end{cases}$$

For the case of 2 independent variable the systems of equations becomes

$$\begin{cases} \dot{x}(t) = a(x(t), y(t)), & x(0) = x_0, \\ \dot{y}(t) = b(x(t), y(t)), & y(0) = y_0, \\ \dot{z}_{x_0,y_0}(t) = c_1(x(t), y(t))z_{x_0,y_0}(t) + c_2(x(t), y(t)), & z(0) = u_0(x_0, y_0). \end{cases}$$

Question 3.2. *How can we obtain a solution for the PDE?*

3.4. **How to find a solution with the method of characteristics.**

To solve the PDE we need to determine the value $u(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$.

From the previous consideration, we know that the solution for the PDE (9) with the given auxiliary condition must obey the characteristics equations.

Hence, if we can find a characteristic curve $\gamma_{\mathbf{x}_0}$ for some initial point $\mathbf{x}_0 \in \Gamma$ such that $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$ for some $t_0 \in [0, \infty)$ then we can solve the characteristic equation for $z_{\mathbf{x}_0} = u \circ \gamma_{\mathbf{x}_0}$

$$\dot{z}_{\mathbf{x}_0}(t) = c_1 \circ \gamma_{\mathbf{x}_0}(t)z_{\mathbf{x}_0}(t) + c_2 \circ \gamma_{\mathbf{x}_0}(t) \ \text{ with } z_{\mathbf{x}_0}(0) = u_0(\mathbf{x}_0)$$

and set $u(\mathbf{x}) = u \circ \gamma_{\mathbf{x}_0}(t_0) = z_{\mathbf{x}_0}(t_0)$.

Hence, if we can solve

$$\gamma_{\mathbf{x}_0}(t_0) =: \Phi(\mathbf{x}_0, t_0) = \mathbf{x} \tag{10}$$

uniquely for every $\mathbf{x} \in \mathbb{R}^n$ we have found a function $u : \mathbb{R}^n \to \mathbb{R}$ that will solve the PDE and satisfies the auxiliary condition (by construction).

The map $\Phi(\mathbf{y}, s)$ is again the flow map of the vectorfield $V : \mathbb{R}^n \to \mathbb{R}^n$ that is in $C^1(\mathbb{R}^n, \mathbb{R}^n)$. Hence, solving (10) uniquely means for any $\mathbf{x}$ we find a unique flow curve $\gamma_{\mathbf{x}_0}$ such that $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$.

A suitable auxiliary condition

$$u(\mathbf{x}) = u_0(\mathbf{x}_0)\ \forall \mathbf{x}_0 \in \Gamma$$

now must satisfy that any flow curve of V intersects Γ exactly once.

Example 3.3. Let us consider (again)

$$(a, b) \cdot \nabla u = 0 \text{ in } \mathbb{R}^2 \text{ with } u(x, 0) = x^3.$$

The characteristics equations are

$$\begin{cases} \dot{x}(t) = a, & x(0) = d_1, \\ \dot{y}(t) = b, & y(0) = d_2, \\ \dot{z}_{x,y}(t) = 0, & z_{x,y}(0) = d_3. \end{cases}$$

We first see that

$$x(t) = at + d_1, \quad y(t) = bt + d_2.$$

How do we choose the constants d_1 and d_2? We choose them such that $(x(0), y(0)) \in \mathbb{R} \times \{0\}$. Hence we set $d_1 = x \in \mathbb{R}$ and $d_2 = 0$.

The equation for $z_{x,y}$ yields $z_{x,y}(t) = d_3$. But since $z_{x,y}(0) = u(x(0), y(0))$, we have $z_{x,y}(0) = x^3$.

The flow map for $(x, 0) \in \Gamma$ is given by

$$\Phi((x, 0), t) = (at + x, bt)$$

Hence, for $(\hat{x}, \hat{y}) \in \mathbb{R}^2$ arbitrary, we set up the equation

$$at + x = \hat{x},\ bt = \hat{y}$$

Hence $t_0 = \hat{y}/b$ and $(x_0, 0)$ with $x_0 = \hat{x} - at_0 = \hat{x} - \frac{a}{b}\hat{y}$ solves

$$\Phi((x_0, 0), t_0) = (\hat{x}, \hat{y})$$

And hence $z_{x_0,0}(t_0) = (\hat{x} - \frac{a}{b}\hat{y})^3$.

Lecture 04.

Last Lecture: We found the general solution of $au_x + bu_y = 0$. Solutions are constant on lines parallel to (a, b).

Example 3.4. As another example we consider

$$u_x + yu_y = 0 \Longleftrightarrow (1, y) \cdot \nabla u = 0 \quad \text{in } \mathbb{R}^2$$

with the auxiliary condition $u(0, y) = g(y)$ for $g \in C^1(\mathbb{R})$.

Instead of straight lines, now we looking for curves $(x, y(x))$ such that

$$\frac{d}{dx}(x, y(x)) = (1, y) \;\Leftrightarrow\; \frac{d}{dx}x = 1 \;\&\; \frac{d}{dx}y = y$$

Hence $y(x) = y_0 e^x$ with $y(0) = y_0$.

Moreover, a solution u of the PDE satisfies along the curve $(x, y(x))$:

$$\frac{d}{dx}u(x, y(x)) = \nabla u \cdot (1, y) = u_x + yu_y = 0$$

and $u(x, y(x)) = u(0, y(0)) = u(0, y_0)$ is independent of x.

Given a point $(\hat{x}, \hat{y}) \in \mathbb{R}^2$ we want to find y_0 and $y(\cdot)$ with $y(0) = y_0$ and $y(\hat{x}) = \hat{y}$.

Then, we know the value of u in $(\hat{x}, \hat{y})$: It is

$$u(\hat{x}, \hat{y}) = u(\hat{x}, y(\hat{x})) = u(0, y_0) = g(y_0).$$

But by the formula for $y(x)$, we can indeed find such y_0: it is $y_0 = \hat{y}e^{-\hat{x}}$. Therefore we can write

$$u(\hat{x}, \hat{y}) = g(\hat{y}e^{-\hat{x}}).$$

This u satisfies the PDE with the given auxiliary condition. (let us drop $\hat{\cdot}$) Indeed

$$u_x + yu_y = g'(ye^{-x})(-ye^{-x}) + yg'(ye^{-x})e^{-x} = 0.$$

Therefore $u(x, y) = g(ye^{-x})$ solves the PDE with auxiliary condition $g(y) = u(0, y)$.

3.5. **Characteristics equations.**

Definition 3.5. Consider linear PDE of order 1 with n independent variables in the form

$$V(\mathbf{x})\nabla u = c_1(\mathbf{x})u + c_2(\mathbf{x}) \quad \text{in } \mathbb{R}^n \quad \text{and } u = u_0 \quad \text{on } \Gamma$$

where Γ is an $n - 1$ dimensional subset.

The corresponding *characteristics equations* are

$$\begin{cases} \dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t) & \gamma_{\mathbf{x}_0}(0) = \mathbf{x}_0, \\ \dot{z}_{\mathbf{x}_0}(t) = c_1(\gamma_{\mathbf{x}_0}(t))z_{\mathbf{x}_0}(t) + c_2(\gamma_{\mathbf{x}_0}(t)) & z_{\mathbf{x}_0}(0) = u_0(\mathbf{x}_0). \end{cases}$$

For the case of 2 independent variable the systems of equations becomes

$$\begin{cases} \dot{x}(t) = a(x(t), y(t)), & x(0) = x_0, \\ \dot{y}(t) = b(x(t), y(t)), & y(0) = y_0, \\ \dot{z}(t) = c_1(x(t), y(t))z(t) + c_2(x(t), y(t)), & z(0) = u_0(x_0, y_0). \end{cases}$$

Question 3.6. *How we obtain a solution for the PDE?*

3.6. **Solving the PDE.** We will determine the value $u(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$ using the method of characteristics.

From the previous consideration, we know that the solution for the PDE (9) must obey the characteristics equations.

Our strategy is:
For $\mathbf{x} \in \mathbb{R}^n$ arbitrary we pick a characteristic $\gamma_{\mathbf{x}_0}$ for an initial point $\mathbf{x}_0 \in \Gamma$ such that $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$ for some $t_0 \in [0, \infty)$.
Then we can solve the characteristic equation for $z_{\mathbf{x}_0} = u \circ \gamma_{\mathbf{x}_0}$

$$\dot{z}_{\mathbf{x}_0}(t) = c_1 \circ \gamma_{\mathbf{x}_0}(t) z_{\mathbf{x}_0}(t) + c_2 \circ \gamma_{\mathbf{x}_0}(t) \quad \text{with } z_{\mathbf{x}_0}(0) = u_0(\mathbf{x}_0)$$

and set

$$z_{\mathbf{x}_0}(t_0) = u \circ \gamma_{\mathbf{x}_0}(t_0) = u(x).$$

Hence, if for every $\mathbf{x} \in \mathbb{R}^n$ we can find a unique $\mathbf{x}_0 \in \Gamma$ and $t_0 \geq 0$ such that

$$\gamma_{\mathbf{x}_0}(t_0) =: \Phi(\mathbf{x}_0, t_0) = \mathbf{x}$$

we can define a function $u : \mathbb{R}^n \to \mathbb{R}$ that will solve the PDE and satisfies the auxiliary condition (by construction). Indeed we have

Proposition 3.7. *Assuming $u : \mathbb{R}^n \to \mathbb{R}$ defined above is a C^1 function, then it solves*

$$\nabla u \cdot V(\mathbf{x}) = c_1(\mathbf{x})u + c_2(\mathbf{x}) \quad \textit{in} \quad \mathbb{R}^n \quad \textit{with} \quad u(\mathbf{x}) = u_0(\mathbf{x}) \textit{ for } \mathbf{x} \in \Gamma.$$

Proof. Let $\mathbf{x} \in \mathbb{R}^n$, assume $\mathbf{x}_0$ is the unique point such that: $\gamma_{\mathbf{x}_0}$ solves $\dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t)$ with $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$.
Then $u(\mathbf{x})$ is defined as $z_{\mathbf{x}_0}(t_0)$.

First, if $x \in \Gamma$, we pick $\mathbf{x}_0 = \mathbf{x}$, $\gamma_{\mathbf{x}_0} = \gamma_{\mathbf{x}}$ and $t_0 = 0$. Then $u(\mathbf{x}) = z_{\mathbf{x}}(0) = u_0(\mathbf{x})$.

For general $\mathbf{x} \in \mathbb{R}^n$ we compute

$$\nabla u(\mathbf{x}) \cdot V(\mathbf{x}) = \nabla u(\gamma_{\mathbf{x}_0}(t_0)) \cdot \dot{\gamma}_{\mathbf{x}_0}(t_0) = \frac{d}{dt} u \circ \gamma_{\mathbf{x}_0}(t_0).$$

Since $z_{\mathbf{x}_0}$ solves the last characteristic equation, the right hand side is equal to

$$c_1(\gamma_{\mathbf{x}_0}(t_0)) u \circ \gamma_{\mathbf{x}_0}(t_0) + c_2(\gamma_{\mathbf{x}_0}(t_0)) = c_1(\mathbf{x})u(\mathbf{x}) + c_2(\mathbf{x}).$$

Hence u indeed solves the equation. □

Remark 3.8. Let us summarize what we assumed here

- We need that any flow curve meets Γ in exactly one point.

  For any $\mathbf{x}$ there exists a unique flow curve $\gamma_{\mathbf{x}_0}$ such that $x_0 \in \Gamma$ and $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$.

  Then, we can solve the initial value problem for $z_{\mathbf{x}_0}$ because the initial value is given by $u(\mathbf{x}_0) = u_0(\mathbf{x}_0)$.

  In other words, we have to solve the equation

  $$\Phi_{t_0}|_\Gamma(\mathbf{x}_0) = \mathbf{x}$$

  where $\Phi_t(\mathbf{y}) = \gamma_{\mathbf{y}}(t)$ is the flow map of V, and $\Phi_t|_\Gamma$ is the restriction of Φ_t to Γ.
- We need that $u \in C^1(\mathbb{R}^n)$.

3.6.1. *Example.* We find the solution for

$$xu_x + 2u_y = 3u \text{ in } \mathbb{R}^2,\ u(x,0) = \sin x,\ \Gamma = \mathbb{R} \times \{0\}.$$

The first two characteristics equations are

$$\dot{x}(t) = x(t)\ x(0) = x_0 \in \mathbb{R}^2,$$
$$\dot{y}(t) = 2\ y(0) = y_0 \in \mathbb{R}^2.$$

The general solutions are $x(t) = x_0 e^t$ and $y(t) = 2t$.

Let $(x, y) \in \mathbb{R}^2$ be arbitrary. Consider the equation

$$x(t_0) = x_0 e^{t_0} = x, \quad y(t_0) = 2t_0 = y \tag{11}$$

The equation (11) has a unique solution. This is $t_0 = \frac{y}{2}$ and $(x_0, y_0) = (xe^{-\frac{y}{2}}, 0)$.

For this initial point $(x_0, y_0) = (xe^{-\frac{y}{2}}, 0)$ and $u_0(x_0, y_0) = \sin\left(xe^{-\frac{y}{2}}\right)$, we consider the third characteristics equation

$$\frac{d}{dt} z(t) = z(t), \quad z(0) = \sin\left(xe^{-\frac{y}{2}}\right).$$

The solution is

$$z(t) = \sin\left(xe^{-\frac{y}{2}}\right) e^t$$

and at $t_0 = \frac{y}{2}$ we get

$$z(t_0) = \sin\left(xe^{-\frac{y}{2}}\right) e^{\frac{y}{2}} =: u(x, y).$$

3.7. **Temporal Equations.** Consider a linear PDE of order 2 of the form

$$u_t + \sum_{i=1}^{n} a_i(\mathbf{x}) u_{x_i} = c_1(\mathbf{x}) u + c_2(\mathbf{x}) \ \text{ in } \mathbb{R}^n. \tag{12}$$

In this case the auxiliary condition is usually given as an initial condition at time $t = 0$:

$$u_0(\mathbf{x}) = g(\mathbf{x}) \ \text{ on } \mathbb{R}^n.$$

The characteristics ODE for the t variable is always $\frac{d}{ds} t(s) = 1,\ t(0) = 0$. Thus $t = s$.

We have $\gamma_{(\mathbf{x}_0, 0)}(t) = (t, x_1(t), \dots, x_n(t))$ and denote $(x_1(t), \dots, x_n(t)) =: \gamma_{\mathbf{x}_0}(t)$ for the characteristics.

If we set $V(\mathbf{x}) = (a_1(\mathbf{x}), \dots, a_n(\mathbf{x}))$, the PDE becomes

$$u_t + V(\mathbf{x}) \cdot \nabla u = c_1(\mathbf{x}) u + c_2(\mathbf{x}).$$

The curves $\gamma_{\mathbf{x}_0}$ solve $\dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t)$ with $\gamma_{\mathbf{x}_0}(0) = \mathbf{x}_0$, hence are the flow curves of V.

If the flow map Φ_t of V is a diffeomorphism of $\mathbb{R}^n$ for every $t \geq 0$, then for $t >$ and for every $\mathbf{x} \in \mathbb{R}^n$ we can solve

$$\Phi_t(\mathbf{x}_0) = \mathbf{x} \iff \Phi_t^{-1}(x) = \mathbf{x}_0$$

uniquely. Hence, $\Phi_t(\mathbf{x}_0) = \gamma_{\mathbf{x}_0}(t) = \mathbf{x}$. In this case we can solve the characteristics equation for z

$$\frac{d}{dt} z_{\mathbf{x}_0}(t) = c_1(\gamma_{\mathbf{x}_0}(t)) z(t) + c_2(\gamma_{\mathbf{x}_0})$$

with initial value $z_{\mathbf{x}_0}(0) = g(x_0)$ and define $u(\mathbf{x}) := z_{\mathbf{x}_0}(t) = z_{\Phi_t^{-1}(\mathbf{x})}(t)$ that is a solution for (13). Note that u is indeed smooth enough.

Lecture 05.

The method of characteristics is a recipe to solve a linear PDE of order 1 in several variables:

$$\nabla u \cdot V(\mathbf{x}) = c_1(\mathbf{x})u + c_2(\mathbf{x}) \text{ in } \Omega \subset \mathbb{R}^n \;\; \& \;\text{ auxiliary condition: } u(\mathbf{x}) = u_0(\mathbf{x}) \text{ on } \Gamma.$$

$V(\mathbf{x}) = (a_1(\mathbf{x}), \dots, a(\mathbf{x})) \in C^1(\Omega, \mathbb{R}^n)$, $c_1, c_2 \in C^1(\Omega)$, and Γ is a $n-1$ dimensional subset in Ω.

The **recipe** goes as follows

- We note that the PDE is a statement about the directional derivatives of a C^1 solution u.
- Assuming the existence of a C^1 solution, we deduced the *characteristics equations*:

$$\begin{cases} \dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t) & \gamma_{\mathbf{x}_0}(0) = \mathbf{x}_0 \in \Gamma, \\ \dot{z}_{\mathbf{x}_0}(t) = c_1(\gamma_{\mathbf{x}_0}(t)) z_{\mathbf{x}_0}(t) + c_2(\gamma_{\mathbf{x}_0}(t)) & z_{\mathbf{x}_0}(0) = u_0(\mathbf{x}_0). \end{cases}$$

  This system of ODEs can be solved uniquely on a maximal intervall $(\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0}) \ni 0$ (General Existence and Uniqueness Theorem for ODEs).
- The number $z_{\mathbf{x}_0}(t)$ tells us the values of u at $\gamma_{\mathbf{x}_0}(t)$ for $t \in (\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0})$.

The only result that this analysis actually proves is:

Proposition 3.9. **If** *$u \in C^1(\Omega)$ solves the PDE and $\gamma_{\mathbf{x}_0} : (\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0}) \to \mathbb{R}^n$ is a flow curve of V with $\mathbf{x}_0 \in \Gamma$,*
then *$u \circ \gamma_{\mathbf{x}_0} : (\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0}) \to \mathbb{R}$ with $u \circ \gamma_{\mathbf{x}_0}(0) = u_0(\mathbf{x}_0)$ must solve the ODE for $z_{\mathbf{x}_0}$.*

This gives us a method to “synthesize” an explicite solution via the following steps:

If we can find a unique solution $(\mathbf{x}_0, t_0)$ for the equation $\gamma_{\mathbf{x}_0}(t_0) = \mathbf{x}$ for every $\mathbf{x} \in \Omega$.
And $(\mathbf{x}_0, t_0)$ depends in a sufficiently smooth way on $\mathbf{x}$.

Remark 3.10. However, this might not be always possible:

There might exist $\mathbf{x} \in \Omega$ for which there exists not solution of $\gamma_{\mathbf{x}_0}(t_0) = x$ with $\mathbf{x}_0 \in \Gamma$ and $t \in (\alpha_{\mathbf{x}_0}, \omega_{\mathbf{x}_0})$.

Temporal Equations, revisited. Given a linear PDE of order 2 of the form

$$u_t + \sum_{i=1}^{n} a_i(\mathbf{x}) u_{x_i} = c_1(\mathbf{x})u + c_2(\mathbf{x}) \;\text{ in } \mathbb{R}^n \times \mathbb{R} \tag{13}$$

The initial condition at time $t = 0$ is

$$u(\mathbf{x}, 0) = g(\mathbf{x}) \;\text{ on } \mathbb{R}^n, \; g \in C^1(\mathbb{R}^n).$$

The characteristics ODE for the t variable is always $\frac{d}{ds} t(s) = 1, \; t(0) = 0$. Thus $t = s$.

If we set $V(\mathbf{x}) = (a_1(\mathbf{x}), \dots, a_n(\mathbf{x}))$, the PDE becomes

$$u_t + V(\mathbf{x}) \cdot \nabla u = c_1(\mathbf{x})u + c_2(\mathbf{x}).$$

The flow curves $\gamma_{\mathbf{x}_0}$ of V are $\dot{\gamma}_{\mathbf{x}_0}(t) = V \circ \gamma_{\mathbf{x}_0}(t)$ with $\gamma_{\mathbf{x}_0}(0) = \mathbf{x}_0$.

The characteristics of the PDE are

$$\gamma_{(\mathbf{x}_0, 0)}(t) = \begin{pmatrix} \gamma_{\mathbf{x}_0}(t) \\ t \end{pmatrix}.$$

Applying our recipe means to solve $\gamma_{\mathbf{x}_0}(t) = \mathbf{x}$ uniquely for every $\mathbf{x} \in \mathbb{R}^n$ and every $t \in \mathbb{R}$.

If the flow map $\Phi_t(\mathbf{x}) = \gamma_{\mathbf{x}}(t)$ of V is a diffeomorphism of $\mathbb{R}^n$ for every $t \geq 0$, then

$$\Phi_t^{-1}(x) = \mathbf{x}_0 \;\text{ solves }\; \Phi_t(\mathbf{x}_0) = \mathbf{x}$$

uniquely. In this case we can define $u(\mathbf{x}) := z_{\mathbf{x}_0}(t) = z_{\Phi_t^{-1}(\mathbf{x})}(t)$ that is a solution for (13).

Example 3.11 (Transport equation with constant coefficients). Consider

$$u_t + \sum_{i=1}^n a_i u_{x_i} = 0 \text{ in } \mathbb{R}^n \times [0,\infty) \quad \text{with} \quad u(\mathbf{x},0) = g(\mathbf{x}) \text{ on } \mathbb{R}^n,\ g, f \in C^1(\mathbb{R}^n).$$

Define $V(\mathbf{x}) \equiv (a_1, \dots, a_n) = v$. The flow curves of V are

$$\gamma_{\mathbf{x}_0}(t) = \mathbf{x}_0 + tv$$

and the flow map $\phi_t(\mathbf{x}_0) = \mathbf{x}_0 + tv$ is a diffeomorphism. Hence

$$\mathbf{x}_0 := \phi_t^{-1}(\mathbf{x}) = \mathbf{x} - tv \ \text{ uniquely solves } \ \phi_t(\mathbf{x}_0) = x.$$

Solving the characteristics equation

$$\frac{d}{dt} z_{\mathbf{x}_0}(t) = 0, \quad z_{\mathbf{x}_0}(0) = u(\mathbf{x}_0, 0) = g(\mathbf{x}_0)$$

yields $z_{\mathbf{x}_0}(t) = g(\mathbf{x}_0)$. Hence $u(\mathbf{x}, t) = g(\mathbf{x} - tv)$ is the solution for the PDE.

3.8. **Semi-linear PDEs.** Consider a semi-linear PDE of order 1

$$V(\mathbf{x}) \cdot \nabla u = c(\mathbf{x}, u) \ \text{ in } \ \Omega \subset \mathbb{R}^n \ \text{ with auxiliary condition } u(\mathbf{x}) = g(\mathbf{x}) \text{ on } \Gamma \subset \Omega.$$

The methods of characteristics applies in the exact same way.

But the equation for $z_{\mathbf{x}_0}$ becomes a nonlinear equation in $z_{\mathbf{x}_0}$:

$$\frac{d}{dt} z_{\mathbf{x}_0}(t) = c(\gamma_{\mathbf{x}_0}(t), z_{\mathbf{x}_0}(t)), \quad z_{\mathbf{x}_0}(0) = g(\mathbf{x}_0),\ \mathbf{x}_0 \in \Gamma.$$

3.9. **Quasi-linear PDEs.** Consider a quasilinear PDE of order 1

$$\sum_{i=1}^n a_i(\mathbf{x}, u) u_{x_i} = c(\mathbf{x}, u) \ \text{ in } \ \Omega \subset \mathbb{R}^n \ \text{ with auxiliary condition } u(\mathbf{x}) = g(\mathbf{x}) \text{ on } \Gamma \subset \Omega.$$

Defining the vector field $V(\mathbf{x}, u) = (a_1(\mathbf{x}, u), \dots, a_n(\mathbf{x}, u))$ the PDE becomes

$$V(\mathbf{x}, u) \cdot \nabla u = c(\mathbf{x}, u).$$

Assuming a sufficiently smooth solution u we can write down the following equations

$$\dot{\gamma}_{\mathbf{x}_0}(t) = V(\gamma_{\mathbf{x}_0}(t), u \circ \gamma_{\mathbf{x}_0}(t))$$
$$\frac{d}{dt} u \circ \gamma_{\mathbf{x}_0}(t) = c(\gamma_{\mathbf{x}_0}(t), u \circ \gamma_{\mathbf{x}_0}(t))$$

Not that in contrast to linear and semi-linear PDEs this is a coupled system of ODEs.

Provided the coefficients are a_i and c are C^1 the solution $(\gamma_{\mathbf{x}_0}(t), z_{\mathbf{x}_0}(t))$ exists and depends in C^1 sense on $(\mathbf{x}_0, t)$.

3.10. **Transversality condition, Existence of local solutions.** Consider again

$$au_x + bu_y = 0 \ \text{ on } \mathbb{R}^2 \quad \text{with } u(0, y) = g(y),\ g \in C^1(\mathbb{R}).$$

We could construct a (unique) solution as long as $a \neq 0$. Or

$$\det \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \neq 0.$$

Consider a general quasi-linear PDE of order 1 in **two independent variables**

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \text{ in } \Omega \subset \mathbb{R}^2 \text{ \& auxiliary condition } u(x, y) = g(x, y) \text{ on } \Gamma$$

where $\Gamma = \mathrm{Im}\eta$ and $\eta : \mathbb{R} \to \mathbb{R}^2, \eta(t) = (k(t), l(t))$, $\eta \in C^1(J, \mathbb{R}^2)$ for an interval $J \subset \mathbb{R}$. Assume Γ is a embedded sumanifold.

The Transversality Condition for this problem is

$$\det \begin{pmatrix} a(k(t), l(t), g(k(t), l(t))) & \dot{k}(t) \\ b(k(t), l(t), g(k(t), l(t)) & \dot{l}(t) \end{pmatrix} \neq 0.$$

Theorem 3.12. *Consider the previous quasi-linear PDE and assume the transversality condition. Then, for every $s_0 \in J$ there exists $\delta > 0$ such that $B_\delta(\eta(s_0)) \subset \Omega$ and*

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \text{ in } B_\delta(\gamma(t)) \quad \text{with } u(x, y) = g(x, y) \text{ on } \Gamma \cap B_\delta(t)$$

has a unique solution u.

Proof. We already mentioned that $\gamma_{x_0,y_0}(t) =: \phi(x_0, y_0, t)$ dependes smoothly on (x_0, y_0, t) for $(x_0, y_0, t) \in \mathbb{R}^2 \times \mathbb{R}$.

We pick $s_0 \in J$ with $\eta(s_0) = x_0$ and define

$$\psi(s, t) = \phi(\eta(s), t), \;\; s \in (s_0 - \delta, s_0 + \delta) \subset J, t \in (-\epsilon, \epsilon),$$

where we choose $\delta > 0$ such that $(-\epsilon, \epsilon) \subset (\alpha_{\eta(s)}, \omega_{\eta(s)})$ for all $s \in (s_0 - \delta, s_0 + \delta)$.

Then $\psi : (s_0 - \delta, s_0 + \delta) \times (-\epsilon, \epsilon) \to \mathbb{R}$ is C^1, since it is a composition of C^1 maps.

We compute

$$\frac{\partial}{\partial t}\psi(s, t)\big|_{s_0,0} = \frac{d}{dt}\Big|_{t=0}\phi(\eta(s_0), t) = (a(k(s_0), l(s_0), z_{k(s_0),l(s_0)}), b(k(s_0), l(s_0), z_{k(s_0),l(s_0)}))$$

and

$$\frac{d}{ds}\psi(s, t)\Big|_{s_0,0} = \frac{d}{ds}\phi(\eta(s_0), 0) = \frac{d}{ds}\eta(s_0) = (\dot{k}(s_0), \dot{l}(s_0)).$$

Now, the transversality condition implies that the differential of the map $(s, t) \mapsto \phi(\eta(s), t)$ in $(s_0, 0)$ is invertible.

Hence, by the inverse function theorem, there exists a smaller $\hat{\delta} > 0$ such that $\psi(s, t)$ is a C^1-diffeomorphism on $(s_0 - \hat{\delta}, s_0 + \hat{\delta}) \times (-\hat{\delta}, \hat{\delta})$.

Hence $\psi((s_0 - \hat{\delta}, s_0 + \hat{\delta}) \times (-\hat{\delta}, \hat{\delta})) =: U \subset \mathbb{R}^2$ is an open domain in $\mathbb{R}^2$ and for all $(x, y) \in U$ there exists a unique pair (s, t) such that $\phi(\eta(s), t) = (x, y)$. □

Example 3.13 (Transport equation with nonlinear right hand side). Consider

$$u_x + u_y = u^2 \text{ on } \Omega \subset \mathbb{R}^2 \;\; \text{with } u(\cdot, 0) = g \in C^1(\mathbb{R}).$$

Here the vector field is $V(x) = (1, 1)$ with the flow $\gamma_{(x_0,0)}(t) = (x_0 + t, y_0 + t)$.

Hence for $(x, y) \in \mathbb{R}^2$ the point $(x_0, 0) = (x - t_0, 0)$ and $t_0 = y$ solves $\gamma_{(x_0,0)}(t_0) = (x, y)$.

The characteristics equation for z_{x_0} is

$$\frac{d}{dt}z_{(x_0,y_0)} = (z_{(x_0,y_0)})^2 \;\; z_{(x_0,y_0)}(0) = g(x_0).$$

The solution of this ODE Is $z_{(x_0,y_0)}(t) = \frac{1}{\frac{1}{g(x_0)} - t}$. So $u(x, y) = u_{(x_0,0)}(t_0) = \frac{1}{\frac{1}{g(x-y)} - y}$.

This yields the following contraint: $g(x - y)y < 1$. Hence, to find a solution it is necessary that $\Omega \subset \{(x, y) \in \mathbb{R}^2 : g(x - y)y \leq 1\}$.

Lecture 06.

3.11. **Burgers' Equations.** We studied the linear transport equation: $u_t + cu_x = 0$ in $\mathbb{R} \times [0, \infty)$, $u(x, 0) = g(x)$.

The solution was given by $u(x, t) = g(x - tc)$.

This is not a good model for describing natural phenomenas like waves, or street traffic.

A better equation is

Definition 3.14 (The inviscid Burgers' equation)**.**

$$u_t + uu_x = 0 \text{ in } \mathbb{R} \times [0, \infty),\ u(x, 0) = g(x),\ g \in C^1(\mathbb{R}).$$

Because of the non-linearity uu_x the equation is a **quasi-linear equation**.

A solution u moves with speed in x that is given by the value of u in x itself.

We can easily write down the characteristics equations:

$$\begin{aligned} \frac{d}{dt}\gamma_{x_0}(t) &= z_{x_0}, \quad \gamma_{x_0}(0) = x_0 \in \mathbb{R}, \\ \frac{d}{dt}z_{x_0}(t) &= 0, \qquad z_{x_0}(0) = g(x_0). \end{aligned}$$

The solution of this system is

$$\begin{aligned} z_{x_0}(t) &\equiv g(x_0) && \text{for } t \geq 0, \\ \gamma_{x_0}(t) &= tg(x_0) + x_0 && \text{for } t \geq 0. \end{aligned}$$

Hence the characteristic that starts in $x_0 (= (x_0, 0))$ is the straight line with slope $g(x_0)$. If g is an increasing function, then the corresponding characteristics will span out the space-time half plane $\mathbb{R} \times [0, \infty)$.

However, if g is not increasing, then characteristics will collide.

Example 3.15.

$$u_t + uu_x = 0 \text{ in } \mathbb{R} \times [0, \infty) \ \ \&\ \ u(x, 0) = g(x),\ g \in C^1(\mathbb{R})$$

with

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ \text{decreasing} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

For simplicity, we set $g(x) = 1 - x$ in $x \in (0, 1)$ (although g is then not C^1).

We can try to solve this equation with the characteristics method:

We observe

- Around $(\hat{x}, 0)$ with $\hat{x} \leq 0$, we have $\gamma_{x_0}(t) = t + x_0$ and the solution $u(x, t) \equiv 1$,
- Around $(\hat{x}, 0)$ with $\hat{x} \geq 1$, we have $\gamma_{x_0}(t) = x_0$ and the solution is $u(x, t) \equiv 0$,
- For $\hat{x} \in (0, 1)$ the following happens:

  If (x, t) satisfies $t < x < 1$, we solve $\gamma_{x_0}(t) = t(1 - x_0) + x_0 = x$: $x_0 = \frac{x-t}{1-t} \in (0, 1)$.

  Then, the slope of $\gamma_{x_0}(t)$ is $g(x_0) = (1 - x_0) = \frac{1-x}{1-t}$. That is also the value of $u(x, t)$.

3.12. **Distributional solutions of scalar conservations laws.** Let us consider a general

Definition 3.16 (Scalar Conversation Law)**.**

$$u_t + (f(u))_x = u_t + f'(u)u_x = 0 \text{ in } \mathbb{R}\times[0,\infty) \ \&\ u(x,0) = g(x) \tag{14}$$

where $g \in C^1(\mathbb{R})$ and $f \in C^2(\mathbb{R})$ with $f'' \geq 0$ (hence f is convex).

From the Burgers Equation we learned that C^1 solutions may be defined only up to some time t^*. Then a **Shock** has developed.

Shocks can have a physical meaning, so it is desirable to extend our concept of solution to include functions u that are **discontinuous and still satisfy the PDE in a generalized (or weak) sense.**

Definition 3.17. We say $u(x,t), (x,t) \in \mathbb{R}\times[0,\infty)$ is piecewise smooth if

- u is C^1 in all points (x,t) except along a C^1 curve $s(t), t \in (a,\infty)$,
- u is discontinuous in $s(t)$ for all $t \in (a,\infty)$.

In addition we assume that for every $t \in (a,\infty)$ the limits

$$u^+(s(t),t) := \lim_{x\downarrow s(t)} u(x,t), \quad u^-(s(t),t) := \lim_{x\uparrow s(t)} u(x,t) \text{ exist.}$$

Question 3.18. *What is a good notion of solution for the conservation law (14) in the class of piecewise smooth functions u?*

Should we call a piecewise smooth function u already a solution if u solves the conversation law in the classical sense in every (x,t) where u is a C^1 function?

- Then answer to the second question is NO!
- When we derived the conversations law, we assumed a priori that a solution would be C^1.
- But eventually the class of C^1 function is to small to capture all physical meaningful events.

Recall we had the integral equation

$$\int_{-\infty}^{\infty} 1_\Omega \left[u_t + (f(u))_x\right] dx = 0$$

for every connected domain with smooth boundary $\Omega \subset \mathbb{R}$.

This motivates the following definition.

Definition 3.19 (Distributional solutions)**.** We say a piecewise smooth function u is a solution of (14) in the distributional sense if

$$\int_0^\infty \int_{-\infty}^{\infty} \left[u\phi_t + f(u)\phi_x\right] dxdt = 0 \text{ for any } \phi \in C_c^\infty(\mathbb{R}\times(0,\infty)).$$

Using integration by parts and the fundamental theorem of calculus we see that for every $(x,t) \neq (s(t),t)$ the function u satisfies $u_t + (f(u))_x = 0$ classically.

Theorem 3.20 (Rankine-Hugoniot jump condition(s))**.** *Let $s(t), t \geq 0$ is a C^1 curve in $\mathbb{R}\times[0,\infty)$ (parametrized as graph). Assume u is piecewise smooth in the sense of the previous definition. Then u is a solution of (14) in the sense of distributions if and only if u is a classical solution in any point where u is C^1 and*

$$s'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} \circ s(t) \quad \text{for every } t \in (0,\infty).$$

Proof. Consider u that is piecwise smooth in the previous sense.

Let ϕ be in $C_c^1(\mathbb{R}\times(0,\infty))$ with compact support in $B_r((s(t),t))$.

We define

$$B^+ = \{(x,t) \in B_r((s(t_0),t_0)) : x \geq s(t)\} \ \&\ B^- = \{(x,t) \in B_r((s(t_0),t_0)) : x \leq s(t)\}$$

Assume u is a solution in distributional sense. Then

$$0=\int_0^\infty\int_{-\infty}^\infty [u\phi_t+f(u)\phi_x]\,dxdt=\int\int_{B^+}[u\phi_t+f(u)\phi_x]\,dxdt+\int\int_{B^-}[u\phi_t+f(u)\phi_x]\,dxdt$$

Now, here comes a trick. The identity

$$u\phi_t+f(u)\phi_x=u\phi_t+f(u)\phi_x+u_t\phi+(f(u))_x\phi$$

holds on $\{(x,t): x>s(t)\}$ and $\{(x,t): x<s(t)\}$. The right hand side in the previous identiy becomes

$$u\phi_t+f(u)\phi_x+u_t\phi+(f(u))_x\phi=(u\phi)_t+(f(u)\phi)_x=\nabla\cdot(f(u)\phi,u\phi)$$

Inserting this back into the integral identy yields

$$\int\int_{B^+}[u\phi_t+f(u)\phi_x]\,dxdt=\int\int_{B^+}\nabla\cdot(f(u)\phi,u\phi)dxdt$$

and the same for the integral over B^-. By the divergenc theorem (for domains with corners)

$$\int\int_{B^+}\nabla\cdot(f(u)\phi,u\phi)dxdt=\int_{\partial B^+}N\cdot(f(u^+)\phi,u^+\phi)dS=\int_{\{(s(t),t):t>0\}}\phi\left(N\cdot(f(u^+),u^+)\right)dS$$

$$\int\int_{B^+}\nabla\cdot(f(u^+)\phi,u^+\phi)dxdt=\int_0^\infty\phi(t)(-s'(t),1)\cdot(f(u^+),u^+)\circ(s(t),t)\sqrt{1+|s'(t)|^2}dt.$$

For the integral that involves B^- this is

$$\int\int_{B^-}\nabla\cdot(f(u^+)\phi,u^+\phi)dxdt=\int_0^\infty\phi(t)(s'(t),-1)\cdot(f(u^+),u^+)\circ(s(t),t)\sqrt{1+|s'(t)|^2}dt.$$

It follows that

$$0=\int_0^\infty\phi(t)\Big[(-s'(t),1)\cdot(f(u^+),u^+)\circ(s(t),t)$$
$$+(s'(t),-1)\cdot(f(u^-),u^-)\circ(s(t),t)\Big]\sqrt{1+|s'(t)|^2}dt.$$

We conclude that

$$0=-s'(t)u^++f(u^+)s'(t)u^--f(u^-)\ \rightarrow\ s'(t)=\frac{f(u^+)-f(u^-)}{u^+-u^-}\circ(s(t),t).$$

This is the jump condition. Now assuming u is a solution when it is C^1 together with jump condition, we can reverse this chain of implications and obtain u is solution in distributional sense. $\square$

3.13. **Non-uniquness of distributional solutions, Lax entropy condition.** Since we know that for certain initial conditions, shocks always develop, and since we have the concept of distributional solution at hand, we consider the following PDE problem for the Burgers equation.

$$u_t+uu_x=0\text{ in }\mathbb{R}\times[0,\infty)\ \ \&\ \ u(x,0)=g(x)=\begin{cases}1 & \text{if } x\le 0,\\ 0 & \text{if } x\ge 0.\end{cases}$$

Let us apply the previous theorem. We want find a C^1 curve $s(t),\ t\ge 0$ that satisfies the jump condition. For the burgers equation we have $f(x)=\frac{1}{2}x^2$. Then the jump condition is

$$\frac{(u^+)^2-(u^-)^2}{2(u^+-u^-)}=\frac{1}{2}(u^+-u^-)=s'(t).$$

Therefore, distributional solutions of the previous PDE with initial condition are

$$u(x,t)=\begin{cases}1 & \text{for } x\le\frac{1}{2}t,\\ 0 & \text{for } x>\frac{1}{2}\text{t}.\end{cases}$$

and

$$v(x,t) = \begin{cases} 0 & \text{for } x \leq \frac{1}{2}t, \\ 1 & \text{for } x > \frac{1}{2}t. \end{cases}$$

We want to choose the solution that is physical meaninful. Which one is it?

Lecture 07.

We consider the Burgers' equation with **discontinuous** initial value:

$$u_t + uu_x = 0 \text{ in } \mathbb{R} \times [0, \infty) \;\; \& \;\; u(x, 0) = g(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Let us apply the previous theorem. We want find a C^1 curve $s(t),\ t \geq 0$ that satisfies the jump condition. For the burgers equation we have $f(x) = \frac{1}{2}x^2$. Then the jump condition is

$$\frac{(u^+)^2 - (u^-)^2}{2(u^+ - u^-)} \circ s(t) = \frac{1}{2}(u^+ + u^-) \circ s(t) = s'(t).$$

Therefore, a distributional solutions of the previous PDE with this initial condition is

$$u(x, t) = \begin{cases} 1 & \text{for } x \leq \frac{1}{2}t, \\ 0 & \text{for } x > \frac{1}{2}t. \end{cases}$$

On the other hand, consider

$$u_t + uu_x = 0 \text{ in } \mathbb{R} \times [0, \infty) \;\; \& \;\; u(x, 0) = g(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

A distributional solution is

$$v(x, t) = \begin{cases} 0 & \text{for } x \leq \frac{1}{2}t, \\ 1 & \text{for } x > \frac{1}{2}t \end{cases}$$

but also

$$w(x, t) = \begin{cases} 0 & \text{for } x \leq 0, \\ \frac{x}{t} & \text{for } 0 < x < \frac{1}{2}t, \\ 1 & \text{for } x \geq \frac{1}{2}t \end{cases}$$

is a solution that is even continuous. This solution is called the **rarefaction wave**.

For this problem there is no uniqueness.

Question 3.21. *Which solution should we pick? Which solution is physically meaninful?*

For the solution v characteristics emanate from the shock.

This is physically unreasonable.

Recall the characteristcs equation

$$\frac{dx}{dt} = f'(u).$$

A sufficient condition such that characteristics do not emanate from the shock is

$$f'(u^+) \geq s' \geq f'(u^-). \tag{15}$$

Since f is convex, 15 implies that $u^+ \geq u^-$.

Definition 3.22. Lax entropy a condition We say a piecewise smooth solution $u(x, t)$ to a conservation law is an entropy solution if the **Lax entropy condition** (15) holds.

Note that a smooth solution is an entropy solution since there is no curve s that describes a discontinuity.

Theorem 3.23. *If an entropy solution exists, then it is the unique distributional solution for the scalar conversation law.*

4. Linear homogeneous second order PDEs

4.1. **Classification of linear second order PDEs.** Consider linear second order PDE for n independent variables has the form

$$\sum_{i,j=1}^{n} a_{i,j} u_{x_i,x_j} + \sum_{k=1}^{n} b_k u_{x_k} + cu = d \text{ on } \Omega. \tag{16}$$

We assume $a_{i,j}, b_k, c, d \in C^0(\Omega)$ and $a_{i,j} = a_{j,i}$. Hence

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \dots & & \dots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \text{ is a symmetric matrix.}$$

Recall form linear algebra that there exists a symmetric matrix B such that

$$BAB^{\top} = \begin{pmatrix} d_1 & 0 & \dots & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots\dots & \dots & \\ 0 & \dots & \dots & d_{n-1} & 0 \\ 0 & \dots & \dots & 0 & d_n \end{pmatrix} =: D$$

with $d_1, \dots, d_n \in C^0(\Omega)$. $\{d_1(\mathbf{x}), \dots, d_n(\mathbf{x})\}$ are the eigenvalues of $A(\mathbf{x})$.

Definition 4.1. The PDE (16) is called

1. **Elliptic** if all the eigenvalues $d_1, \dots, d_n$ are positive. That is equivalent to say that A is positive definite,
2. **Parabolic** if exactly one eigenvalue is 0 and the other eigenvalues have the same sign,
3. **Hyperbolic** if exactly one eigenvalue is negative and the other eigenvalues are positive,
4. **Ultrahyperbolic** if there are more thatn one negative eigenvalues and the other eigenvalues are positive.

Consider linear second order PDE for 2 independent variables has the form

$$a_{1,1} u_{x_1,x_1} + 2a_{1,2} u_{x_1,x_2} + a_{2,2} u_{x_2,x_2} + b_1 u_{x_1} + b_2 u_{x_2} + cu = d. \tag{17}$$

Definition 4.2. The PDE (17) is

1. Elliptic $\iff$ $a_{1,1}a_{2,2} - a_{1,2}^2 > 0$,
2. Parabolic $\iff$ $a_{1,1}a_{2,2} - a_{1,2}^2 = 0$,
3. Hyperbolic $\iff$ $a_{1,1}a_{2,2} - a_{1,2}^2 < 0$.

Let B the $n \times n$ matrix such that

$$BAB^{\top} = D$$

We can introduce new coordinates $(y_1, \dots, y_n) = \mathbf{y}$ via $B\mathbf{x} = \mathbf{y}$.

Lemma 4.3. *The PDE (16) writes w.r.t. the coordinates* $\mathbf{y}$ *as*

$$\sum_{i=1}^{n} d_i u_{x_i,x_i} + \sum_{k=1} b_k u_{x_k} + cu = d.$$

After rescaling with $\frac{1}{\sqrt{|d_i|}}$ *in* y_i *for every* $i = 1, \dots, n$ *as long as* $d_i \neq 0$ *this becomes*

$$\Delta u + \nabla u \cdot (b_1, \dots, b_k) + cu = d.$$

Proof. We compute

$$u_{x_i}(\mathbf{x}) = \frac{\partial u}{\partial x_i}\Big|_{\mathbf{x}} = \frac{\partial(u \circ B^{-1} \circ B)}{\partial x_i}\Big|_{\mathbf{x}} = \nabla_y(u \circ B^{-1})\big|_{B\mathbf{x}} \cdot (B_{1,i}, \ldots, B_{n,i}) = \sum_{k=1}^{n} \frac{\partial u \circ B^{-1}}{\partial y_k}\Big|_{B\mathbf{x}} B_{k,i}$$

We set $u(\mathbf{y}) := u \circ B^{-1}\mathbf{y}$ and $u_{y_i} := \frac{\partial(u \circ B^{-1})}{\partial y_k}$. Therefore

$$u_{x_j,x_i} = \sum_{k,l=1}^{n} u_{y_k,y_l} B_{k,i} B_{l,j} \implies \sum_{i,j=1}^{n} A_{i,j} u_{x_i,x_j} = \sum_{k,l=1}^{n} \underbrace{\sum_{i,j=1}^{n} B_{k,i} A_{i,j} B^{\top}_{j,l}}_{d_k \delta_{k,l}} u_{y_k,y_l} = \sum_{k=1}^{n} d_k u_{y_k,y_l}.$$

□

Consider linear second order PDE for 2 independent variables has the form

$$a_{1,1}u_{x_1,x_1} + 2a_{1,2}u_{x_1,x_2} + a_{2,2}u_{x_2,x_2} + b_1 u_{x_1} + b_2 u_{x_2} + cu = d \text{ on } \mathbb{R}. \tag{18}$$

Example 4.4. Consider the PDE (18) for 2 independent variables. Let $d = c = b_2 = 0$. Applying the transformations of the previous lemma yields

(1) **Elliptic**: $u_{x_1,x_1} + u_{x_2,x_2} + b_1 u_{x_1} = 0$. If $b_1 = 0$, we have the **Laplace equation**:

$$u_{x_1,x_1} + u_{x_2,x_2} = 0.$$

(2) **Parabolic**: Assume $d_2 = 0$ and set $x_1 = x$ and $x_2 = t$. Then $u_{x,x} + b_1 u_1$. If $b_1 = 1$ we have the **diffusion equation**:

$$u_{x,x} + u_t = 0.$$

(3) **Hyperbolic**: Assume $d_2 < 0$. Then $u_{x,x} - u_{t,t} + b_1 u_t$. If $b_1 = 0$ we have the **wave equation**:

$$u_{x,x} - u_{t,t} = 0.$$

Lecture 08.

4.2. **Deriving the wave equation in** $1D$**.** Consider a flexible, elastic homogenous string or thread of length l, that undergoes relatively small transverse vibrations.

We think of the string as the graph of function $u(x,t)$ on $[0,l]$ that depends also on $t \in [0,\infty)$.

Let $T(x,t)$ be the magnitude of the tension force that pulls in $(x, u(x,t))$ along the string at time t. And let us assume T does not depend on t. Moreover there are no other forces.

Because the string is perfect flexible the tension force is directed tangential to the string.

And let $\rho(x)$ be the mass density of the string as distribution on $[0,l]$. Since the string is homogeneous, we assume $\rho(x) \equiv \textit{constant}$.

We consider an interval $[x_0,x_1] \subset [0,l]$ that gives a section $\{(x,u(x)) : x \in [x_0,x_1]\}$ of the string.

We apply Newton's law: the Force F is given by mass times accelaration.

This yields the following two equations

$$\frac{T(x_1)}{\sqrt{1+u_x(x_1)^2}} - \frac{T(x_0)}{\sqrt{1+u_x(x_0)^2}} = 0 \quad \text{for horizontal forces}$$

and

$$\frac{T(x_1)u_x(x_1)}{\sqrt{1+u_x(x_1)^2}} - \frac{T(x_0)u_x(x_0)}{\sqrt{1+u_x(x_0)^2}} = \int_{x_0}^{x_1} \rho u_{t,t}(x)dx \quad \text{for vertical forces.}$$

We assume the magnitude of the motion is small compared to 1. By that we mean that the slope $u_x(x,t)$ of $u(x,t)$ w.r.t. x at time t is small compared to 1.

If Taylor expand the $x \mapsto \sqrt{x}$ around 1 we get

$$\sqrt{1+u_x^2} = (1+u_x^2)^{\frac{1}{2}} = \sum_{i=0}^{\infty} \binom{1/2}{i} (u_x)^i = 1 + \frac{1}{2}u_x^2 + \dots .$$

(Binomial series) where

$$\binom{\alpha}{i} = \frac{\alpha \cdot (\alpha-1) \cdot \ldots \cdot (\alpha - i + 1)}{1 \cdot 2 \cdot \dots \cdot i} \text{ for } i \in \mathbb{N} \text{ and } \alpha \in \mathbb{R}, \alpha \geq 0.$$

Hence, we make the assumption that Newton's laws for the string reduces to

$$T(x_1) - T(x_0) = 0 \ \text{ for horizontal forces.}$$

and

$$T(x_1)u_x(x_1) - T(x_0)u_x(x_0) = \int_{x_0}^{x_1} \rho u_{t,t}(x)dx \ \text{ for vertical forces.}$$

The first equation says that $T(x) \equiv T$ is constant along $[0,l]$.

In the second equation we can apply the fundamental theorem of calculus. Hence

$$T\int_{x_0}^{x_1} u_{x,x}(x,t)dx = \int_{x_0}^{x_1} \rho u_{t,t}(x,t)dx.$$

Since this euqation holds for every $x_0 < x_1$ with x_0 and x_1 close to each other, it follows $Tu_{x,x} = \rho u_{t,t}$. Now, let us also assume the mass ditribution $\rho(x)$ along the string is constant and set $c = \sqrt{\frac{T}{\rho}}$.

Definition 4.5 (Wave equation in 1D)**.**

$$u_{t,t} = c^2 u_{x,x} \text{ on } \mathbb{R} \times [0,\infty)$$

for $c \neq 0$.

Remark 4.6 (Modifications).
(1) If there is an air resistence r present, one has an extra term proportional to the speed u_t:

$$u_{x,x} - c^2 u_{t,t} + r u_t = 0 \text{ where } r > 0.$$

(2) If there is transversal elastic force, we have an extra term proportional to the magnitude of the displacement u:

$$u_{x,x} - c^2 u_{t,t} + k u = 0 \text{ where } k > 0.$$

(3) If there is an external force, an extra term f independent of u appears:

$$u_{x,x} - c^2 u_{t,t} + f(x,t) = 0 \text{ where } f(x,t) \text{ is a time dependent function.}$$

4.3. **General solution of the wave equation in** $1D$**.** The wave equation in 1D factors nicely in the following way:

$$0 = u_{t,t} - c^2 u_{x,x} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u.$$

We recognize that this yields a coupled system of two first order equations:

$$u_t + c u_x = v$$
$$v_t - c v_x = 0.$$

This idea allows us to prove the following

Theorem 4.7. *The general C^2 solution of the wave equation $u_{t,t} - c^2 u_{x,x} = 0$ on $\mathbb{R}$ is of the form*

$$u(x,t) = f(x+ct) + g(x-ct)$$

for arbitrary functions $f, g \in C^2(\mathbb{R})$.

Proof. We first consider

$$v_t - c v_x = 0 \text{ on } \mathbb{R} \times [0,\infty). \tag{19}$$

We know the general solution of (19):

$$v(x,t) = h(x+ct)$$

for $h \in C^1(\mathbb{R})$ arbitrary where v satisfies $v(x,0) = h(x)$. Then we can consider

$$u_t + c u_x = h(x+ct) \text{ on } \mathbb{R} \times [0,\infty) \text{ with } u(x,0) = \tilde{g}(x),\ g \in C^2(\mathbb{R}). \tag{20}$$

Lets solve this. The characterisitcs equations are

$$\frac{d}{dt} x_{x_0}(t) = c \text{ with } x(0) = x_0 \ \ \& \ \ \frac{d}{dt} z_{x_0}(t) = h(x_{x_0}(t) + ct) \text{ with } z_{x_0}(0) = \tilde{g}(x_0).$$

It follows that $x_{x_0}(t) = ct + x_0$. Hence for $x \in \mathbb{R}$ and $t > 0$ we set $x_0 = x - ct$.
Moreover

$$z_{x_0}(t) = \int_0^t h(x_{x_0}(s) + cs) ds + \tilde{g}(x_0) = \int_0^t h(cs + x_0 + cs) ds + \tilde{g}(x_0).$$

Then the solution u in (x,t) is given by $u(x,t) = \displaystyle\int_0^t h(cs + x_0 + cs) ds + \tilde{g}(x_0)$.

Applying the substituion ruel $\int_a^b f \circ \phi(s) \phi'(s) ds = \int_{\phi(a)}^{\phi(b)} f(s) ds$ with $\phi(s) = x_0 + 2cs$ gives

$$u(x,t) = \int_{x_0}^{x_0+2ct} \frac{1}{2c} h(\tau) d\tau + \tilde{g}(x-ct) = \int_{x-ct}^{x+ct} \frac{1}{2c} h(\tau) d\tau + \tilde{g}(x-ct).$$

Then the claim follows with $f(s) := \int_0^s \frac{1}{2c} h(\tau)\tau$ and $g(s) = \int_s^0 \frac{1}{2c} h(\tau) d\tau + \tilde{g}(s)$ where $f, g \in C^2(\mathbb{R})$. □

An alternative proof (without characteristics). We can check that $f(x+ct)$ for

$$f(s) = \int_0^s \frac{1}{2c} h(\tau) d\tau$$

solves the equation $u_t + cu_x = h(x+ct)$. Indeed

$$\frac{\partial}{\partial t} f(x+ct) = f'(x+ct)c = \frac{1}{2} h(x+ct), \quad \frac{\partial}{\partial x} f(x+ct) = f'(x+ct) = \frac{1}{2c} h(x+ct).$$

Therefore

$$\frac{\partial}{\partial t} f(x+ct) + c\frac{\partial}{\partial x} f(x+ct) = h(x+ct).$$

On the other hand $g(x-ct)$ for $g \in C^2(\mathbb{R})$ solves the homogeneous equation $u_t + cu_x = 0$.

But we learned before that the sum of a solution of the homogeneous equation and of a solution of the inhomogeneous equation, still solves the inhomogeneous equation $u_t + cu_x = h(x+ct)$.

Therefore, $f(x+ct) + g(x-ct)$ also solves the wave equation.

Remark 4.8. It seems we found two different expression for g (depending on the proof), but for the first expression we fixed a initial condition $\tilde{g}$ and found g depending on $\tilde{g}$.

The Initial Value Problem for the wave equation in $1D$. Now we consider

$$\begin{cases} u_{t,t} - c^2 u_{x,x} = 0 & \text{on } \mathbb{R} \times [0,\infty) \\ u(x,0) = \phi(x) \ \ \& \ \ u_t(x,0) = \phi(x) & \phi \in C^2(\mathbb{R}),\ \psi \in C^1(\mathbb{R}). \end{cases} \tag{21}$$

Theorem 4.9 (D'Alembert's formula)**.** *The unique solution of the initial value problem* (21) *is given by*

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Proof. From the formula for the general solution, we get

$$\phi(x) = f(x) + g(x) \ \ \& \ \ \frac{1}{c}\psi(x) = f'(x) - g'(x).$$

Differentiating ϕ yields $\phi' = f' + g'$. Adding and substracting these identities yields

$$f'(x) = \frac{1}{2}\left(\phi'(x) + \frac{1}{c}\psi(x)\right), \ \ \& \ \ g'(x) = \frac{1}{2}\left(\phi'(x) - \frac{1}{c}\psi(x)\right).$$

Integrating from

$$f(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int_0^x \psi(s)ds\right) + A_1 \ \& \ g(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int_0^x \psi(s)ds\right) + A_2.$$

Since $\phi(x) = f(x) + g(x)$ we have $A_1 + A_2 = 0$.

Now, we can write

$$\begin{aligned} f(x+ct) + g(x-ct) =& \frac{1}{2}\left(\phi(x+ct) + \frac{1}{c}\int_0^{x+ct}\psi(s)ds\right) + \frac{1}{2}\left(\phi(x-ct) - \frac{1}{c}\int_0^{x-ct}\psi(s)ds\right) \\ &+ A_1 + A_2 \\ =& \frac{1}{2}\left(\phi(x+ct) + \frac{1}{c}\int_0^{x+ct}\psi(s)ds\right) + \frac{1}{2}\left(\phi(x-ct) + \frac{1}{c}\int_{x-ct}^{0}\psi(s)ds\right) \\ =& \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds \end{aligned}$$

This is what was to prove. □

- **Semigroup property**: Define

$$W(t)(\phi, \psi) := (u, u_t)$$

where u is the unique solution of the wave equation

$$u_{t,t} = c^2 u_{xx} \text{ in } \mathbb{R} \times [0, \infty) \text{ with } \phi(x) = u(x, 0),\ \psi(x) = u_t(x, 0) \tag{22}$$

given by d'Alembert's formula.

Corollary 4.10. *The following semi-group property holds*

$$W(s + \tau)(\phi, \psi) = W(s)(W(t)(\phi, \psi))$$

Proof. Let $P_1(x, y) = x$ be the projection map.

Then $s \geq 0 \mapsto v(\cdot, s) := P_1 \circ W(s + \tau)(\phi, \psi)$ is a solution of the wave equation with initial conditions

$$v(x, 0) = u(x, \tau) \ \&\ v_t(x, 0) = u_t(x, \tau)$$

By d'Alembert's formula we have $v(x, s) = W(s)(u(\cdot, \tau), u_t(\cdot, \tau)) = W(s)(W(\tau)(\phi, \psi))$. □

- **Causality**: For a point $(x, t) \in \mathbb{R} \times (0, \infty)$ the solution u given by d'Alembert's formula

$$u(x, t) = \frac{1}{2}\left[\phi(x - ct) + \phi(x + ct)\right] + \frac{1}{2}\int_{x-ct}^{x+ct} \psi(s) d$$

depends only on the values of ψ on $[x - ct, x + ct]$ and the values of ϕ in $x - ct$ and $x + ct$.

Moreover, for $s \in (0, t)$ we have by the semi-group property

$$W(t)(\phi, \psi) = W(t - s)(W(s)(\phi, \psi)).$$

So $u(x, t)$ also only depends on the values of $W(s)(\phi, \psi) = (u(\cdot, s), u_\tau(\cdot, s))$ on $[x - c(t - s), x + c(t - s)]$.

Hence, the domain of dependence is space-time triangle in $\mathbb{R} \times [0, \infty)$.

Similar, the domain of influence for $(x, t) \in \mathbb{R} \times [0, \infty)$ is a space time triangle in $\mathbb{R} \times [0, \infty)$.

lecture9

Example 4.11 (Plucked String). Consider the initial value problem

$$u_{t,t} = c^2 u_{x,x} \text{ in } \mathbb{R} \times [0,\infty) \;\;\&\;\; u(x,0) = \phi(x) = \begin{cases} b - \frac{b|x|}{a} & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a. \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right].$$

We note that the initial condition is not C^2-differentiable. Hence also the formula doesn't give us a C^2 solution. Nevertheless, this u is still a solution in a "weak" sense, similar like a distributional solution for conservation laws.

For $t = \frac{a}{2c}$ the solution has the form

$$u(x,t) = \begin{cases} 0 & \text{if } |x| \geq \frac{3a}{2}, \\ \frac{1}{2}\phi(x+\frac{1}{2}a) & \text{if } x \in (-\frac{3a}{2}, -\frac{a}{2}), \\ \frac{1}{2}\left[\phi(x+\frac{1}{2}a) + \phi(x-\frac{1}{2}a)\right] & \text{if } |x| \leq \frac{a}{2} \\ \frac{1}{2}\phi(x-\frac{1}{2}a) & \text{if } x \in (\frac{a}{2}, \frac{3a}{2}). \end{cases}$$

where for instance

$$\frac{1}{2}\phi(x+\frac{1}{2}a) = \frac{1}{2}\left[b - \frac{b}{a}|x+\frac{1}{2}a|\right] = \frac{1}{2}\left[b + \frac{b}{a}x + \frac{1}{2}b\right] = \frac{3}{4}b + \frac{b}{2a}x \text{ for } x \in (-\frac{3a}{2}, -\frac{a}{2}).$$

and

$$\frac{1}{2}\phi(x-\frac{1}{2}a) = \frac{1}{2}\left[b - \frac{b}{a}|x-\frac{1}{2}a|\right] = \frac{1}{2}\left[b - \frac{b}{a}x + \frac{1}{2}a\right] = \frac{3}{4}b - \frac{b}{a}x \text{ for } x \in (\frac{a}{2}, \frac{3a}{2}).$$

and

$$\begin{aligned} \frac{1}{2}\phi(x+\frac{1}{2}a) + \frac{1}{2}\phi(x-\frac{1}{2}a) &= \frac{1}{2}\left[b - \frac{b}{a}|x+\frac{1}{2}a|\right] + \frac{1}{2}\left[b - \frac{b}{a}|x-\frac{1}{2}a|\right] \\ &= \frac{1}{2}\left[b - \frac{b}{a}x + \frac{1}{2}b\right] + \frac{1}{2}\left[b + \frac{b}{a}x - \frac{1}{2}b\right] = b \end{aligned}$$

For $t = \frac{3a}{c}$ the solution has the form

$$u(x,t) = \begin{cases} 0 & \text{if } |x| \geq 4a, \\ \frac{1}{2}\phi(x+\frac{1}{2}a) & \text{if } x \in (-4a, -2a), \\ 0 & \text{if } |x| \leq 2a \\ \frac{1}{2}\phi(x-\frac{1}{2}a) & \text{if } x \in (2a, 4a). \end{cases}$$

4.4. **Preservation of Energy.** Imagine an infinite string with uniform mass distribution ρ, uniform tension force T given by the graph of $u(x,t)$ for $x \in \mathbb{R}$. The string behaves according to the wave equation

$$\rho u_{t,t} = T u_{x,x}$$

with initial conditions $\phi(x) = u(x,0)$ and $\psi(x) = u_t(x,0)$ on $\mathbb{R}$. We set again $c = \sqrt{\frac{T}{\rho}}$.

The kinetic energy is defined as

$$KE = \frac{1}{2}\int_{-\infty}^{\infty} \rho \cdot (u_t)^2(x)dx$$

This integral and the following ones are evaluated from $-\infty$ to $+\infty$.

To be sure that the integral converge we assume that $\phi(x) = u(x,0)$ & $\psi(x) = u_t(x,0)$ vanish outside of $[-R,R]$ for some $R > 0$.

Then $u_t(x)$ vanishes outside of $[-R-ct, R+ct]$ and

$$\frac{1}{2}\int_{-\infty}^{\infty}\rho(u_t)^2dx = \frac{1}{2}\int_{-R-ct}^{R+ct}\rho(u_t)^2dx.$$

We differentiate the kinetic energy in t:

$$\frac{dKE}{dt} = \frac{1}{2}\rho\int_{-\infty}^{\infty}\frac{d}{dt}(u_t)^2dx = \rho\int_{-\infty}^{\infty}u_tu_{t,t}dx.$$

Here we apply that we can differentiate under the integral. At this point we use that u satisfies the wave equation $u_{t,t} = c^2u_{x,x}$:

$$\frac{dKE}{dt} = T\int_{-\infty}^{\infty}u_tu_{x,x}dx = Tu_tu_x\Big|_{-\infty}^{\infty} - T\int_{-\infty}^{\infty}u_{t,x}u_xdx.$$

The first term on the right hand side vanishes since u_t vanishes outside $[-R-ct, R+ct]$.

In the second term we can write $u_{t,x}u_x = \frac{1}{2}\left((u_x)^2\right)_t$. Hence

$$\frac{dKE}{dt} = -\frac{d}{dt}\int\frac{1}{2}T(u_x)^2dx.$$

We call $\frac{1}{2}\int_{-\infty}^{\infty}T(u_x)^2dx =: PE$ the *potential energy.*
We see that

$$\frac{d}{dt}[KE+PE] = 0$$

and therefore

$$KE+PE = \frac{1}{2}\int_{-\infty}^{\infty}\left(\rho(u_t)^2 + T(u_x)^2\right)dx =: E$$

is constant. This is the law of **preservation of enery**.

Application: We can use the Preservation of Enery to show uniquness of solutions of the wave equation.

Let u^1, u^2 be two solutions of

$$\rho u_{t,t} = Tu_{x,x} \quad (\Leftrightarrow \rho u_{t,t} - Tu_{x,x} = 0) \tag{23}$$

with the same initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

The wave equation (23) is a linear, homogeneous PDE.

Hence, the difference $v = u^1 - u^2$ is also solution with initial condition $u(x,0) = 0$ and $u_t(x,0)$. The task is to show that $v(x) \equiv 0$, then clearly we have $u^1 = u^2$.

By Preservation of Energy it follows

$$0 = \frac{1}{2}\int_{-\infty}^{\infty}\left(\rho(v_t(x,0))^2 + T(v_x(x,0))^2\right)dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(\rho(v_t(x,t))^2 + T(v_x(x,t))^2\right)dx \geq 0.$$

It follows

$$0 = \frac{1}{2}\int_{-\infty}^{\infty}\left(\rho(v_t(x,t))^2 + T(v_x(x,t))^2\right)dx.$$

This is only possible if $v_t \equiv 0$ and $v_x \equiv 0$ ($\Leftrightarrow \nabla v = 0$).

It follows that $v = const$ on $\mathbb{R}\times[0,\infty)$. Since $v(t,0) = 0$, it follows $v(x) \equiv 0$.

4.5. **The diffusion equation: Maximum principle and consequences.** Recall the Diffusion Equation

(24) $$u_t = ku_{x,x}.$$

First we study this equation without considering initial or boundary conditions.

Though similar to the wave equation, its mathematical properties are completely different.

The constant $k > 0$ is called the diffusion constant or volatility.

To solve this equation is harder than to solve the wave equation. Therefore we start by assuming we have a solution and studying its properties.

Theorem 4.12 (Maximum Principle). *Let $u(x,t)$ be a C^2 solution of (24) on $[0,l] \times [0,T] \subset \mathbb{R} \times [0,\infty)$. Then*

(1) *The maximum of $u(x,t)$ is assumed either on $[0,l] \times \{0\}$ or on $\{0\} \times [0,T] \cup \{l\} \times [0,T]$:*
$$\max_{(x,t)\in[0,l]\times[0,T]} u(x,t) = \max_{(x,t)\in[0,l]\times\{0\}\cup\{0\}\times[0,T]\cup\{l\}\times[0,T]} u(x,t).$$
(Weak Maximum Principle)

(2) *If there exists $(x_0,t_0) \in (0,l) \times (0,T)$ such that*
$$\max_{(x,t)\in[0,l]\times[0,T]} u(x,t) = u(x_0,t_0) =: M.$$
Then $u(x,t) \equiv M$ on $[0,l] \times [0,T]$. (Strong Maximum Principle)

Theorem 4.13 (Maximum Principle, short). *Let $M := \max_{(x,t)\in\partial([0,l]\times[0,T])\setminus[0,l]\times\{T\}} u(x,t)$.*

(1) *$u \le M$ on $(0,l) \times (0,T)$.*
(2) *If there exists $(x_0,t_0) \in (0,l) \times (0,T)$ then $u \equiv M$ on $[0,l] \times [0,T]$.*

In this formulation we can see the weak maximum principle as a geometric inequality, and the strong maximum principle as the characterization of the equality case.

Physical Interpretation: Imagine a rod of length $l > 0$ with no internal heat source. Then then the weak maximum principle tells us that the hottest and the coldest spot can only occur at th initial time $t = 0$ or at one of the two ends of the rod.

On the other hand, by the strong maximum principle if the coldest or the hottest spot occur inside of the rod away from the ends at some positive time $t > 0$, then, the temperature distribution must be constant along the rod.

Proof of the weak maximum principle. Here, we will only prove the weak maximum principle.
The proof of the strong maximum principle is much more difficult and requires tools that currently are not at our disposal.

Idea for the proof of the weak maximum principle:
If there exists $(x_0,t_0) \in (0,l) \times (0,T)$ such that $u(x_0,t_0) = \max_{[0,l]\times[0,T]} u$ then $u_t = u_x = 0$ and $u_{x,x} \le 0$. If we would even know that $u_{x,x} < 0$, this would contradict the heat equation. For a rigorous proof we need to work a little bit more.

The trick is to consider $v(x,t) = u(x,t) + \epsilon\frac{1}{2}x^2$ for some $\epsilon > 0$. Then
$$v_{x,x} = u_{x,x} + \epsilon = ku_t + \epsilon = kv_t + \epsilon$$
Hence, the Partial Differential Inequality

(25) $$v_{x,x} > kv_t.$$

Let $M = \max_{(x,t)\in\partial([0,l]\times[0,T])\setminus[0,l]\times\{T\}} u(x,t)$. Then it is clear that
$$v(x,t) \le M + \epsilon l^2 \quad \text{on} \quad \partial([0,l] \times [0,T])\setminus[0,l] \times \{T\}$$

Now suppose that there is $(x_0, t_0) \in (0, l) \times (0, T]$ such that $u(x_0, t_0) > M$. Then

$$v(x_0, t_0) = u(x_0, t_0) + \epsilon \frac{1}{2} x_0^2 > M.$$

In particular, the maximum of v is occurs in a point $(x_1, t_1) \in (0, l) \times (0, T]$ and $v(x_1, t_1) > M$.

If $(x_1, t_1) \in (0, l) \times (0, T)$ then $v_t = 0$ and $v_{x,x} \geq 0$ ($\Rightarrow kv_{x,x} \geq 0$) what contradicts (25). If $(x_1, t_1) \in (0, l) \times \{T\}$, then we still have $v_{x,x}(x_1, T) \geq 0$.

However $u_t(x_1, T) = 0$ does eventually not hold.

But we now that $v(x_1, T - \delta) \leq v(x_1, T)$. Hence

$$u_t(x_1, T) = \lim_{\delta \downarrow 0} \frac{v(x_1, T - \delta) - v(x_1, T)}{-\delta} \geq 0.$$

So we have

$$v_t(x_1, T) \geq v_{x,x}(x_1, T)$$

what again contradicts (25): $v_{t,t} \leq k u_{x,x}$. □

Application: Uniqueness of solutions to the diffusion equation Dirichlet Problem for the diffusion equation We consider the diffusion equation

$$u_t = k u_{x,x} \quad \text{on } [0, l] \times [0, T]$$

with the initial condition

$$u(x, 0) = \phi(x) \quad \text{for } x \in [0, l] \quad \text{and } \phi \in C^2([0, l])$$

and boundary values

$$u(0, t) = g(t) \ \& \ u(l, t) = h(t) \quad \text{for } t \in [0, T] \quad \text{and } h, g \in C^2([0, T]).$$

Corollary 4.14. *There exists at most one C^2 solution to the Dirichlet problem for the diffusion equation.*

Proof. The first step of the proof is similar to what we saw for the wave equation.

Assume u^1, u^2 are two solutions to the Dirichlet problem with ϕ, g, h given as above. Then, by linearity of the equation $w = u^1 - u^2$ is a solution of the Dirichlet problem with $\phi = g = h \equiv 0$.

We need to show that $w = 0$.

By the weak maximum principle $w \leq 0$. Since also $-w$ solves the Dirichlet problem with $\phi, g, h \equiv 0$ we also have $-w \leq 0$. Hence $w = 0$. □

Alternative Proof of uniqueness via energy method: Let us investigate an alternative method to prove uniqueness of solution of the diffusion equation. This method is similar to the strategy that we applied for the corresponding statement for solutions of the wave equation.

Let u be a solution to the previous Dirichlet Problem with $g = h \equiv 0$ and consider

$$E(t) = \int_0^l \frac{1}{2} (u(x, t))^2 dx.$$

Proposition 4.15. *The qantity $E(t)$ is positive and monotone decreasing in $t \in [0, \infty)$.*

Proof. Let us compute the derivative in t.

$$\frac{dE}{dt} = \int_0^l u_t(x, t) u(x, t) dx = \int_0^l u_{x,x}(x, t) u(x, t) dx.$$

By integration by parts the right hand side becomes

$$u_x(l, t) u(l, t) - u_x(0, t) u(0, t) - \int_0^l (u_x(x, t))^2 dx = - \int_0^l (u_x(x, t))^2 dx \leq 0.$$

□

Now, if u^1, u^2 are two solution for the Dirichlet problem with $\phi \in C^2([0,l])$ and $g, h \in C^2([0,T])$, then $u = u^1 - u^1$ is a solution of the Dirichlet problem with $\phi, g, h \equiv 0$.

Since $E(0) = 0$, the previous proposition implies $E(t) = 0$ for all $t \geq 0$ and therefore $u(x,t) \equiv 0$.

4.6. **Stability.** Stability was the third property for well-posedness of a PDE: "Small changes of the date imply only small changes for the corresponding solutions".

The energy method shows the following: If u^1 and u^2 are solutions of the Dirichlet problem with the same g, h and but for ϕ^1 and ϕ^2 that are eventually different, then

$$\int \frac{1}{2}(u^1(x,t) - u^2(x,t))^2 dx \leq \int_0^l \frac{1}{2}(\phi^1(x) - \phi^2(x))^2 dx.$$

This is **Stability in the square integral sense**.

Alternative we can use the maximum principle again. Let u^1 and u^2 be solutions of the Dirichlet problem with ϕ^1, g^1, h^1 and ϕ^2, g^2, h^2 respectively. Set $u = u^1 - u^2$. Then

$$u^1 - u^2 =\leq \max\{\max(\phi^1 - \phi^2), \max(g^1 - g^2), \max(h^1 - h^2)\}.$$

But we also get

$$u^2 - u^1 \leq \max\{\max(\phi^2 - \phi^1), \max(g^2 - g^1), \max(h^2 - h^1)\}.$$

Then

$$|u^1 - u^2| \leq \max\{\max|\phi^2 - \phi^1|, \max|g^2 - g^1|, \max|h^2 - h^1|\}.$$

This is called **stability in the uniform sense.**

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4.7. **Solving the Diffusion Equation on the real line.** Consider the diffusion equation on the real line

$$u_t = ku_{x,x} \text{ on } \mathbb{R} \times (0, \infty) \tag{26}$$

More precisely, we look for $u \in C^2(\mathbb{R} \times (0, \infty))$ that solves the initial value problem

$$\begin{aligned} u_t = ku_{x,x} \qquad & \text{on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = \phi(x) \qquad & \text{for } x \in \mathbb{R}. \end{aligned} \tag{27}$$

$\phi \in C^1(\mathbb{R})$ and $k > 0$.

The initial condition is understood in the sense that $\lim_{t\downarrow 0} u(x, t) = \phi(x)$.

For ϕ we assume that $\phi(x) \to 0$ if $|x| \to \infty$. The method to find a solution will be very different from the previous techniques that we used.

Let us collect some general properties of solutions of the diffusion equation $u_t = ku_{x,x}$.

(a) **Translation invariance:** If $u(x, t)$ solves (26), then also $u(x - y, t)$ solves (26) for any $y \in \mathbb{R}$.

(b) If $u(x, t)$ is a smooth (C^k) solution (26), any derivative ($u_t, u_x, u_{x,x}$, ect.) if it exists, solves (26) as well.

(c) **Superposition:** Any linear combinations of solutions of (26) is again a solutions of (26):

$$u^i(x, t),\ i = 1, \dots, n \text{ solves (26)} \implies \sum_{i=1}^{n} \lambda_i u^i(x, t) =: u(x, t) \text{ solves (26).}$$

(d) **An integral of a solution is again a solution:** If $u(x, t)$ solves (26) and $\phi \in C^0(\mathbb{R})$ then

$$v(x, t) = \int u(x - y, t)\phi(y)dy \text{ solves (26).}$$

Proof. We calculate

$$\begin{aligned} v_t(x, t) = \int \frac{\partial}{\partial t} u(x - y, t)\phi(y)dy &= \int \frac{\partial^2}{\partial z^2}\Big|_{z=x-y} u(z, t)\phi(y)dy \\ &= \int \frac{\partial^2}{\partial x^2} [u(x - y)] \phi(y)dy = v_{x,x}(x, t). \qquad \square \end{aligned}$$

(e) **Scaling property:** If $u(x, t)$ is a solution of (26), so is $u(\sqrt{a}x, at)$ for any $a > 0$.

Remark 4.16. Of course these transformations do not preserve the initial value problem

Let us consider the following *special initial condition*:

$$\psi(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

and arbitrary value in $x = 0$.

We consider this ψ because it is scaling invariant: $\psi(ax) = \psi(x)\ \forall a > 0$.

We say $u(x, t)$ in $C^2(\mathbb{R} \times (0, \infty))$ solves the diffusion equation with initial condition ψ if

$$u_t = ku_{x,x} \quad \mathbb{R} \times (0, \infty)$$

and $\lim_{t\downarrow 0} u(x, t) = \psi(x)$ for all $x \in \mathbb{R}$.

If $u(x, t)$ solves the diffusion equation on $\mathbb{R} \times (0, \infty)$ with initial condition $\psi(x)$, then by the scaling property also $u(\sqrt{a}x, at)$ is a solution with the same initial condition $\psi(x)$.

Moreover, we expect uniqueness of solutions for a given initial value function. Hence, it should hold

$$u(x,t) = u(\sqrt{a}x, at).$$

From this we make the following **Ansatz**:

$$Q(x,t) = g(x/\sqrt{t})$$

Why do we choose this Q?

Because Q satisfies $Q(\sqrt{a}t, a) = Q(x,t)$.

Lemma 4.17. *$Q(x,t) = g(x/\sqrt{t})$ solves (26) if and only if g satisfies $g''(r) = -\frac{1}{2k} r g'(r)$.*

Proof. We calculate

$$\begin{aligned} Q_t(x,t) &= \left[g\left(\frac{x}{\sqrt{t}}\right)\right]_t = -\frac{1}{2}\frac{x}{(\sqrt{t})^3} g'\left(\frac{x}{\sqrt{t}}\right) \\ Q_x(x,t) &= \left[g\left(\frac{x}{\sqrt{t}}\right)\right]_x = \frac{1}{\sqrt{t}} g'\left(\frac{x}{\sqrt{t}}\right) \\ Q_{x,x}(x,t) &= \left[g\left(\frac{x}{\sqrt{t}}\right)\right]_{x,x} = \frac{1}{t} g''\left(\frac{x}{\sqrt{t}}\right). \end{aligned}$$

Hence

$$0 = \frac{1}{t}\left[\frac{1}{2}\frac{x}{\sqrt{t}} g'\left(\frac{x}{\sqrt{t}}\right) + k g''\left(\frac{x}{\sqrt{t}}\right)\right].$$

Since $t > 0$ and by substitution of $\frac{x}{\sqrt{t}}$, it follows that g must satisfy

$$g''(r) = -\frac{1}{2k} r g'(r). \tag{28}$$

On the other hand, if g satisfies (28), then $Q(x,t) = g\left(\frac{x}{\sqrt{t}}\right)$ satisfies the $u_t = k u_{x,x}$. ☐

Lemma 4.18. *The general solution of $g''(r) = -\frac{1}{2k} r g'(r)$ is $g(r) = c_1 \int_{r_0}^{r} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau + c_2,\ c_1, c_2 \in \mathbb{R}$.*

Proof. We set $h = g'$ and consider the ODE $h'(r) = -\frac{1}{2k} r h(r)$.

We can easily solve this equation by standard techniques. The general solution is given by

$$h(r) = c_1 e^{-r^2}. \quad \text{And therefore } g(r) = c_1 \int_{r_0}^{r} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau + c_2. \quad ☐$$

Corollary 4.19. *The function $Q(x,t) = c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)} d\tau + c_2$ is a solution of $u_t = k u_{x,x}$ on $\mathbb{R} \times (0, \infty)$.*

We want to choose the constants $c_1, c_2 \in \mathbb{R}$ such that

$$\lim_{t \downarrow 0} Q(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases}$$

We can compute the following limits

$$\begin{aligned} x > 0, \quad \lim_{t\downarrow 0} Q(x,t) &= \lim_{t \downarrow 0} c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)} d\tau + c_2 \\ &= \lim_{t\downarrow 0} c_1 \sqrt{4k} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\tau^2} d\tau + c_2 = c_1 \frac{\sqrt{4k\pi}}{2} + c_2. \end{aligned}$$

Hence, we require

$$1 = c_1 \frac{\sqrt{4k\pi}}{2} + c_2.$$

Here we used that $\int_0^\infty e^{-\tau^2} d\tau = \frac{\sqrt{\pi}}{2}$. Similar

$$x < 0, \quad \lim_{t\downarrow 0} Q(x,t) = \lim_{t\downarrow 0} c_1 \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau + c_2 = -c_1 \frac{\sqrt{4k\pi}}{2} + c_2$$

and hence we require also

$$0 = -c_1 \frac{\sqrt{4k\pi}}{2} + c_2.$$

We can solve this system of two linear equations for c_1 and c_2 and obtain

$$c_1 = \frac{1}{\sqrt{4k\pi}} \text{ and } c_2 = \frac{1}{2}.$$

Corollary 4.20. $Q(x,t) = \frac{1}{\sqrt{4k\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau + \frac{1}{2}$ *solves* $u_t = ku_{x,x}$ *on* $\mathbb{R} \times (0,\infty)$ *and*

$$\lim_{t\downarrow 0} u(x,t) = \psi(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases}$$

Definition 4.21 (Fundamental solution of the diffusion equation on the real line)**.**

$$S(x,t) = \frac{\partial}{\partial x} Q(x,t) = \frac{\partial}{\partial x} \frac{1}{\sqrt{4k\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{-\left(\frac{\tau}{\sqrt{4k}}\right)^2} d\tau = \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{x}{\sqrt{4kt}}\right)^2}.$$

Note that Q (and S) are C^∞ functions on $\mathbb{R} \times (0,\infty)$ (because e^x is C^∞).

The function $S(x,t)$ is called the fundamental solution of $u_t = ku_{x,x}$ on the real line.

Theorem 4.22. *The unique solution of the initial value problem* (27) *:*

$$\begin{aligned} u_t &= ku_{x,x} && \text{on } \mathbb{R} \times [0,\infty) \\ u(x,0) &= \phi(x) && \text{for } x \in \mathbb{R} \end{aligned}$$

where $\phi \in C^1(\mathbb{R})$ *with* $\phi(x) \to 0$ *if* $|x| \to \infty$ *and* $k > 0$ *is*

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy.$$

Proof. We already saw that $u(x,t)$ is indeed a solution of $u_t = ku_{x,x}$ on $\mathbb{R} \times (0,\infty)$.

We only need to check the initial value condition. For that we compute the following:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial}{\partial z}\Big|_{z=x-y} Q(z,t)\phi(y)dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y} \left[Q(x-y,t)\right] \phi(y)dy \\ &= -Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=\infty} + \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy. \end{aligned}$$

Since $\phi(x) \to 0$ for $|x| \to \infty$, it follows

$$u(x,t) = \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy, \quad t > 0.$$

Moreover

$$\begin{aligned}\lim_{t\downarrow 0} u(x,t) &= \int_{-\infty}^{\infty} \lim_{t\downarrow 0} Q(x-y,t)\phi'(y)dy \\ &= \int_{-\infty}^{\infty} 1_{[0,\infty)}(x-y)\phi'(y)dy = \int_{-\infty}^{\infty} 1_{(-\infty,0]}(y-x)\phi'(y)dy \\ &= \int_{-\infty}^{x} \phi'(y)dy = \phi(x).\end{aligned}$$

Uniqueness follows by the enery method. □

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Properties of the fundamental solution

- $S(x,t) > 0$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$.
- We compute that
$$\int_{-\infty}^{\infty} S(x-y,t)dy = \int_{-\infty}^{\infty} S(y,t)dy = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4kt}} e^{-\left(\frac{y}{\sqrt{4kt}}\right)^2} dy = \frac{1}{\sqrt{\pi}}\int e^{-x^2}dx = 1.$$
- The solution $u(x,t) = \int S(x-y,t)\phi(y)$ is in $C^\infty(\mathbb{R} \times (0,\infty))$.
- We have
$$\max_{\{x\in\mathbb{R}:|x|\geq\delta\}} S(x,t) \leq \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{\delta}{\sqrt{4kt}}\right)^2} \to 0 \text{ when } t \downarrow 0.$$
  In particular $S(x,t) \to 0$ for all $x \neq 0$ as $t \downarrow 0$.
- $S(x,0)$ is not defined.

  But we computed
$$\lim_{t\downarrow 0} u(x,t) = \lim_{t\downarrow 0} \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \phi(x).$$
  Hence, we can interpret $y \mapsto S(x-y,0)$ not as function but as a linear operators δ_x on $C^1(\mathbb{R})$:
$$\delta_x(\phi) = \phi(x).$$
  The operator δ_x is an example for a **distribution**, the *Dirac* δ_x *distribution.*

4.8. **Distributions.** Distributions are generalized functions.

We define $\mathcal{D} = C_c^\infty(\mathbb{R})$, the set of C^∞ functions ϕ with $\phi(x) = 0$ for $|x| \geq R$ for some $R > 0$. We say ϕ has compact support.

Definition 4.23 (Distributions)**.** A distribution is continuous linear map $\mathcal{F} : \mathcal{D} \to \mathbb{R}$.

What means continuous in this context?
Let us first define a notion of convergence on $\mathcal{D}$. Consider $\phi_i, \phi \in \mathcal{D}$, $i \in \mathbb{N}$.

We say $\phi_i \to \phi$ in $\mathcal{D}$ if
$$\max_{x\in\mathbb{R}} |\phi_i(x) - \phi(x)| \to 0 \text{ and } \max_{x\in\mathbb{R}} \left|\frac{d^k}{dr}\phi_i(x) - \frac{d^k}{dr}\phi(x)\right| \to 0, \quad \forall k \in \mathbb{N}.$$
Then we say a linear map $\mathcal{F} : \mathcal{D} \to \mathbb{R}$ is continuous if
$$\mathcal{F}(\phi_n) \to \mathcal{F}(\phi) \;\text{ whenever } \phi_n \to \phi \text{ in } \mathcal{D}.$$

The concept of distribution allows us to make sense of $''S(x,0)''$ as the distribution δ_0.

Examples.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Then
$$\mathcal{F}(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$
is a distribution.

We see $\mathcal{F}$ is linear. Let's check continuity of $\mathcal{F}$. Consider $\phi_n \to \phi \in \mathcal{D}$. Then
$$|F(\phi_n) - \mathcal{F}(\phi)| \;\leq \int |f(x)|\,|\phi_n(x) - \phi(x)|\,dx \;\leq \max_{x\in\mathbb{R}} |\phi_n(x) - \phi(x)| \int |f(x)|dx \to 0.$$
Hence $\mathcal{F}$ is indeed a distribution.

The example shows that we can think of functions as distributions.

(2) Let f be as before. Then

$$\mathcal{G}(\phi) = \int_{-\infty}^{\infty} f(x)\phi'(x)dx$$

is a distribution. Continuity follows as in the previous example.

Now, if $f \in C^1(\mathbb{R})$, then $\mathcal{G}(\phi) = -\int_{-\infty}^{\infty} f'(x)\phi(x)dx$.

The function f' represents the distribution $\mathcal{G}$.

The concept of distributions now allows us to define derivatives for functions that are not differentiable in the classical sense.

4.8.1. *Derivatives in distributional sense.*

Definition 4.24. We say a locally integrable function $f : \mathbb{R} \to \mathbb{R}$ has a derivative in distributional sense if there exists a distribution $\mathcal{G} : \mathcal{D} \to \mathbb{R}$ such that

$$\int f(x)\phi'(x)dx = \mathcal{G}(\phi) \ \forall \phi \in \mathcal{D}.$$

Example 4.25.

- Every C^1 function has a derivative in distributional sense:

$$\int f(x)\phi'(x)dx = -\int f'(x)\phi(x)dx$$

and the right hand side defines a distribution.

- The function $f(x) = 0$ for $x < 0$ & $f(x) = x$ for $x \geq 0$ has a derivative in distributional sense:

$$\int f(x)\phi'(x)dx = \int_0^{\infty} x\phi(x)dx \ = x\phi(x)\big|_0^{\infty} - \int_0^{\infty} \phi(x)dx = -\int_{-\infty}^{\infty} \psi(x)\phi(x)dx$$

and derivative is represented by ψ.

Question: Has the function

$$\psi(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases}$$

a distributional derivative? If yes, what is it?

We can compute the distributional derivative using the $S(x,t)$:

$$\int \psi(x)\phi'(x)dx \ = \int_0^{\infty} \phi'(x) = \lim_{t\downarrow 0} \int Q(x,t)\phi'(x)dx \ = \lim_{t\downarrow 0} \int S(x,t)\phi(x)dx = \phi(0) = \delta_0(\phi)$$

So indeed ψ has a derivative in distributional sense but it cannot be represented as function!

Physical Intepretations of $S(x,t)$. The fundamental solution $S(x-y,t)$ describes the diffusion of a substance.

For any time $t > 0$ the total mass is 1.

Initally at time $t = 0$ the substance completely concentrated in y.

We can see the convolution $\int S(x-y,t)\phi(y)dy$ also as follows. For $t > 0$ we can approximate the integral via a Riemann sum:

$$\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \sim \sum_{i=1}^{n} S(x-y_i,t)\phi(y_i)\Delta y_i$$

where $\{y_0 \leq y_1 \leq \cdots \leq y_n\} \subset \mathbb{R}$ with $n \in \mathbb{N} \uparrow \infty$ and $\Delta y_i = y_i - y_{i-1}$.

On the right hand side we have a sum that is the mean value in space of the family

$$S(x-y_i,t) \text{ weighted with } \phi(y_i), i = 1, \ldots, n.$$

Consequently, we can interpret $\int S(x-y,t)\phi(y)dy$ as the limit of these mean values when we let the number of points go to infinity.

Probabilistic interpretation of $S(x,t)$. .
The fundamental solution is the transition probability density of Brownian motion in $\mathbb{R}$.

What does that mean? If a particle in 0 at time $t=0$ follows a "random path" then

$$\int_a^b S(-y,t)dy$$

is the probability that we will find this particle at time $t>0$ in the interval $[a,b]$.

4.9. **Back to Uniqueness.**

Lemma 4.26. *Let $\phi \in C^1(\mathbb{R})$ with $\phi(x) \to 0$ for $|x| \to \infty$. We consider*

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy.$$

Then, for $t>0$ fixed, $u(x,t) \to 0$ for $|x| \to \infty$.

Proof. We pick $\epsilon > 0$. Let $R(\epsilon) > 0$ such that $|\phi(x)| \leq \epsilon$ for $|x| \geq R$ for $R \geq R(\epsilon)$. We fix such an $R > R(\epsilon)$. We pick $R > R(\epsilon)$ such that

$$S(R,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{R}{\sqrt{4kt}}\right)^2} \leq \epsilon.$$

Consider a sequence $(x_n)_{n\in\mathbb{N}}$ with $|x_n| \to \infty$, and let $N \in \mathbb{N}$ such that $|x_n| \geq 2R$ for $n \geq N$.

Let $n \geq \mathbb{N}$. Then

$$u(x_n,t) = \int_{x_n-R}^{x_n+R} S(x_n-y)\phi(y)dy + \int_{\{y\in\mathbb{R}:|x_n-y|>R\}} S(x_n-y)\phi(y)dy.$$

We write $\{y \in \mathbb{R} : |x_n - y| > R\} = \{|x_n - y| > R\}$ in the following. Hence

$$|u(x_n,t)| \leq \left|\int_{x_n-R}^{x_n+R} S(x_n-y)\phi(y)dy\right| + \left|\int_{\{|x_n-y|>R\}} S(x_n-y)\phi(y)dy\right|.$$

The second integral on the right hand side can be estimated as follows

$$\begin{aligned}\left|\int_{\{|x_n-y|>R\}} S(x_n-y)\phi(y)dy\right| &\leq \int_{\{|x_n-y|>R\}} \frac{1}{\sqrt{4k\pi t}} e^{-\left(\frac{R}{\sqrt{4kt}}\right)^2} |\phi(y)|\, dy \\ &\leq \epsilon \int_{\{|x_n-y|>R\}} |\phi(y)|\, dy \leq \epsilon \int |\phi(y)|\, dy.\end{aligned}$$

If $|x_n - y| \leq R$, then $|y| \geq |x_n| - |x_n - y| \geq |x_n| - R \geq 2R - R = R$. Therefore, the first integral on the right hand side becomes

$$\left|\int_{x_n-R}^{x_n+R} S(x_n-y)\phi(y)dy\right| \leq \epsilon \int_{x_n-R}^{x_n+R} S(x_n-y)dy \ \leq \epsilon \int S(x_n-y,t)dy \leq \epsilon.$$

We can conclude that for $n \geq N$ it follows that

$$|u(x_n,t)| \leq \epsilon + \epsilon \int |\phi(y)|dy$$

Therefore

$$\limsup_{n\to\infty} |u(x_n,t)| \leq \epsilon(1 + \int |\phi(y)|dy) \ \forall \epsilon > 0 \Rightarrow \ \limsup_{n\to\infty} |u(x_n,t)| \leq 0 \ \Rightarrow \ \lim_{n\to\infty} |u(x_n,t)| = 0.$$

□

Theorem 4.27 (Existence and Uniqueness)**.** *The initial value problem*

$$\begin{aligned} u_t &= k u_{x,x} \quad \text{on } \mathbb{R}\times(0,\infty) \\ u(x,0) &= \phi(x) \quad \text{for } x\in\mathbb{R} \end{aligned} \tag{29}$$

for $\phi \in C^1(\mathbb{R})$ with $\phi(x) \to 0$ if $|x| \to \infty$ and $k > 0$ has a unique solution $u(x,t)$ with $u(x,t) \to 0$ if $|x| \to \infty$.

Proof. Assume there are 2 solutions $u^1(x,t)$ and $u^2(x,t)$ with $u^1(x,t), u^2(x,t) \to 0$ for $|x| \to \infty$.

Then we consider $u = u^1 - u^2$ and also $u(x,t) \to 0$ if $|x| \to \infty$.

Now we apply the energy method

$$\frac{d}{dt}\int \frac{1}{2}[u(x,t)]^2 dx = \int u_t(x,t)u(x,t)dx = \int k u_{x,x}(x,t)u(x,t)dx.$$

Hence

$$\frac{d}{dt}\int \frac{1}{2}[u(x,t)]^2 dx = k u_x(x,t)u(x,t)\Big|_{x=-\infty}^{x=\infty} - \int (u_x(x,t))^2\, dx \le 0.$$

It follows that

$$\int \frac{1}{2}[u(x,t)]^2 dx \le \int \frac{1}{2}[u(x,\epsilon)]^2 \to 0.$$

Hence $u = 0$ and $u^1 = u^2$. ☐

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5. Second order equations: Sources and Reflections

5.1. **Diffusion equations with a source term.** In the following we will study inhomogeneous, linear second order PDEs

For instance, consider the initial value problem for diffusion equation with a source term: Diffusion equation with a source term Let $f \in C^0(\mathbb{R} \times (0,\infty))$.

$$\begin{aligned} u_t - k u_{x,x} &= f(x,t) \quad \text{on} \quad \mathbb{R} \times (0,\infty) \\ u(x,0) &= \phi(x) \quad \text{on} \quad \mathbb{R}. \end{aligned}$$

The physical interpretation of this equation is, for instance, the heat evolution of an infinitely long rod with an initial temperatur ϕ and a source (or sink) of heat at later times.

Remark 5.1. If we define $A = k\frac{\partial^2}{\partial x^2}$, then A is linear operator that goes from $C^2(\mathbb{R})$ to $C^0(\mathbb{R})$.

Then the inhomogeneous diffusion equation then takes the form

$$\frac{d}{dt}u(t) = Au(t) + f(t) \ t > 0 \text{ and } u(0) = \phi \in C^1(\mathbb{R})$$

where $u(t) = u(\cdot, t) \in C^2(\mathbb{R})$ and $f(t) = f(\cdot, t) \in C^0(\mathbb{R})$.

5.1.1. *Structural similarities with inhomogeneous ODEs.* Recall the following ODE problem. Let $A \in \mathbb{R}^{n\times n}$.

$$\frac{d}{dt}v(t) = Av(t) + f(t), \quad v(0) = v_0$$

where $t \in [0,\infty) \mapsto v(t), f(t) \in \mathbb{R}^n$.

For $f \equiv 0$ this is a homogeneous, linear ODE with constant coefficients.

The solution is given by $t \in [0,\infty) \mapsto e^{tA}v_0$.

e^{tA} is called the solution operator.

Recall: In case $A = BDB^{-1}$ for a diagonal matrix $D = (d_1, \dots, d_n)$ then

$$e^{tA} = B(e^{td_1}, \dots, e^{td_n})B^{-1}$$

More general, one can find the operator e^{tA} by means of the Jordan form for the matrix A.

The solution formula for the inhomogeneous problem with $f \neq 0$ is given by

$$u(t) = e^{tA}v_0 + \int_0^t e^{(t-s)A} f(s)ds.$$

5.1.2. *Dunhamel's principle.* The solution formula for the inhomogeneous ODE is derived via Dunhamel's principle.

Assume $v(t)$ solves the inhomeogeneous problem. Assume $S(-t) = e^{-tA} = \left[e^{tA}\right]^{-1}$ exists.

Then we can compute

$$S(-t)f(t) = S(-t)\left[\frac{d}{dt}v(t) - Av(t)\right] = S(-t)\frac{d}{dt}v(t) - S(-t)Av(t) = \frac{d}{dt}\left[S(-t)v(t)\right].$$

The last equality is the product rule. Integrating from 0 to $t > 0$ gives

$$\int_0^t S(-s)f(s)ds = S(-t)v(t) - v_0$$

Hence

$$v(t) = S(t)v_0 + S(t)\int_0^t S(-s)f(s)ds = S(t)v_0 + \int_0^t S(t-s)f(s)ds.$$

We can then check that this $v(t)$ indeed satisfies the inhomogeneous ODE

$$\begin{aligned}\frac{d}{dt}v(t) &= \frac{d}{dt}S(t)v_0 + \frac{d}{dt}\int_0^t S(t-s)f(s)ds\\ &= AS(t)v_0 + S(0)f(t) + \int_0^t AS(t-s)f(s)ds\\ &= A\left[S(t)v_0 + \int_0^t S(t-s)f(s)ds\right] + f(t) = Av(t) + f(t).\end{aligned}$$

The solution formula for ODEs gives us an idea how a solution formula for PDEs should look like.

We saw a version of this formula before in the case of inhomogeneous first order PDEs of the form

$$u_t - au_x = f(x,t)$$

Here, the operator is given by $A = a\frac{\partial}{\partial x}$. Then the PDE takes the form

$$u_t = Au + f(x,t)$$

Recall the solution of the homogeneous equation was given by

$$\phi(x+ta) = [S(t)\phi](x)$$

Dunhamel's principle suggests the solution formula

$$v(t) = S(t)\phi + \int_0^t S(t-s)f(s)ds = \phi(x+at) + \int_0^t f(x+a(t-s),s)ds.$$

for the inhomogeneous problem.

This is exactly the formula that we already derived from the method of characteristics.

Back to the inhomogeneous diffusion equation The unique solution of the initial value problem

$$\begin{aligned} u_t &= ku_{x,x} \quad \text{on } \mathbb{R}\times(0,\infty)\\ u(x,t)\to 0, &\quad |x|\to\infty\\ u(x,0) &= \phi(x) \quad \text{on } \mathbb{R}.\end{aligned}$$

was given by

$$\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy =: \mathbf{S}(t)\phi(x) \;\text{ where } S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\left(\frac{x}{\sqrt{4kt}}\right)^2}.$$

We can see $\mathbf{S}(t): C^1(\mathbb{R}) \to C^2(\mathbb{R})$ as a family of solution operators.

Now we consider the same problem but with a source term $f \in C^0(\mathbb{R}\times(0,\infty))$:

$$u_t = ku_{x,x} + f(x,t) \quad \text{on } \mathbb{R}\times(0,\infty).$$

We also write $f(s)$ for $f(\cdot,s)$. We assume $|f(x,t)| \le C$. We prove the following theorem.

Theorem 5.2. *The unique solution ofthe inhomogeneous problem is given by the formula*

$$v(x,t) = [\mathbf{S}(t)\phi](x) + \int_0^t [\mathbf{S}(t-s)f(s)](x)ds.$$

Proof. We only check the existence statement.

First we compute $v_t = [\mathbf{S}(t)\phi]_t + \frac{d}{dt}\int_0^t \mathbf{S}(t-s)f(s)ds = k[\mathbf{S}(t)\phi]_{x,x} + \frac{d}{dt}\int_0^t \mathbf{S}(t-s)f(s)ds.$

We consider the second term on the right hand side

$$\frac{d}{dt}\int_0^t [\mathbf{S}(t-s)f(s)](x)ds = \frac{d}{dt}\int_0^t\int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds = \frac{d}{dt}\int_0^t g(s,t)ds.$$

$s \in (-\infty, t] \mapsto g(s,t)$ is continuous with

$$g(t,t) = \lim_{s\uparrow t} g(s,t) = \delta_x[f(\cdot,t)] = f(x,t).$$

More precisely, we can compute

$$\begin{aligned} g(s,t) &= \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dy \\ &= \int_{-\infty}^{\infty} S(x-y,t-s)f(y,t)dy + \int_{-\infty}^{\infty} S(x-y,t-s)(f(y,s)-f(y,t))dy \end{aligned}$$

For the first term it follows by computation as we did before that

$$\lim_{s\rightarrow t}\int_{-\infty}^{\infty} S(x-y,t-s)f(y,t)dy = \delta_x[f(t)] = f(x,t).$$

For the second we get

$$\begin{aligned} \int_{-\infty}^{\infty} S(x-y)[f(y,s)-f(y,t)]dy = &\int_{x-\tilde{C}}^{x+\tilde{C}} S(x-y,t-s)[f(y,s)-f(y,t)]dy \\ &+ \int_{\{y:|x-y|>\tilde{C}} S(x-y,t-s)[f(y,s)-f(y,t)]dy \end{aligned}$$

Since $f \in C^0(\mathbb{R}\times(0,\infty))$, f is uniformily continuous on $[x-C,x+C]\times[t-\eta,t+\eta]$ for $t>0$ and $\eta>0$ sufficiently small such that $t-\eta>0$.

In particular, given $\epsilon>0$ there exists $\delta(\tilde{C},\epsilon)>0$ such $|f(y,s)-f(y,t)|<\epsilon$ for $|s-t|<\delta$.

Therefore, for the first term on the right hand side in the last formula we have

$$-\epsilon \le \int_{x-\tilde{C}}^{x+\tilde{C}} S(x-y,t-s)[f(y,s)-f(y,t)]dy \le \epsilon.$$

For the second term on the right hand side in the last formula we have

$$-2CKe^{-\tilde{C}^2} \le \int S(x-y,t-s)[f(y,s)-f(y,t)] \le 2CKe^{-\tilde{C}^2}$$

because $|f(y,s)-f(y,t)| \le C$ and $S(x-y,t-s) \le Ke^{-\tilde{C}^2}$ on $\{y : |x-y| \ge C\}$ for a constant $K>0$. So we can choose $\tilde{C}$ such that $2CKe^{-\tilde{C}} \le \epsilon$.

This considerations together imply that

$$\lim_{s\rightarrow t} g(s,t) = f(x,t) \pm 2\epsilon$$

and since $\epsilon>0$ was arbitrary, the limit is $f(x,t)$.

Hence $\tau \in [0,\infty) \mapsto \int_0^\tau g(s,t)ds = \int_0^\tau \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds$ is differentiable in t with

$$\frac{d}{dt}\int_0^\tau g(s,t)ds = g(t,t) = f(x,t).$$

Therefore

$$\frac{d}{dt}\int_0^t g(s,t)ds = f(x,t) + \int_0^t \frac{\partial}{\partial t} g(s,t)ds$$

For the second term on the right hand side we calculate

$$\begin{aligned} \int_0^t \frac{\partial}{\partial t} g(s,t)dt = &\int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s)f(s)dyds = \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s)f(s)dyds \\ &= k\frac{\partial^2}{\partial x^2}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(s)dyds = k\left[\int_0^t [\mathbf{S}(t-s)f(s)](x)ds\right]_{x,x}. \end{aligned}$$

So we computed $v_t = f(x,t) + k\left[\mathbf{S}(t)\phi + \int_0^t \mathbf{S}(t-s)f(s)ds\right]_{x,x}$. □

Lecture13

5.2. **Diffusion on the half line, Reflection method.** We consider the Dirichlet problem for the diffusion equation:

$$\begin{aligned} u_t &= ku_{x,x} && \text{on } (0,\infty)\times(0,\infty) \\ u(x,0) &= \phi(x) && \text{on } [0,\infty) \\ u(0,t) &= 0 && \text{for } t>0. \end{aligned} \tag{30}$$

To find a solution formula for this equation we apply the **reflection method**:

Consider the odd extension of ϕ to the real line:

$$\phi_{odd}(x) = \begin{cases} \phi(x) & x \geq 0 \\ -\phi(-x) & x < 0. \end{cases}$$

The corresponding initial value problem has the solution:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{odd}(y)dy.$$

Since ϕ_{odd} is odd, also $x \mapsto u(x,t)$ is odd, that is $u(x,t) = -u(-x,t)$ (Exercise).

Hence $u(0,t) = 0$ and the restriction v of u to $[0,\infty)\times[0,\infty)$ solves the Dirichlet problem for the diffusion equation with initial condition ϕ.

A solution formula of v that only depends on ϕ is derived as follows

$$v(x,t) = \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_{-\infty}^{0} S(x-y,t)\phi_{odd}(y)dy$$

$$= \int_0^{\infty} \left[S(x-y,t)\phi(y) + S(x+y,t)\phi_{odd}(-y)\right] dy = \int_0^{\infty} \left[S(x-y,t) - S(x+y,t)\right]\phi(y)dy.$$

The solution of the problem (30) is given by the formula

$$v(x,t) = \int_0^{\infty} \left[S(x-y,t) - S(x+y,t)\right]\phi(y)dy.$$

Similar, we can consider the Neumann problem for the diffusion equation:

$$\begin{aligned} u_t &= ku_{x,x} && \text{on } [0,\infty)\times(0,\infty) \\ u(x,0) &= \phi(x) && \text{on } [0,\infty) \\ u_x(0,t) &= 0 && \text{for } t>0. \end{aligned}$$

To derive a solution formula we apply the same strategy as for the Dirichlet problem.
We consider the following initial value problem for the diffusion equation on the real line:

$$\begin{aligned} u_t &= ku_{x,x} && \text{on } \mathbb{R}\times(0,\infty) \\ u(x,0) &= \phi_{even}(x) && \text{on } \mathbb{R} \end{aligned}$$

where ϕ_{even} is the even extension of ϕ to $\mathbb{R}$:

$$\phi_{even}(x) = \begin{cases} \phi(x) & x \geq 0 \\ \phi(-x) & x < 0 \end{cases}$$

The solution of this initial value problem will be again even in x: $u(x,t) = u(-x,t)$.

5.2.1. *Diffusion with source term on the half line.* Now we consider

$$
\begin{aligned}
u_t - ku_{x,x} &= f(x,t) \quad \text{on } (0,\infty)\times(0,\infty)\\
u(x,0) &= \phi(x) \quad \text{on } [0,\infty)\\
u(0,t) &= h(t) \quad \text{for } t>0.
\end{aligned}
\tag{31}
$$

for a boundary source function $h:[0,\infty)\to\mathbb{R}$ in $C^1([0,\infty))$.

A strategy to solve this problem is the *Substraction method*:

We consider $v(x,t) = u(x,t) - h(t)$. If $u \in C^2((0,\infty)\times(0,\infty))$ solves the previous problem, then $v \in C^2((0,\infty)\times(0,\infty))$ solves

$$
\begin{aligned}
v_t - kv_{x,x} &= f(x,t) - h'(t) \quad \text{on } [0,\infty)\times(0,\infty)\\
v(x,0) &= \phi(x) - h(0) \quad \text{on } [0,\infty)\\
u(0,t) &= 0 \quad \text{for } t>0.
\end{aligned}
$$

To solve this problem we can apply the reflection method as we did for the equation with $f \equiv 0$.

Then one can check that $v(x,t) + h(t) =: u(x,t)$ solves the problem (31).

5.3. **Wave equation with a source term.** Consider $\phi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$ and $f \in C^0(\mathbb{R}\times(0,\infty))$ and the inital value problem

$$
\begin{aligned}
u_{t,t} - c^2u_{x,x} &= f(x,t) \quad \text{on } \mathbb{R}\times(0,\infty),\\
u(x,0) &= \phi(x) \quad \text{on } \mathbb{R},\\
u_t(x,0) &= \psi(x) \quad \text{on } \mathbb{R}.
\end{aligned}
\tag{32}
$$

We can interpret f as an external force that acts on an infinitely long vibrating string.

We will prove

Theorem 5.3. *The unique solution of the initial value problem* (32) *is*

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy + \frac{1}{2c}\iint\limits_{\Delta_{x,t}} f(y,s)dyds.$$

The double integral in the formula is on the characteristic space-time triangle $\Delta_{x,t}$ corresponding to the point $(x,t)\in\mathbb{R}\times(0,\infty)$. More precisely

$$\frac{1}{2c}\iint_{\Delta_{x,t}} f(y,s)dyds = \frac{1}{2c}\int_0^t\int_{x-ct}^{x+ct} f(y,s)dyds.$$

5.3.1. *Deriving the solution formual via the operator method.* We follow the same ideas as for the diffusion equation.

Defining the operator $A = c\frac{\partial}{\partial x}$ the PDE takes the form

$$
\begin{aligned}
\tfrac{d}{dt}u - A^2u &= f(t) \quad \text{on } \mathbb{R}\times(0,\infty),\\
u(0) &= \phi \quad \text{on } \mathbb{R},\\
\tfrac{d}{dt}u(0) &= \psi \quad \text{on } \mathbb{R}.
\end{aligned}
$$

where $u(t) = u(\cdot,t)\in C^2(\mathbb{R})$ and $f(t) = f(\cdot,t)\in C^0(\mathbb{R})$ for $t>0$.

This equation has again the structure of an ODE of the form

$$
\begin{aligned}
\tfrac{d^2}{dt^2}u - a^2u &= f(t) \quad \text{on } (0,\infty),\\
u(0) &= \phi \in \mathbb{R}\\
u'(0) &= \psi \in \mathbb{R}.
\end{aligned}
\tag{33}
$$

where $f \in C^0([0,\infty))$. Let us first consider the case $f \equiv 0$.

We consider the solutions u_1 and u_2 for problem with the following initial conditions

$$\begin{array}{rcl} u_1(0) &=& 0 \\ u_1'(0) &=& \psi \end{array} \quad \text{and} \quad \begin{array}{rcl} u_2(0) &=& \phi \\ u_2'(0) &=& 0 \end{array}$$

and the sum $u_1 + u_2 = u$ is a solution of (33).

Precisely $u_1(t) = \psi\frac{1}{a}\sin(at)$, $u_2(t) = \phi\cos(at)$and $u = \psi\frac{1}{a}\sin(at) + \phi\cos(at)$. We can define the solution operator

$$S(t)\psi = \psi\frac{1}{a}\sin(at) = u_1(t) \ \text{ and } \ \frac{d}{dt}[S(t)\phi] = \phi\cos(at) = u_2(t).$$

We note that $S(0)\psi = 0$ and $\frac{d}{dt}\big|_{t=0}[S(t)\phi] = \phi$.

By Dunhammel's principle the general solution for the inhomogeneous ODE

$$\begin{array}{rcll} \frac{d^2}{dt^2}u - a^2u &=& f(t) \neq 0 & \text{on } \mathbb{R}\times(0,\infty), \\ u(0) &=& 0 \in \mathbb{R} & \\ u'(0) &=& \psi \in \mathbb{R}. & \end{array} \tag{34}$$

is given by the formula

$$\tilde{u}_1(t) = S(t)\psi + \int_0^t S(t-s)f(s)ds.$$

Indeed, since we can check that $\frac{d^2}{dt^2}[S(t)\psi] - a^2S(t)\psi = 0$ and

$$\frac{d^2}{dt^2}\int_0^t S(t-s)f(s)ds = f(t) - a^2\left[\int_0^t S(t-s)f(s)ds\right]$$

Now, by linearity

$$\tilde{u}_1 + u_2 = S(t)\psi + \int_0^t S(t-s)f(s)ds + \frac{d}{dt}S(t)\phi = v$$

solves the inhomogeneous problem with $v(0) = \phi$ and $v'(0) = \psi$. The same method works for the wave equation with source. First we solve the IVP (32) with $f \equiv 0$. By d'Alembert's formula the solution is

$$u_1(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy = [\mathbf{S}(t)\psi](x) \text{ for } \phi = 0 \text{ and } \psi \in C^1(\mathbb{R})$$

where $[\mathbf{S}(0)\psi](x) = 0$ and

$$u_2(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] \text{ for } \phi \in C^2(\mathbb{R}) \text{ and } \psi = 0.$$

Observe that $u_2(x,t) = \frac{d}{dt}\mathbf{S}(t)\phi(x)$ with $\frac{d}{dt}\big|_{t=0}[\mathbf{S}(t)\phi](x) = \phi(x)$. Then

$$u_1 + u_2 = \frac{d}{dt}\mathbf{S}(t)\phi + \mathbf{S}(t)\psi = u$$

solves the initial value problem for the homogeneous wave equation and

$$v(x,t) = \frac{d}{dt}\mathbf{S}(t)\phi + \mathbf{S}(t)\psi + \int_0^t \mathbf{S}(t-s)f(s)(x)ds$$

is the "candidate" for a solution of the initial value problem of the inhomogeneous wave equation.

This is the formula that shows up in the theorem before. Indeed

$$\int_0^t \mathbf{S}(t-s)f(s)(x)ds = \int_0^t \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t+s)} f(y,s)dyds.$$

5.3.2. *Proof of the theorem.* By linearity of the PDE we only need to check that the function

$$(x,t)\in\mathbb{R}\times(0,\infty)\mapsto \frac{1}{2c}\iint_{\Delta_{x,t}} f(y,s)dyds = \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

satisfies the inhomogeneous wave equation with initial conditions $\phi=\psi=0$.

We apply the coordinate change

$$\xi = x+ct,\quad \eta = x-ct.$$

First, we note that the operator $\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}$ becomes $4c^2\frac{\partial^2}{\partial\xi\partial\eta}$. Indeed let $\tilde g(\xi,\eta)$ be defined by $\tilde g(\xi,\eta)=\tilde g(x+ct,x-ct)=g(x,t)$. We compute

$$\frac{\partial}{\partial x}g(x,t)=\frac{\partial}{\partial\xi}\tilde g(\xi,\eta)+\frac{\partial}{\partial\eta}g(\xi,\eta),\qquad \frac{\partial}{\partial t}g(x,t)=c\frac{\partial}{\partial\xi}\tilde g(\xi,\eta)-c\frac{\partial}{\partial\eta}g(\xi,\eta).$$

It is straightforward to confirm that

$$g_{t,t}-c^2g_{x,x}=-4c^2\frac{\partial^2}{\partial\xi\partial\eta}\tilde g(\xi,\eta).$$

Using the transformation formula we compute the integral

$$\frac{1}{2c}\iint_{\Delta_{x,t}} f(y,s)dyds=\frac{1}{2c}\iint_{\Delta_{x.t}}\tilde f(y+cs,y-cs)dyds.$$

The Jacobian determinant of the transformation $\Phi(x,t)=(x+ct,x-ct)$ is

$$|\det D\Phi(x,t)|=\left|\det\begin{pmatrix}1 & c\\ 1 & -c\end{pmatrix}\right|=2c.$$

Hence

$$\frac{1}{2c}\iint_{\Delta_{x,t}} f(y,s)dyds=\frac{1}{4c^2}\iint_{\Delta_{x,t}}\tilde f\circ\Phi(y,s)J\Phi(y,s)dyds=\frac{1}{4c^2}\iint_{\Phi(\Delta_{x,t})}\tilde f(\xi,\eta)d\xi d\eta.$$

where

$$\frac{1}{4c^2}\iint_{\Phi(\Delta_{x,t})}\tilde f(\xi,\eta)d\eta d\xi=\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi}\tilde f(\xi,\eta)d\eta d\xi=-\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\xi}^{\eta_0}\tilde f(\xi,\eta)d\xi d\eta.$$

Hence

$$-4c^2\frac{\partial^2}{\partial\eta_0\partial\xi_0}\frac{-1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\xi}^{\eta_0}\tilde f(\xi,\eta)d\xi d\eta=\frac{\partial}{\partial\eta}\int_{\xi_0}^{\eta_0}f(\xi_0,\eta)d\eta=\tilde f(\xi_0,\eta_0)=f(x,t).$$

Hence, we confirmed the PDE. □

5.3.3. *Consequences: Wellposedness of the wave equation with a source term.*

Existence follows from the solution formula.

Uniqueness Let u be a C^2 solution on $\mathbb{R}\times[0,\infty)$ for the wave equation with source and initial values $\phi=\psi=0$. Then

$$\frac{1}{2c}\iint_{\Delta_{x,t}} f(y,s)dyds=\frac{1}{2c}\iint_{\Delta_{x,t}}[u_{t,t}-c^2u_{x,x}]dyds.$$

By the divergence theorem it follows

$$=\frac{1}{2c}\int_{\partial\Delta_{x,t}}N\cdot(-c^2u_x,u_t)dL=\frac{1}{2c}\int_{\partial\Delta_{x,t}}N\cdot(-c^2u_x,u_t)dL.$$

This line integral has 3 components: the bottome side

$$\frac{1}{2}\int_{x-ct}^{x+tc}-cu_t(y,0)dy=0$$

and the side formed by curve $s \in [0,t] \mapsto x + c(t-s)$. Note that the normal vector on this side is $\frac{1}{\sqrt{c^2+1}}(1,c)$ and line integral along this curve comes with a weight $\sqrt{c^2+1}$.

$$\begin{aligned}&\frac{1}{2c}\int_0^t cu_t(x+c(t-s) - c^2 u_x(x+c(t-s),s)ds\\ &\quad = \frac{1}{2}\int_0^t \frac{d}{ds}[u(x+c(t-s),s)]ds = \frac{1}{2}[u(x,t) - u(x+ct,0)] = \frac{1}{2}u(x,t).\end{aligned}$$

Similar for the remaining term. Hence

$$u(x,t) = \frac{1}{2c}\iint_{\Delta_{x,t}} f(y,s)dyds.$$

Stability: We claim the wave equation with source is stable. That means small perturbations of the data functions f, ϕ and ψ result in small perturbations of the solution u.

How do we measure smallness?

Definition 5.4 (Maximums Norm on $\mathbb{R}$ and $\mathbb{R}\times[0,\infty)$)**.** Let $v \in C^0(\mathbb{R})$ and $w \in C^0(\mathbb{R}\times[0,\infty))$. Define the Maximumnorms:

$$\|v\| = \max_{x\in\mathbb{R}}|v(x)|, \qquad \|w\|_T = \max_{(x,t)\in\mathbb{R}\times[0,T]}|w(x,t)|$$

From the solution formula we have the following a priori estimate for the solution u on $\mathbb{R}\times[0,T]$:

$$\begin{aligned}|u(x,t)| &\le \frac{1}{2}|\phi(x+ct) - \phi(x-ct)| + \frac{1}{2c}\int_{x-ct}^{x+ct}|\phi(y)|dy + \frac{1}{2c}\iint_{\Delta_{x,t}}|f(y,s)|dyds\\ &\le \|\phi\| + T\|\psi\| + T^2\|f\|_T\end{aligned}$$

Hence

$$\|u\|_T \le \|\phi\| + T\|\psi\|_T + T^2\|f\|_T\,.$$

If we have to solution u_1 and u_2 with corresponding data $\phi_1,\phi_2,\psi_1,\psi_2,f_1,f_2$, then $u_1 - u_2$ is a solution with data $\phi_1-\phi_2, \psi_1-\psi_2, f_1-f_2$ by linearity of the problem.

Hence the estimate for the norm yields stabiltiy w.r.t. the Maximums Norm.

lecture 14

5.4. **Reflection method for wave equations.** We will study the following Dirichlet problem for the wave equation on the half-line:

$$\begin{array}{rclcc} v_{t,t} & = & c^2 v_{x,x} & \text{on} & (0,\infty)\times\mathbb{R} \\ v(x,0) & = & \phi(x) & \text{on} & (0,\infty) \\ v_t(x,0) & = & \psi(x) & \text{on} & (0,\infty) \\ v(0,t) & = & 0 & \text{on} & \mathbb{R}. \end{array} \tag{35}$$

The reflexion method works the same way as for the diffusion equation.

We consider *odd* extensions ϕ_{odd} and ψ_{odd} of ϕ and ψ respectively.

Let $u(x,t)$ be the solution of the initial value problem for the wave equation on $\mathbb{R}$. We have the formula

$$u(x,t) = \frac{1}{2}\left[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)dy.$$

Then $u(x,t)$ is once again odd. In particular we have $u(0,t) = 0$ for $t > 0$ and we can define the solution v on $[0,\infty)\times\mathbb{R}$ of (35) by restriction of u to $[0,\infty)$.

We observe that for $x \geq c|t|$ it follows that $x-ct, x+ct \geq 0$. Hence

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy \quad x \geq c|t|.$$

For $0 < x < c|t|$ we have $\phi_{odd}(x-ct) = -\phi(-x+ct)$. Hence

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(-x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{0} [-\psi(-y)]dy + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)dy \quad 0 < x < c|t|.$$

We can apply a change of variable $y \mapsto -y$ to the first integral term. We obtain

$$\begin{aligned} v(x,t) &= \frac{1}{2}\left[\phi(ct+x) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{0} \psi(y)dy + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)dy \\ &= \frac{1}{2}\left[\phi(ct+x) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)dy \quad 0 < x < c|t|. \end{aligned}$$

Remark 5.5. The complete solution is given by

$$v(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \psi(y)dy & \text{if } x \geq c|t| \\ \frac{1}{2}\left[\phi(ct+x) - \phi(ct-x)\right] + \dfrac{1}{2c}\displaystyle\int_{ct-x}^{ct+x} \psi(y)dy & \text{if } 0 < x < c|t|. \end{cases}$$

5.4.1. *Finite Interval.* Similarly we can also study the problem

$$\begin{array}{rclcc} v_{t,t} & = & c^2 v_{x,x} & \text{on} & (0,l)\times\mathbb{R} \\ v(x,0) & = & \phi(x) & \text{on} & (0,l) \\ v_t(x,0) & = & \psi(x) & \text{on} & (0,l) \\ v(0,t) = v(l,t) & = & 0 & \text{on} & \mathbb{R}. \end{array} \tag{36}$$

5.5. **Diffusion equation with continuous initial data.** Let us consider once more

$$\begin{array}{rcll} u_t & = & ku_{x,x} & \text{on } \mathbb{R}\times(0,\infty) \\ \lim_{t\downarrow 0} u(x,t) & = & \phi(x) & \text{on } \mathbb{R} \end{array}$$

This time we assume $\phi \in C^0(\mathbb{R})$ and $|\phi(x)| \leq M \ \forall x \in \mathbb{R}$.

The convolution formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-y}{\sqrt{4kt}}\right)^2} \phi(y)dy$$

still makes sense. Indeed, since $|\phi(x)| \leq M$ the integral is finite and bounded from above by M:

$$|u(x,t)| = \left| \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(\frac{z}{\sqrt{4kt}}\right)^2} \phi(x-z)dz \right| \leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\left(\frac{z}{\sqrt{4kt}}\right)^2} Mdz \leq M.$$

Remark 5.6. A refined satement is that for $m \leq \phi(x) \leq M$ it follows

$$m \leq \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \leq M \quad \forall t > 0 \qquad \textit{(Maximum Principle)}.$$

Theorem 5.7. *Let $\phi(x)$ and $u(x,t)$ be as above. Then $u \in C^\infty(\mathbb{R}\times(0,\infty))$ such that $u_t = ku_{x,x}$ on $\mathbb{R}\times(0,\infty)$ and $\lim_{t\downarrow 0} u(x,t) = \phi(x)$ for every $x \in \mathbb{R}$.*

5.5.1. *Proof of the theorem.* We check that u is in $C^\infty(\mathbb{R}\times(0,\infty))$. Let $S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\left(\frac{x}{\sqrt{4\pi kt}}\right)^2}$.

We show that

$$\frac{\partial}{\partial x} \int S(x-y,t)\phi(y)dy = \int \frac{\partial}{\partial x} S(x-y,t)\phi(y)dy.$$

Recall that

$$\frac{\partial}{\partial x} S(x-y,t)\phi(y) = \lim_{h\to 0} \frac{1}{h} \left[S(x+h-y,t) - S(x-y,t) \right] \phi(y).$$

By the *dominated convergence theorem* for integrals we can pull this limit inside the integral if the modulus of the limit is bounded by an integrable function. This is indeed the case

$$\left| \frac{\partial}{\partial x} S(x-y,t)\phi(y) \right| \leq \left| -\frac{1}{\sqrt{4\pi kt}} \frac{x-y}{2kt} e^{-\frac{(x-y)^2}{4kt}} \right| M \leq \frac{M}{\sqrt{4\pi kt}} \frac{|x-y|}{2kt} e^{-\frac{|x-y|^2}{4kt}}.$$

The term on the right hand side has a finite integral on $\mathbb{R}$. Hence

$$\frac{\partial}{\partial x} u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} S(x-y,t)\phi(y)dy.$$

All other derivatives of higher order in x and t will work the same way: we always get an estimate by function of the form

$$C|y-x|^n e^{-\tilde{C}(x-y)^2}$$

that has finite integral on $\mathbb{R}$.

5.5.2. *Checking the initial condition.* We also know that u satisfies $u_t = ku_{x,x}$ because $S(x,t)$ does.

Hence, we only need to prove that u satisfies the initial condition for $t \downarrow 0$.

Consider

$$\begin{aligned} u(x,t) - \phi(x) &= \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy - \int_{-\infty}^{\infty} S(x-y,t)\phi(x)dy \\ &= \int_{-\infty}^{\infty} S(x-y,t)(\phi(y)-\phi(x))dy. \end{aligned}$$

Since ϕ is continous in x, for $\epsilon > 0$ we can choose $\delta > 0$ such that

$$|y-x| \leq \delta \ \Rightarrow |\phi(x) - \phi(y)| \leq \epsilon$$

Hence

$$
\begin{aligned}
|u(x,t)-\phi(x)| \leq & \int_{\{y\in\mathbb{R}:|x-y|>\delta\}} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{2kt}} \underbrace{|\phi(x)-\phi(y)|}_{\leq 2M} dy \\
& + \int_{\{y\in\mathbb{R}:|x-y|\leq\delta\}} S(x-y,t) \underbrace{|\phi(x)-\phi(y)|}_{\leq \epsilon} dy \\
\leq & \frac{2M}{\sqrt{4\pi}} \int_{\{z\in\mathbb{R}:|z|\geq \frac{\delta}{\sqrt{kt}}\}} e^{-\frac{z^2}{4}} dz + \epsilon.
\end{aligned}
$$

It follows that

$$\limsup_{t\downarrow 0} |u(x,t)-\phi(x)| \leq \epsilon$$

Since $\epsilon > 0$ was arbitrary, it follows that $\lim_{t\downarrow 0} |u(x,t)-\phi(x)| = 0$. □

5.5.3. *Additional Remarks.*

- Decay of the solution for $t \to \infty$.
  For $\phi \in C^0(\mathbb{R})$ with $|\phi| \leq M$ we have
  $$|u(x,t)| \leq \int_{-\infty}^{\infty} S(x-y,t)|\phi(y)|dy \leq \frac{M}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} dy \leq \frac{M}{\sqrt{4\pi kt}} \to 0.$$
  In particular, this means the backwards diffusion equation
  $$u_t = -k u_{x,x} \text{ on } \mathbb{R}\times(0,\infty)$$
  is not well-posed because stability fails.
- About uniqueness again: Let $\phi_1, \phi_2 \in C^0(\mathbb{R})$ with $|\phi_1|, |\phi_2| \leq M$.
  We saw that in the class of solutions with $u(x,t) \to 0$ for $|x| \to \infty$ we find a unique solution.
  But if we drop this assumption uniquness might fail: There are solutions of the heat equation with $u(x,t) \to 0$ for $t \downarrow 0$ for all $x \in \mathbb{R}$.
  See also exercise 10 on page 399 in Choksi's Lecture Notes for an example that hints to nonuniquness.

Lecture 15

6. Separtion of Variables

6.1. **Separation of Variables: Wave equation.** Consider the wave equation on an interval $[0, l]$:

$$(37)\qquad \begin{array}{rclcc} u_{t,t} & = & c^2 u_{x,x} & \text{on} & (0,l)\times\mathbb{R} \\ u(x,0) & = & \phi(x) & u_t(x,0)=\psi(x) & \text{for } x\in(0,l). \end{array}$$

We assume **Dirichlet boundary conditions**

$$\text{DC:}\quad u(0,t) = u(l,t) = 0 \text{ for } t \in \mathbb{R}$$

Remark 6.1. Recall that the PDE is linear and homogeneous. Therefore, if u_1 and u_2 are solutions to (37), then also $u_1 + u_2 = u$ is a solution to (37).
This is called *superpositon principle.*

We will build the general solution for (37) from special ones that are easier to find.

The easier solutions we want to find have the following structure:

$$u(x,t) = X(x) \cdot T(t)$$

(*Separation of variables*). Assuming this particular structure the PDE reduces to

$$X(x)T''(t) = c^2 X''(x)T(t)$$

This yields

$$-\frac{T''}{c^2 T} = -\frac{X''}{X} = \lambda.$$

for a constant $\lambda \in \mathbb{R}$.
The last equation yields two separate differential equations for T and X:

$$-\frac{T''}{c^2 T} = \lambda \quad \text{and} \quad -\frac{X''}{X} = \lambda.$$

For the moment let us assume $\lambda > 0$.
Why can we do that?
If $\lambda = 0$, we have that $X'' = 0$. It follows that $X(x) = C + Dx$.
By the boundary condition $X(0) = X(l) = 0$ it follows $X \equiv 0$ and $u \equiv 0$.
If $\lambda > 0$ we set $\beta = \sqrt{\lambda} > 0$:

$$T'' + \beta^2 c^2 T = 0 \quad \& \quad X'' + \beta^2 X = 0.$$

We can easily see that the last two equations have the following general solution

$$T(t) = A\cos(\beta ct) + B\sin(\beta ct) \quad \& \quad X(x) = C\cos(\beta x) + D\sin(\beta x).$$

for real constants $A, B, C, D \in \mathbb{R}$.
In particular, any $u = T \cdot X$ with such T and X solves $u_{t,t} = c^2 u_{x,x}$.
Now, we would like to choose the constants A, B, C, D accordingly to given initial and boundary conditions.

For a given time $t_0 \in \mathbb{R}$ a solution $u(t_0, x) = T(t_0)X(x)$ must satify the boundary condition:

$$0 = X(0) = C \qquad 0 = X(l) = D\sin(\beta l)$$

We are not interested in the trivial solution with $D = C = 0$. Hence $\beta l = n\pi$ for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, the roots of the sine function. Or equivalently

$$\lambda_n = (\beta_n)^2 = \left(\frac{n\pi}{l}\right)^2.$$

Hence

$$X_n(x) = \sin\left(\frac{n\pi}{l}x\right), \qquad n \in \mathbb{N},$$

is a family of distinct solutions where $D = 1$.

Note that each sine function may be multiplied with a function that is contant in x to obtain another solution.

We obtain an infinite number of solutions of the form

$$u_n(x,t) = \left(A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right) \sin\left(\frac{n\pi}{l}x\right)$$

for constants $A_n, B_n \in \mathbb{R}$.

Moreover, any **finite** sum of these solutions is also a solution:

$$u(x,t) = \sum_{i=1}^{k} \left(A_{n_i} \cos\left(\frac{n_i\pi}{l}ct\right) + B_{n_i} \sin\left(\frac{n_i\pi}{l}ct\right)\right) \sin\left(\frac{n_i\pi}{l}x\right).$$

Now, assume $\lambda < 0$. We will rule out this case. We set $\beta = \sqrt{-\lambda}$.

Again we can easily see that the general solutions for $T'' + \lambda T = 0$ and $X'' + \lambda X = 0$ are given by

$$T(t) = A\cosh(\beta ct) + B\sinh(\beta ct) \;\;\&\;\; X(x) = C\cosh(\beta x) + D\sinh(\beta x)$$

The boundary condition again implies $0 = X(l) = D\sinh(\beta x)$.

This can only occur if $D = 0$.

A similar argument also rules out the case $\lambda \in \mathbb{C}\backslash(0,\infty) \times \{0\}$ (complex numbers).

Hence, the relevant numbers λ in the problem are positive.

We note that we also could assume **Neumann boundary conditions**

$$\text{NC: } u_x(t,0) = u_x(t,l) = 0 \text{ on } \mathbb{R}.$$

for the PDE in the beginning.

Then the considerations for λ are similar as for Dirichlet case. We can rule out that $\lambda < 0$.

In the case $\lambda = 0$, the equation for X becomes $X'' = 0$. Again we have

$$X(x) = C + Dx, \quad C, D \in \mathbb{R}$$

Together with the Neumann boundary condition $X_x(0) = X_x(l) = 0$ we see that for any $C \in \mathbb{R}$ the constant function $X(x) = C$ is a solution. For $\lambda = \beta^2 > 0$ we have the solutions

$$X(x) = C\cos(\beta x) + D\sin(\beta X)$$

The Neumann boundary condition imply that

$$0 = X_x(0) = -C\beta\sin(\beta 0) + D\beta\cos(\beta 0) = D$$

Hence $D = 0$ and $X_x(l) = -C\beta\sin(\beta l)$. Hence, we have again $\beta l = n\pi$ and we define a family of solutions

$$\tilde{X}_n(x) = \cos\left(\frac{n\pi}{l}x\right)$$

where we set $C = 1$.

A family of solutions for the PDE with Neumann boundary conditions is then

$$u_n(x,t) = \left(A\cos(\frac{n\pi}{l}ct) + B\sin(\frac{n\pi}{l}ct)\right)\cos\left(\frac{n\pi}{l}x\right)$$

And again finite sums of these solutions are also solultions

$$u(x,t) = \sum_{i=1}^{k} \left(A \cos(\frac{n_i \pi}{l} ct) + B \sin(\frac{n_i \pi}{l} ct) \right) \cos\left(\frac{n_i \pi}{l} x\right).$$

Finally, we want to bring the inital conditions ϕ and ψ into play.

For this we go back to the Dirichlet condition.

The solution given by the previous formula solves the initial value problem if

$$\phi(x) = u(x,0) = \sum_{i=1}^{k} A_{n_i} \sin\left(\frac{n_i \pi}{l} x\right)$$

and

$$\psi(x) = u_t(x,0) = \sum_{i=1}^{k} \frac{n_i \pi c}{l} B_{n_i} \sin\left(\frac{n_i \pi}{l} x\right)$$

Question 6.2. *Can we approximate any continuous function ϕ with $\phi(0) = \phi(l) = 0$ by trigonometric polynomials of the form*

$$\tilde{\phi}(x) = \sum_{i=1}^{k} A_{n_i} \sin\left(\frac{n_i \pi}{l} x\right)$$

What does approximation mean in this context?

And do the solutions w.r.t. $\tilde{\phi}$ approximate the solution w.r.t. ϕ?

Or can we maybe write any continuous function ϕ as series of the form

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l} x\right).$$

Lecture 16

Last Lecture. The wave equation on $[0, l]$

$$(38)\qquad \begin{array}{rcll} u_{t,t} & = & c^2 u_{x,x} & \text{on } (0,l)\times\mathbb{R} \\ u(x,0) & = & \phi(x) & \text{on } [0,l] \\ u_t(x,0) & = & \psi(x) & \text{on } [0,l] \end{array}$$

and with **Dirichlet boundary conditions (DC)**

$$u(0,t) = u(l,t) = 0 \quad \forall t \in \mathbb{R}$$

or with **Neumann boundary conditions (NC)**

$$u_x(0,t) = u_x(l,t) = 0 \quad \forall t \in \mathbb{R}.$$

Via separation of variable we found a family of special solutions.

For (38) with DC we found special solutions of the form $u_n(x,t) = T_n(t)X_n(t)$, $n \in \mathbb{N}$, where

$$T_n(t) = A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)$$

and the functions $X_n(x) := \sin\left(\frac{n\pi}{l}x\right)$ solve the following ODE boundary problem

$$X_n'' + \left(\frac{n\pi}{l}\right)^2 X_n = 0 \quad \text{with } X_n(0) = X_n(l) = 0, \quad n \in \mathbb{N}.$$

6.2. **Superposition principle.** Any finite linear combination of u_n is also a solution of (38) with DC:

$$u(x,t) := \sum_{i=1}^{k}\left(A_{n_i}\cos\left(\frac{n_i\pi}{l}ct\right) + B_{n_i}\sin\left(\frac{n_i\pi}{l}ct\right)\right)\sin\left(\frac{n_i\pi}{l}x\right) \quad \text{where } n_1,\dots,n_k \in \mathbb{N}.$$

u has initial conditions

$$\phi(x) = u(x,0) = \sum_{i=1}^{k} A_{n_i}\sin\left(\frac{n_i\pi}{l}x\right), \quad \psi(x) = u_t(x,0) = \sum_{i=1}^{k}\frac{n\pi c}{l}B_{n_i}\sin\left(\frac{n_i\pi}{l}x\right).$$

For (38) with NC we found that

$$\tilde{u}_n(x,t) = T_n(t)\tilde{X}_n(x)\ \forall n \in \mathbb{N}\cup\{0\}.$$

is a solution.

The functions $\tilde{X}_n(x) := \cos\left(\frac{n\pi}{l}x\right)$, $n \in \mathbb{N}$, solve the following ODE boundary problem

$$X_n'' + \left(\frac{n\pi}{l}\right)^2 X_n = 0 \quad \text{with } (X_n)_x(0) = (X_n)_x(l) = 0, \quad n \in \mathbb{N}\cup\{0\}.$$

where we set $X_0(x) = 1$.

Again we have that $T_n(t)$ solves $T'' + \left(\frac{n\pi}{l}\right)^2 cT = 0$. Therefore

$$T_n(t) = A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right) \text{ for } n \in \mathbb{N} \text{ and for } A_n,\ B_n \in \mathbb{R}.$$

But also $T_0(t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t$ for $A_0, B_0 \in \mathbb{R}$.

6.3. **Eigenvalues and Eigenfunction.** The constants $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ are called eigenvalues.

The functions $X_n(x)$ are called eigenfunctions of the the differential operator

$$L : V \to C^0([0,l]), \quad L\phi = -\frac{\partial^2}{\partial x^2}\phi \quad \text{for } V = \left\{\phi \in C^2([0,l]) : \phi(0) = \phi(l) = 0\right\}.$$

The differential equality that determines X_n has the form of an **eigenvalue equation**

$$LX_n = \lambda_n X_n.$$

Similar, the functions $\tilde{X}_n(x)$ are called eigenfunctions for the differential operator

$$\tilde{L} : \tilde{V} \to C^0([0,l]), \quad \tilde{L}\phi = -\frac{\partial^2}{\partial x^2}\phi \quad \text{for } \tilde{V} = \left\{\phi \in C^2([0,l]) : \phi_x(0) = \phi_x(l) = 0\right\}.$$

The terminology is motivated from Linear Algebra:

Consider a matrix $A \in \mathbb{R}^{n\times n}$ we say $\lambda \in \mathbb{R}$ is an eigenvalue of A if there exists $v \neq 0$ such that

$$Av = \lambda v$$

Given an eigenvalue λ for A the set of eigenvectors E_λ is a vector space.

If we can find n different eigenvalue $\lambda_1, \dots, \lambda_n$ then

$$E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_n} = \mathbb{R}^n$$

Hence, for every vector w there are unique eigenvectors $v_i \in E_{\lambda_i}$ such that

$$W = v_1 + \cdots + v_n.$$

Let us go back to (38) with the Dirichlet Boundary Condition. Consider an infinite serie of the form

$$u(x,t) := \lim_{N\to\infty} \sum_{n=1}^{N} u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right) \sin(\lambda_n x). \tag{39}$$

Remark 6.3. When does such a series converge uniformily?

Since

$$\left|\sum_{n=1}^{N} u_n(x,t)\right| \leq \sum_{n=1}^{N} (|A_n| + |B_n|) \leq \sum_{n=1}^{\infty} |A_n| + \sum_{n=1}^{\infty} |B_n|$$

the series (39) converges uniformily provided $\sum_{n=1}^{\infty} |A_n|, \sum_{n=1}^{\infty} |B_n| < \infty$. Indeed

$$\max_{x\in[0,l]} \left|\sum_{n=1}^{M} u_n(x,t) - \sum_{n=1}^{N} u_n(x,t)\right| = \max_{x\in[0,l]} \left|\sum_{n=N+1}^{M} u_n(x,t)\right| \leq \sum_{n=N+1}^{M} (|A_n| + |B_n|) \to 0$$

if $N < M \to \infty$. Now also recall the following theorem about differentiation of series

Theorem 6.4. *Let $f_n(x)$ a sequence of functions on $[0,l]$ that are differentiable. Assume $\sum_{n=1}^{\infty} f_n(x)$ is converging uniformily.*

If $\sum_{n=1}^{\infty} f_n'(x)$ is uniformily convergent then it follows that f is differentiable on $[0,l]$ and

$$f'(x) = \sum_{n=1}^{\infty} f'(x).$$

Hence, the partial derivatives u_x and u_t exist and satisfy

$$u_x(x,t) = \sum_{n=1}^{\infty} \lambda_n \left(A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right) \cos(\lambda_n x)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \lambda_n c\left(-A_n \sin(\lambda_n ct) + B_n \cos(\lambda_n ct)\right) \sin(\lambda_n x)$$

provided $\sum_{n=1}^{\infty} \lambda_n |A_n|, \sum_{n=1}^{\infty} \lambda_n |B_n| < \infty$.

Similar, the second partial derivatives $u_{x,x}$ and $u_{t,t}$ exist and satisfy

$$u_{x,x}(x,t) = \sum_{n=1}^{\infty} \lambda_n^2 \left(A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right) \cos(\lambda_n x)$$

$$u_{t,t}(x,t) = \sum_{n=1}^{\infty} \lambda_n^2 c^2 \left(-A_n \sin(\lambda_n ct) + B_n \cos(\lambda_n ct)\right) \sin(\lambda_n x)$$

provided $\sum_{n=1}^{\infty} \lambda_n^2 |A_n|, \sum_{n=1}^{\infty} \lambda_n^2 |B_n| < \infty$.

Consequently, since each function

$$u_n = (A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)) \sin(\lambda_n x)$$

satisfies the PDE $(u_n)_{t,t} = c^2 (u_n)_{x,x}$ with the Dirichlet boundary condition, the series u satisfies the same PDE also with Dirichlet boundary condition $u(0,t) = u(l,t) = 0$.

In the same way we can construct solutions to the PDE with the Neumann boundary condition. We only have to replace

$$X_n, n \in \mathbb{N} \quad \text{with } \tilde{X}_n, n \in \mathbb{N} \cup \{0\}.$$

Moreover u satisfies the following initial condition

$$u(x,0) = \lim_{N\to\infty} \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi}{l} x\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l} x\right) =: \phi(x) \tag{40}$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin\left(\frac{n\pi}{l} x\right) =: \psi(x) \tag{41}$$

Note that these series are converge uniformily and hence are well-defined because we assumed

$$\sum_{n=1}^{\infty} |A_n|, \ \sum_{n=1}^{\infty} |B_n|, \ \sum_{n=1}^{\infty} \lambda_n |A_n|, \ \sum_{n=1}^{\infty} \lambda_n |B_n|, \ \sum_{n=1}^{\infty} \lambda_n^2 |A_n|, \ \sum_{n=1}^{\infty} \lambda_n^2 |B_n| < \infty.$$

Question 6.5. *What kind of data pairs ϕ, ψ can be expanded as series for coefficients A_n and B_n as above?*

In the same way we can find solutions for the PDE with NC:

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty} (A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)) \cos(\lambda_n x).$$

The initial conditions are $\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda_n x)$ and $\frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \lambda_n c \cos(\lambda_n x)$.

Let us consider the analogous problem for diffusion on $[0,l]$:

$$\begin{array}{rcll} u_t & = & k u_{x,x} & \text{on } (0,l) \times (0,\infty) \\ u(x,0) & = & \phi(x) & \text{on } (0,l) \end{array}$$

with Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0 \quad \forall t \in \mathbb{R}$$

or with **Neumann boundary conditions**

$$u_x(0,t) = u_x(l,t) = 0 \quad \forall t \in \mathbb{R}.$$

We can again apply the methode of Separation-of-Variables: We consider a solution of the form

$$u(x,t) = T(t)X(x).$$

This leads to

$$-\frac{T'}{kT} = -\frac{X''}{X} = \lambda.$$

Again we see easily that λ must be constant and T and X solve

$$T' + \lambda k T = 0 \quad \& \quad X'' + \lambda X = 0.$$

The general solution for X is the same as before. In particular, we can have $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ and the set of solutions $X_n(x) = \sin\left(\frac{n\pi}{l}x\right)$. The general solution for T in this case is

$$T(t) = Ae^{-\left(\frac{n\pi}{l}\right)^2 kt} \text{ for } A \in \mathbb{R}.$$

Hence, as before a family of special solutions of the diffuison equation with DC is given by

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \underbrace{\sin\left(\frac{n\pi}{l}x\right)}_{=:X_n(x)}, \ n \in \mathbb{N} \text{ and } A_n \in \mathbb{R}.$$

where X_n are as before. Then

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 kt} \sin(\lambda_n x)$$

solves the diffusion equation with DC and inital data $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)$ provided

$$\sum_{n=1}^{\infty} |A_n|, \quad \sum_{n=1}^{\infty} \lambda_n |A_n|, \quad \sum_{n=1}^{\infty} \lambda_n^2 |A_n| < \infty.$$

For NC we consider $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ with $n \in \mathbb{N} \cup \{0\}$ and the corresponding solutions $\tilde{X}_n$ of $\tilde{X}_n'' + \lambda X_n = 0$. Precisely, we set $\tilde{X}_n(x) = \cos(\lambda_n x)$ and $X_0(x) = 1$.

Again we also have to consider $\lambda_0 = 0$. In particular, for T we also consider the solutions of

$$T' = 0 \ \Leftrightarrow T_0(0) = \frac{1}{2}A_0 \in \mathbb{R}$$

Then, a family of special solutions of the diffusion equation with NC is given by

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \underbrace{\cos\left(\frac{n\pi}{l}x\right)}_{=:\tilde{X}_n(x)}, \ n \in \mathbb{N} \text{ and } A_n \in \mathbb{R}.$$

andn $u_0(x,t) = \frac{1}{2}A_0$. Again

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 kt} \cos(\lambda_n x)$$

is a solution to the diffusion equation with NC for the initial data

$$u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda x).$$

provided

$$\sum_{n=1}^{\infty} |A_n|, \quad \sum_{n=1}^{\infty} \lambda_n |A_n|, \quad \sum_{n=1}^{\infty} \lambda_n^2 |A_n| < \infty.$$

7. Fourier Series

We encouter the following question

Question 7.1. *Given a function ϕ on $[0,l]$ can we find a sequence $(A_n)_{n\in\mathbb{N}}$ such that*

$$\phi(x) = \lim_{N\to\infty}\sum_{n=1}^{N} A_n \sin(\lambda_n x) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x),\ \ x\in[0,l]\ ?$$

(where $\lambda_n = \left(\frac{n\pi}{l}\right)^2$). We call the series on the right hand side the Fourier sine series.
Or can we find a sequence $(B_n)_{n\in\mathbb{N}}$ such that

$$\phi(x) = \frac{1}{2}A_0 + \lim_{N\to\infty}\sum_{n=1}^{N} A_n \cos(\lambda_n x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda_n x), x\in[0,l]\ ?$$

We call the series on the right hand side the Fourier cosine series.

7.1. **How can we determine the coefficients A_n?** We perform the following forma calculations:

$$\int_0^l \phi(x)\sin(\lambda_m x)dx \ = \int_0^l \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)\sin(\lambda_m x)dx \ = \sum_{n=1}^{\infty}\int_0^l A_n \sin(\lambda_n x)\sin(\lambda_m x)dx$$

Let consider a single term in the sum on the right hand side:

$$A_n\int_0^l \sin(\lambda_n x)\sin(\lambda_m x)dx = A_n\int_0^l \frac{1}{2}\left(\cos((\lambda_n-\lambda_m)x) - \cos((\lambda_n+\lambda_m)x)\right)dx$$

Here we used the first of the following identities

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y), \quad \sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y).$$

The first identity gives

$$\begin{aligned}\cos(x+y) - \cos(x-y) &= \cos(x)\cos(y) - \sin(x)\sin(y) - \cos(x)\cos(-y) + \sin(x)\sin(-y)\\ &= -2\sin(x)\sin(y)\end{aligned}$$

Then, if $n\neq m$, we compute

$$\begin{aligned}A_n\int_0^l \sin(\lambda_n x)\sin(\lambda_m x)dx &= A_m\frac{l}{2\pi}\int_0^l \frac{\pi}{l}\cos\left(\frac{(n-m)\pi}{l}x\right) - \cos\left(\frac{(n+m)\pi}{l}\right)dx\\ &= \frac{l}{2\pi}A_n\int_0^\pi \left(cos((n-m)x) - \cos((n+m)x)\right)dx\\ &= \frac{l}{2\pi}A_n\left[\frac{1}{n-m}\sin((n-m)x) - \frac{1}{n+m}\sin((n+m)\right]_0^\pi = 0\end{aligned}$$

If $n=m$, then

$$A_m\int_0^l \sin(\lambda_m x)^2dx = A_m\frac{l}{2\pi}\int_0^\pi [1-\cos(2mx)]\,dx = A_m\frac{l}{2} - A_m\frac{l}{2\pi}\left[\frac{1}{2m}\sin(2mx)\right]_0^\pi = \frac{l}{2}A_n$$

Hence

$$\begin{aligned}\int_0^l \phi(x)\sin(\lambda_m x)dx &= \int_0^l \sum_{n=1}^{\infty} A_n\sin(\lambda_n x)\sin(\lambda_m x)dx\\ &= \sum_{n=1}^{\infty} A_n\int_0^l \sin(\lambda_n x)\sin(\lambda_m x)dx = \frac{l}{2}A_m.\end{aligned}$$

Remark 7.2. $A_m = \frac{2}{l}\int_0^l \phi(x)\sin(\lambda_m x)dx = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{m\pi}{l}x\right)dx.$

This is the Fourier sine coefficient for ϕ.

By the same formal calculation we also compute the Fourier cosine coefficient for ϕ.

Precisely:

$$\int_0^l \phi(x)\cos(\lambda_m x)dx = \int_0^l \sum_{n=1}^{\infty} \tilde{A}_n \cos(\lambda_n x)\cos(\lambda_m x)dx$$
$$= \sum_{n=1}^{\infty} \tilde{A}_n \int_0^l \cos(\lambda_n x)\cos(\lambda_m x)dx$$

Let us again consider a single term in the sum on the right hand side with $n, m \geq 1$:

$$\tilde{A}_n \int_0^l \cos(\lambda_n x)\cos(\lambda_m x)dx = \tilde{A}_n \frac{1}{2}\int_0^l \cos\left(\frac{(n+m)\pi}{l}x\right) + \cos\left(\frac{(n-m)\pi}{l}x\right)dx$$
$$= \tilde{A}_n \frac{l}{2\pi}\int_0^\pi \cos((n+m)x) + \cos((n-m)x)dx$$

For $n \neq m$ the right hand side in the last term is

$$= \tilde{A}_n \frac{l}{2\pi}\left[\frac{1}{n+m}\sin((n+m)x) + \frac{1}{n-m}\sin((n-m)x)\right]_0^\pi = 0.$$

For $n = m$ we obtain

$$= \frac{l}{\pi 2}\int_0^\pi \left[\cos(2mx) + 1\right]dx = \frac{l}{2\pi}\tilde{A}_m\left[\frac{1}{2n}\sin(2mx)\right]_0^\pi + \frac{l}{2}\tilde{A}_m = \frac{l}{2}\tilde{A}_m.$$

A computation yields the same conclusion even when $n = 0$ or $m = 0$.

We obtain that

Remark 7.3. $\tilde{A}_0 = \frac{2}{l}\int_0^l \phi(x)dx, \quad \tilde{A}_m = \frac{2}{l}\int_0^l \phi(x)\cos(\lambda_m x)dx = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{m\pi}{l}x\right)dx.$

Definition 7.4. The Fourier sine series of ϕ is defined

$$\sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi}{l}x\right)dx\right]\sin\left(\frac{n\pi}{l}x\right) =: \mathcal{S}(\phi)$$

Similar the Fourier cosine series of ϕ is defined

$$\sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi}{l}x\right)dx\right]\cos\left(\frac{n\pi}{l}x\right) =: \mathcal{C}(\phi)$$

Finally the **full** Fourier coefficients are (we abuse notation at this point)

Remark 7.5. $B_m = \frac{1}{l}\int_{-l}^l \phi(x)\cos\left(\frac{m\pi}{l}x\right)dx.$

$$A_0 = \frac{1}{l}\int_{-l}^l \phi(x)dx, \quad A_m = \frac{1}{l}\int_{-l}^l \phi(x)\cos\left(\frac{m\pi}{l}x\right)dx.$$

Definition 7.6. The Fourier series of ϕ is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n \sin(\frac{n\pi}{l}x)dx + B_n\cos(\frac{n\pi}{l}x)\right] = \mathcal{F}(\phi)$$

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7.2. **Orthogonality and General Fourier Series.** Consider two **continuous functions** $f, g : [a, b] \to \mathbb{R}$ that are **square integrable:**

$$\|f\|_2^2 = \int_a^b |f(x)|^2 dx, \quad \|g\|_2^2 = \int_a^b |g(x)|^2 dx < \infty$$

We define the **inner product** between f and g as the integral of their product:

$$(f, g) = \int_a^b f(x)g(x)dx \tag{42}$$

The product $g(x)f(x)$ is integrable because of the *Cauchy-Schwartz inequality*:

$$\int_a^b |f(x)g(x)|dx \leq \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx} = \|f\|_2 \|g\|_2 < \infty.$$

We say that two square integrable functions f and g are **orthogonal** if $(f, g) = 0$.

Note that a real valued continuous function f is never orthogonal to itself unless $f = 0$.

Recall the case of an inner product (v, w) on $\mathbb{R}^n$, for instance $v_1 w_1 + \cdots + v_n w_n$.
The number $\|v\| = \sqrt{(v, v)}$.

A basis $v_1, \dots, v_n$ of V is orthonormal if $\|v_i\| = 1$, $i = 1, \dots, n$, and $(v_i, v_j) = 0$, $i \neq j$. Then

$$w = \sum_{i=1}^n (v_i, w) v_i \text{ and } \|w\|^2 = \sum_{i=1}^n |(v_i, w)|^2.$$

For instance, $v_1, \dots, v_n$ can be the eigenvectors of a symmetric operator $A : \mathbb{R}^n \to \mathbb{R}^n$. The theory of Fourier series translates this idea to an infinite dimensional context.

Let $[a, b] = [0, l]$. Let us go back to the operator

$$Lf = -\frac{\partial^2}{\partial x^2} f \quad \text{for} \quad f \in C^2([0, l]).$$

We saw that

$$\sin\left(\frac{n\pi}{l} x\right), \; n \in \mathbb{N}$$

was a set of eigenfunctions for the operator L with Dirichlet boundary conditions, and

$$1, \quad \cos\left(\frac{n\pi}{l} x\right), \; n \in \mathbb{N}$$

was a set of eigenfunctions for the same operator with Neumann boundary conditions.

To determine Fourier sine coefficients we computed that

$$\int_0^l \sin\left(\frac{n\pi}{l} x\right) \sin\left(\frac{m\pi}{l} x\right) dx = 0 \quad \text{for} \quad n \neq m \in \mathbb{N}.$$

Also we can compute that

$$\int_0^l \cos\left(\frac{n\pi}{l} x\right) \cdot 1 dx = \int_0^l \cos\left(\frac{n\pi}{l} x\right) \cos\left(\frac{m\pi}{l} x\right) dx = 0 \quad \text{for} \quad n \neq m \in \mathbb{N}.$$

Hence, these eigenvectors are orthogonal w.r.t. $(\cdot, \cdot)$.

7.2.1. *General Fourier series.* Let us consider two eigenfunctions X_1 and X_2 of $L = -\frac{d^2}{dx^2}$ on $[a, b]$ for eigenvalues $\lambda_1 \neq \lambda_2$.

We don't specify boundary conditions yet. We can compute the following

$$(-X_1'X_2 + X_1X_2')' = -X_1''X_2 + X_1X_2''$$

Integration over $[a, b]$ yields

$$\int_a^b [-X_1''(x)X_2(x) + X_1(x)X_2''(x)]\,dx = -X_1'(x)X_2(x) + X_1(x)X_2'(x)\Big|_a^b$$
$$= -X_1'(b)X_2(b) + X_1(b)X_2'(b) + X_1'(a)X_2(a) - X_1(a)X_2'(a).$$

If the right hand side is 0 we have that

$$0 = -\int_a^b X_1''(x)X_2(x)dx - \int_a^b X_1(x)X_2''(x)dx = (LX_1, X_2) - (X_1, LX_2) = (\lambda_1 - \lambda_2)(X_1, X_2)$$

Since $\lambda_1 \neq \lambda_2$, $(X_1, X_2) = 0$. Hence X_1 and X_2 are othogonal.

Remark 7.7. **Question:** When do we have

$$-X_1'(b)X_2(b) + X_1(b)X_2'(b) + X_1'(a)X_2(a) - X_1(a)X_2'(a) = 0 \text{ ?}$$

Remark 7.8. For instance, for Dirichlet or Neumann boundary conditions on $[0, l] = [a, b]$.

But also for periodic boundary conditions: $f \in C^1(\mathbb{R})$ satisfies a periodic boundary conditions with period $l > 0$ if $f(x + nl) = f(x)$ for all $x \in \mathbb{R}$. Hence

$$f(a) = f(b) \;\&\; f'(a) = f'(b).$$

In general, we could consider boundary conditions of the form

$$\begin{Bmatrix} \alpha_1 f(a) + \beta_1 f(b) + \gamma_1 f'(a) + \delta_1 f'(b) = 0 \\ \alpha_2 f(a) + \beta_2 f(b) + \gamma_2 f'(a) + \delta_2 f'(b) = 0 \end{Bmatrix} \tag{43}$$

for 8 independent constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$.

Definition 7.9. The set of boundary conditions (43) are called symmetric if

$$f'(x)g(x) - f(x)g'(x)\Big|_a^b = f'(b)g(b) - f(b)g'(b) - f'(a)g(a) + f(a)g'(a) = 0$$

for any pair of functions that satisfy (43).

Hence, we proved the following theorem.

Theorem 7.10. *Eigenfunctions of $-\frac{\partial^2}{\partial x^2}$ with symmetric boundary conditions for eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal.*

By explicite compuations we saw that this is true for L with Dirichlet boundary conditions on $[0, l]$ where the eigenfunctions are $\sin\left(\frac{n\pi}{l}x\right)$, $n \in \mathbb{N}$.

Remark 7.11. If there are 2 eigenfunctions X_1 and X_2 for the same eigenvalue λ, then either $X_1 = cX_2$ for some constant c, or they can be made orthogonal by the Gram-Schmidt orthogonalization procedure.

Considering $L = -\frac{\partial^2}{\partial x^2}$ with periodic boundary conditions on $[-l, l]$. There are eigenfunctions

$$\sin\left(\frac{n\pi}{l}x\right), \quad \cos\left(\frac{n\pi}{l}x\right)$$

for the same eigenvalue $\left(\frac{n\pi}{l}\right)^2$ that are orthogonal.

But also any linear combination is again an eigenfunction for the same eigenvalue. In particular

$$\sin\left(\frac{n\pi}{l}x\right), \quad \cos\left(\frac{n\pi}{l}x\right) + \sin\left(\frac{n\pi}{l}x\right).$$

But they are not orthogonal.

7.2.2. *General Fourier coefficients.* If a continuous and integrable function ϕ is given by an infinite converging serie $\sum_{n=1}^{\infty} A_n X_n$ for eigenfunctions X_n of $L = -\frac{\partial^2}{\partial x^2}$ on $[a, b]$ with symmetric boundary conditions, then the coefficients are determined by the formula

$$A_m = \frac{1}{\|X_m\|_2^2}(X_m, \phi) = \frac{1}{\int_a^b (X_m)^2(x)dx} \int_a^b \phi(x) X_m(x) dx.$$

Indeed

$$(\phi, X_m) = \left(\sum_{n=1}^{\infty} A_n X_n, X_m\right) = \sum_{n=1}^{\infty} A_n (X_n, X_m) = A_m (X_m, X_m) = A_m \|X_m\|_2^2.$$

For instance, if we consider the set $\sin\left(\frac{n\pi}{l}x\right)$ of eigenfunctions $L = -\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions, we computed

$$A_m = \frac{1}{\int_0^l \left(\sin\left(\frac{m\pi}{l}x\right)\right)^2 dx} \int_0^l \phi(x) \sin\left(\frac{m\pi}{l}x\right) dx \text{ where } \int_0^l \left(\sin\left(\frac{m\pi}{l}x\right)\right)^2 dx = \frac{l}{2}.$$

For periodic boundary conditions on $[-l, l]$ the eigenfunctions are $1, \cos\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right)$ and the Fourier coefficients are

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x) \sin\left(\frac{n\pi}{l}x\right), \ n \in \mathbb{N}, \ \tilde{A}_0 = \frac{1}{l}\int \phi(x)dx, \ \tilde{A}_n = \frac{1}{l}\int_{-l}^{l} \phi(x) \cos\left(\frac{n\pi}{l}x\right) dx, \ n \in \mathbb{N}.$$

Problem/Questions: In which sense does $\sum_{n=1}^{\infty} A_n X_n$ converge? And why does the second equality hold in the previous equation?

7.3. **Notions of convergence.**

Definition 7.12 (Pointwise and uniform convergence)**.** We say an infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to a function f in (a, b) if

$$\left|f(x) - \sum_{n=1}^{N} f_n(x)\right| \to 0 \quad \text{as} \quad N \to \infty \text{ for all } x \in (a, b).$$

We say the series converges uniformly to f in $[a, b]$ if

$$\max_{x\in[a,b]} \left|f(x) - \sum_{n=1}^{N} f_n(x)\right| \to 0 \quad \text{as} \quad N \to \infty$$

Note that for the notion of uniform convergence we include a and b.

Definition 7.13 (Mean square convergence)**.** The serie $\sum_{n=1}^{\infty} f_n(x)$ converges in mean square (or L^2) sense to f in (a, b) if

$$\int_a^b \left|f(x) - \sum_{n=1}^{N} f_n(x)\right|^2 dx \to 0 \quad \text{as} \quad N \to \infty.$$

Remark 7.14. We have: uniform convergence $\Rightarrow$ pointwise and mean square convergence.

But in general not the other way.

Example 7.15. Consider $f_n(x) = (1-x)x^{n-1}$ on $[0,1]$. Then

$$\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} \left(x^{n-1} - x^n\right) = 1 - x^N \to 1 \text{ as } N \to \infty \quad \text{for all } x \in [0,1].$$

But the convergence is not uniform because

$$\max_{x\in[0,1]} \left|1 - (1 - x^N)\right| = 1 \quad \text{for all} \quad N \in \mathbb{N}.$$

On the other hand, we still have mean square convergence because

$$\int_0^1 \left|1 - (1 - x^N)\right| dx = \int_0^1 x^{2N} dx = \frac{1}{2N+1} \to 0 \text{ as } N \to \infty.$$

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Definition 7.16 (Pointwise and uniform convergence)**.** We say an infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to a function f in (a,b) if

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as} \quad N \to \infty \text{ for all } x \in (a,b).$$

We say the series converges uniformly to f in $[a,b]$ if

$$\max_{x\in[a,b]} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as} \quad N \to \infty$$

Note that for the notion of uniform convergence we include a and b.

Definition 7.17 (Mean square convergence)**.** The serie $\sum_{n=1}^{\infty} f_n(x)$ converges in mean square (or L^2) sense to f in (a,b) if

$$\int_a^b \left| f(x) - \sum_{n=1}^{N} f_n(x) \right|^2 dx \to 0 \quad \text{as} \quad N \to \infty.$$

Remark 7.18. We have: uniform convergence $\Rightarrow$ pointwise and mean square convergence.

But in general not the other way.

Example 7.19. Consider $f_n(x) = (1-x)x^{n-1}$ on $[0,1]$. Then

$$\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} \left(x^{n-1} - x^n\right) = 1 - x^N \to 1 \text{ as } N \to \infty \quad \text{for all } x \in [0,1].$$

But the convergence is not uniform because

$$\max_{x\in[0,1]} \left|1 - (1-x^N)\right| = 1 \quad \text{for all} \quad N \in \mathbb{N}.$$

On the other hand, we still have mean square convergence because

$$\int_0^1 \left|1-(1-x^N)\right| dx = \int_0^1 x^{2N} dx = \frac{1}{2N+1} \to 0 \text{ as } N \to \infty.$$

Consider

$$f_n(x) = \frac{n}{1+n^2x^2} - \frac{n-1}{1+(n-1)^2x^2} \quad \text{on } (0,l)$$

$$\sum_{n=1}^{N} f_n(x) = \frac{N}{1+N^2x^2} = \frac{1}{N\left[\frac{1}{N^2}+x^2\right]} \to 0 \quad \text{as } N \to \infty \text{ if } x > 0.$$

So the series converges pointwise to 0.

On the other hand

$$\int_0^l \frac{N^2}{(1+N^2x^2)^2} dx = N \int_0^{Nl} \frac{1}{(1+y^2)^2} dy \to \infty \quad \text{where } y = Nx$$

because

$$\int_0^{Nl} \frac{1}{(1+y^2)^2} dy \to \int_0^{\infty} \frac{1}{(1+y^2)^2} dy$$

Hence the series does not converge in mean square sense to 0.

Also it does not converge uniformily because

$$\max_{x\in(0,l)} \frac{N}{1+N^2x^2} = N \to \infty$$

Recall we have an inner product

$$(f,g) = \int_a^b f(x)g(x)dx \text{ for } f,g \in C^0([a,b])$$

and a norm given by $\|f\|_2 = \sqrt{(f,f)}$. Convergence of $\sum_{n=1}^N f_n(x)$ to $f(x)$ in L^2 sense means that

$$\left\| f - \sum_{n=1}^N f_n \right\|_2^2 \to 0$$

that is convergence w.r.t. the norm $\|\cdot\|_2$.

Theorem 7.20 (Least Square Approximation)**.** *Let X_n, $n \in \mathbb{N}$, be a set of eigenfunctions for the operator $-\frac{\partial^2}{\partial x^2}$ on $[a,b]$ with symmetric boundary condition. In particular, we have*

$$(X_n, X_m) = \int_a^b X_n(x)X_m(x)dx = 0$$

Let $f:[a,b] \to \mathbb{R}$ be continuous and hence $\|f\|_2 < \infty$ and let $N \in \mathbb{N}$ be fixed.

Among all possible choices of N constants $c_1, c_2, \dots, c_N \in \mathbb{R}$ the choice that minimizes

$$E_N := E_N(c_1,\dots,c_N) := \left\| f - \sum_{n=1}^N c_n X_n \right\|_2^2 = \int_a^b \left(f(x) - \sum_{n=1}^N c_n X_n(x) \right)^2 dx$$

is $c_1 = A_1, \dots, c_N = A_N$ where $A_n = \frac{1}{\|X_n\|_2^2}(f, X_n)$.

Proof.
We expand E_N:

$$\begin{aligned} E_N &= \int_a^b \left(f(x) - \sum_{n=1}^N c_n X_n(x) \right)^2 dx \\ &= \int_a^b |f(x)|^2 dx - 2\sum_{n=1}^N c_n \int_a^b f(x)X_n(x)dx + \sum_{n,m=1}^N c_n c_m \int_a^b X_n(x)X_m(x)dx. \end{aligned}$$

By orthogonality of the eigenfunctions the last term reduces to $\sum_{n=1}^N c_n^2 \int_a^b |X_n(x)|^2 dx$. Hence

$$\begin{aligned} 0 \le E_N &= \|f\|_2^2 - \sum_{n=1}^N \frac{(f,X_n)^2}{\|X_n\|_2^2} + \sum_{n=1}^N \frac{(f,X_n)^2}{\|X_n\|_2^2} - 2\sum_{n=1}^N c_n(f,X_n) + \sum_{n=1}^N c_n^2 \|X_n\|_2^2 \\ &= \|f\|_2^2 - \sum_{n=1}^N \frac{(f,X_n)^2}{\|X_n\|_2^2} + \sum_{n=1}^N \|X_n\|_2^2 \left(\frac{(f,X_n)^2}{\|X_n\|_2^4} - 2c_n \frac{(f,X_n)}{\|X_n\|_2^2} + c_n^2 \right) \\ &= \|f\|_2^2 - \sum_{n=1}^N \frac{(f,X_n)^2}{\|X_n\|_2^2} + \sum_{n=1}^N \|X_n\|_2^2 \left(\frac{(f,X_n)}{\|X_n\|_2^2} - c_n \right)^2 \end{aligned}$$

The coefficients appear only in one place and we see that the right hand side is minimal if

$$c_n = \frac{1}{\|X_n\|_2^2}(f, X_n) = A_n.$$

□

Corollary 7.21 (Bessel's Inequality)**.** $\displaystyle\sum_{n=1}^N \frac{(f,X_n)^2}{\|X_n\|_2^2} = \sum_{n=1}^N A_n^2 \|X_n\|^2 \le \|f\|_2^2.$

In particular, if $\|f\|_2^2 = \int_a^b |f(x)|^2 dx$ *is finite then the series*

$$\sum_{n=1}^{\infty} A_n^2 \|X_n\|^2 = \sum_{n=1}^{\infty} A_n \int_a^b |X_n(x)|^2 dx \quad \textit{converges absolutely.}$$

By the theorem we have for any collection $c_1, \dots, c_N \in \mathbb{R}$:

$$\left\| f - \sum_{n=1}^{N} A_n X_n \right\|_2 = E_N(A_1, \dots, A_N) \le E_N(c_1, \dots, c_N)$$

If we can find a sequence of finite linear combinations

$$g_i = \sum_{n=1}^{N_i} c_n^i X_n \text{ with } N_i \to \infty \text{ for } i \to \infty$$

such that $g_i \to f$ in L^2 sense, that is $\|g_i - f\| = E_N(c_1^i, \dots, c_{N_i}^i) \to 0$, then

$$\sum_{n=1}^{N} A_n X_n \to f \text{ in } L^2 \text{ sense,} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(f, X_n)^2}{\|X_n\|_2^2} = \sum_{n=1}^{\infty} A_n^2 \|X_n\|_2^2 = \|f\|_2^2 .$$

We say eigenfunctions X_n, $n \in \mathbb{N}$, are complete if this holds for every function $f \in C^0([a,b])$.

7.4. **Pointwise convergence.** We will prove pointwise convergence of the full Fourier series on $[-l, l] = [-\pi, \pi]$.

That is we consider the set of eigenfunctions $\sin(nx)$, 1, $\cos(nx)$ with periodic boundary condition on $[-\pi, \pi]$, that is the functions are periodic with period 2π: $X_n(x) = X_n(x + 2\pi)$ for all $x \in \mathbb{R}$.

Given $\phi \in C^0(\mathbb{R})$ that is periodic with period 2π, its full Fourier serie is

$$\frac{1}{2}\tilde{A}_0 + \sum_{n=1}^{\infty} \left(\tilde{A}_n \cos(nx) + A_n \sin(nx) \right), \ x \in [-\pi, \pi]$$

with Fourier coefficients

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \sin(nx) dx, \ \tilde{A}_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) dx, \ \tilde{A}_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(x) \cos(nx) dx, \ n \in \mathbb{N}.$$

We denote

$$S_N(x) = \frac{1}{2}\tilde{A}_0 + \sum_{n=1}^{N} \left(\tilde{A}_n \cos(nx) + A_n \sin(nx) \right), \ x \in [-\pi, \pi], \ N \in \mathbb{N}$$

the Nth partial sum. We can rewrite this as

$$S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2 \sum_{n=1}^{N} \left(\cos(ny)\cos(nx) + \sin(ny)\sin(nx) \right) \right] \phi(y) dy$$

This simplifies as

$$S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left[1 + 2 \sum_{n=1}^{N} \cos(n(x-y)) \right]}_{=: K_N(x-y)} \phi(y) dy$$

Lemma 7.22. $\displaystyle\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(\theta) d\theta = 1$ *and* $\displaystyle K_N(\theta) = 1 + 2 \sum_{n=1}^{N} \cos(n\theta) = \frac{\sin\left[(N + \frac{1}{2})\theta\right]}{\sin\left(\frac{1}{2}\theta\right)}.$

Proof of the Lemma.

$$\int_{-\pi}^{\pi} K_N(\theta)d\theta \ = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1d\theta + \sum_{n=1}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(n\theta)d\theta \ = 1.$$

This proves the first claim.

$$1 + 2\sum_{n=1}^{N}\cos(n\theta) \ = 1 + \sum_{n=1}^{N}\left(e^{in\theta} + e^{-in\theta}\right) = \sum_{n=-N}^{N} e^{iN\theta}.$$

Now consider for some $x \in \mathbb{C}$

$$\left(x^{-N} + x^{-(N-1)} + \cdots + 1 + \cdots + x^{N-1} + x^N\right)(1-x) = x^{-N} + \ldots x^N - \left(x^{-(N-1)} + \ldots x^{N+1}\right).$$

Hence

$$x^{-N} + \cdots + x^N = \frac{x^{-N} - x^{N+1}}{1-x} \ = \frac{x^{-N-\frac{1}{2}} - x^{N+\frac{1}{2}}}{x^{\frac{1}{2}} - x^{\frac{1}{2}}}.$$

If we set $x = e^i$, it follows $K_N(\theta) = \dfrac{e^{i(N+\frac{1}{2})\theta} - e^{-i(N+\frac{1}{2})\theta}}{e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}} = \dfrac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}.$ □

Theorem 7.23. *If $\phi \in C^0(\mathbb{R})$ with periodic boundary condition with period 2π, that is $\phi(x+2\pi) = \phi(x)$ for all $x \in \mathbb{R}$ and if ϕ is differentiable (not necessarily $\phi \in C^1(\mathbb{R})$) then*

$$\frac{1}{2}\tilde{A}_0 + \sum_{n=1}^{\infty}\left(A_n\sin(nx) + \tilde{A}_n\cos(nx)\right) = \phi(x) \text{ for all } x \in \mathbb{R}.$$

Proof of pointwise convergence.

We want to show that $S_N(x) \to \phi(x)$ for all $x \in \mathbb{R}$. We write

$$S_N(x) - \phi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y-x)\left(\phi(y) - \phi(x)\right)dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin((N+\frac{1}{2})(y-x))\frac{\phi(y)-\phi(x)}{\sin(\frac{1}{2}(y-x))}dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\sin((N+\frac{1}{2})\theta)}_{=Y_n(\theta)}\underbrace{\frac{\phi(x+\theta)-\phi(x)}{\sin(\frac{1}{2}\theta)}}_{=g(\theta)}d\theta$$

The functions Y_n, $n \in \mathbb{N}$, are eigenfunction for $-\frac{\partial^2}{\partial x^2}$ on $[0,\pi]$ with mixed boundary conditions $Y_n(0) = 0$ and $\frac{d}{d\theta}Y_n(\pi) = 0$. Mixed boundary conditions are symmetric. Hence, Y_n are orthogonal w.r.t. $(\cdot,\cdot)$ on $[0,\pi]$:

$$\int_0^{\pi} Y_n(\theta)Y_m(\theta)d\theta = 0, \quad \int_0^{\pi}(Y_n(\theta))^2 d\theta = \frac{\pi}{2}.$$

Since $Y(-\theta) = -Y(\theta)$, they are also orthogonal on $[-\pi,\pi]$:

$$\int_{-\pi}^{\pi} Y_n(\theta)Y_m(\theta)d\theta = 0, \quad \int_{-\pi}^{\pi}(Y_n(\theta))^2 d\theta = \pi.$$

Therefore

$$S_N(x) - \phi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y_n(\theta)g(\theta)d\theta = \frac{1}{2}C_n \text{ for } g(\theta) = \frac{\phi(x+\theta)-\phi(x)}{\sin(\frac{1}{2}\theta)},$$

C_n in fact the Fourier coefficient of g w.r.t. the set of orthogonal eigenfunctions Y_n on $[-\pi,\pi]$.

If we can show that $\int_{-\pi}^{\pi}|g(\theta)|^2 d\theta = \|g\|_2^2 < \infty$, then by the Bessel inequality the serie

$$0 \leq \sum_{n=1}^{\infty} C_n^2 \underbrace{\|Y_n\|}_{=\pi} \leq \|g\|_2^2 < \infty$$

converges and hence $C_n \to 0$. The claim is true, if g is continuous on $[-\pi, \pi]$. For that we only need continuity at $\theta = 0$ of

$$g(\theta) = \frac{\phi(x+\theta) - \phi(x)}{\sin(\frac{1}{2}\theta)} = \frac{\phi(x+\theta) - \phi(x)}{\theta} \frac{\theta}{\sin(\frac{1}{2}\theta)} \to 2\phi'(\theta).$$

□

7.5. **Uniform convergence.**

Theorem 7.24. *The full Fourier serie of $\phi \in C^1(\mathbb{R})$ periodic converges uniformily on $[-\pi, \pi]$.*

Proof of uniform convergence.

Since we assume $\phi \in C^1(\mathbb{R})$ with periodic boundary condition, the function ϕ' is continuous and periodic. Hence, the full Fourier coefficients A'_n and $\tilde{A}'_n$ of ϕ are defined. By integration by parts

$$A_n = \int_{-\pi}^{\pi} \phi(x) \sin(nx) dx = -\frac{1}{n} \phi(x) \cos(nx) \Big|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \cos(nx) dx = \frac{1}{n} \tilde{A}'_n$$

Similar $\tilde{A}_n = -\frac{1}{n} A'_n$.

On the other hand we know that $\|\phi\|_2, \|\phi'\|_2 < \infty$ because ϕ and ϕ' are continuous functions on $[-\pi, \pi]$. In particular

$$\sum_{n=1}^{\infty} \left(|A'_n|^2 + |\tilde{A}'_n|^2 \right) < \infty$$

It follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(|A_n| + |\tilde{A}_n| \right) &= \sum_{n=1}^{\infty} \frac{1}{n} |A'_n| + \sum_{n=1}^{\infty} \frac{1}{n} |\tilde{A}'_n| \\ \text{(Cauchy-Schwartz)} \quad &\leq \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \sqrt{\sum_{n=1}^{\infty} \left(|A'_n| + |\tilde{A}'_n| \right)^2} \\ (a+b)^2 \leq 2a^2 + 2b^2 \quad &\leq \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \sqrt{\sum_{n=1}^{\infty} 2 \left(|A'_n|^2 + |\tilde{A}'_n|^2 \right)} \end{aligned}$$

Hence

$$\max_{x \in [-\pi, \pi]} |f(x) - S_N(x)| \leq \sum_{n=N+1}^{\infty} |A_n \cos(nx) + \tilde{A}_n \sin(nx)| \leq \sum_{n=N+1}^{\infty} |A_n| + |\tilde{A}_n| \to 0 \text{ as } N \to \infty.$$

□

In fact the following stronger theorems is true

Theorem 7.25. *For every $f \in C^0(\mathbb{R})$ with periodic boundary conditions and period π its full Fourier serie converges uniformily to f on $[-\pi, \pi]$.*

Lecture 19

Last Lecture

Theorem 7.26. *Let $f \in C^1(\mathbb{R})$ be periodic with period 2π.*

Then **the full Fourier serie of** f

$$\frac{1}{2}\tilde{A}_0 + \sum_{n=1}^{\infty} \Big[A_n \sin(nx) + \tilde{A}_n \cos(nx) \Big]$$

converges on $[-\pi, \pi]$ uniformily to f. Recall that the coefficients are given by

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx,\ \tilde{A}_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,\ \tilde{A}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx.$$

We also showed that

$$\frac{1}{2}|\tilde{A}_0| + \sum_{n=1}^{\infty}\Big(|A_n| + |\tilde{A}_n|\Big)$$

Note that

$$A_n = \frac{1}{\|X_n\|^2}(f, X_n), \qquad \tilde{A}_n = \frac{1}{\left\|\tilde{X}_n\right\|^2}(f, \tilde{X}_n) \quad \text{for } n \in \mathbb{N}$$

where $X_n(x) = \sin(nx)$, $\tilde{X}_n(x) = \cos(nx)$ and $(f, g) = \int_{-\pi}^{\pi} f(x)g(x)dx$ for $f, g \in C^0(\mathbb{R})$ periodic.

But for $\tilde{X}_0(x) \equiv 1$ the definition says

$$\frac{1}{2}A_0 = \frac{1}{\left\|\tilde{X}_0\right\|^2}(f, \tilde{X}_0).$$

7.6. **Application to Fourier sine and cosine serie.** Consider $f \in C^1([0, \pi])$ with Dirichlet boundary conditions: $f(0) = f(\pi) = 0$.

Let f_{odd} be the odd periodic extension of f to $\mathbb{R}$. Then $f_{odd} \in C^1(\mathbb{R})$ because

$$\lim_{h\downarrow} \frac{f(h) - f(0)}{h} = \lim_{h\downarrow 0}\frac{f(h)}{h} = \lim_{h\downarrow 0}\frac{-f(h)}{-h} = \lim_{h\uparrow 0}\frac{f_{odd}(h)}{h}.$$

We know that the full Fourier series of f_{odd} has the form

$$\sum_{n=1}^{\infty} A_n \sin(nx) = \mathcal{F}(f)$$

and converges uniformily on $[-\pi, \pi]$ to f_{odd}.

Recall that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} \underbrace{f_{odd}(x)}_{-f(-x)}\sin(nx)dx + \frac{1}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx.$$

Hence $A_n = A_n^{sin}$ where A_n^{sin} is the Fourier sine coefficient.

Therefore, the full Fourier series $\mathcal{F}(f_{odd})$ coincides with the Fourier sine series $\mathcal{S}(f)$ of f on $[0, \pi]$ and we have the following

Corollary 7.27. *Let $f \in C^1([0, \pi])$ with $f(0) = f(\pi) = 0$.*

Then, the Fourier sine series converges uniformily to f on $[0, \pi]$.

For $f \in C^1([0,\pi])$ with Neumann boundary conditions ($f'(0) = f'(\pi) = 0$) we consider the even periodic extension f_{even}.

Then agan $f_{even} \in C^1(\mathbb{R})$, the full Fourier serie converges uniformily to f and $\tilde{A}_n = A_n^{cos}$ where A_n^{cos} are the coefficients of the Fourier cosine series.

We obtain

Corollary 7.28. *For $f \in C^1([0,\pi])$ with Neumann boundary conditions, the Fourier cosine series of f converges uniformily to f on $[0,\pi]$.*

Recall that more generally one has

Theorem 7.29. *Let $f \in C^0(\mathbb{R})$ be periodic with period 2π.*

Then the full Fourier serie of f converges uniformily on $[-\pi,\pi]$ to f.

Corollary 7.30. *Let $f \in C^0([0,\pi])$ with $f(0) = f(\pi) = 0$. Then, the Fourier sine series converges uniformily to f on $[0,\pi]$.*

For $f \in C^1([0,\pi])$ with Neumann boundary conditions, the Fourier cosine series of f converges uniformily to f on $[0,\pi]$.

7.7. **Application: Heat equation with Dirichlet boundary conditions on $[0,\pi]$.** Let $f \in C^1([0,\pi])$ with Dirichlet boundary conditions and let f_{odd} be the odd periodic extenstion. Then f has an expansion as Fourier sine serie:

$$\sum_{n=1}^{\infty} A_n \sin(nx) = f(x).$$

Theorem 7.31. *The series*

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2 t} \sin(nx)$$

converges uniformily on $[0,\pi] \times [0,\infty)$.

We have $u \in C^2([0,\pi] \times (0,\infty)) \cap C^0([0,\pi] \times [0,\infty))$ and u solves

$$\begin{aligned} u_t &= k u_{x,x} \quad \text{on } [0,\pi] \times (0,\infty) \\ u(x,0) &= f(x) \quad \text{on } [0,\pi] \\ u(0,t) &= 0 \text{ and } u(\pi,t) = 0 \ \forall t > 0. \end{aligned}$$

7.7.1. *Proof of the theorem.* First, we have for $N, M \in \mathbb{N}$ and $M > N$

$$\begin{aligned} &\max_{(x,t)\in[0,\pi]\times[0,\infty)} \left| \sum_{n=1}^{M} A_n e^{-ktn^2} \sin(nx) - \sum_{n=1}^{N} A_n e^{-kn^2 t} \sin(nx) \right| \\ &= \max_{(x,t)\in[0,\pi]\times[0,\infty)} \left| \sum_{n=N+1}^{M} A_n e^{-kn^2 t} \sin(nx) \right| \\ &\leq \sum_{n=N+1}^{M} |A_n| e^{-kn^2 t} | \sin(nx)| \to 0. \end{aligned}$$

We use that $\sum_{n=1}^{\infty} |A_n| < \infty$ is finite, what was part of the proof of uniform convergence of the Fourier serie for $f \in C^1(\mathbb{R})$ periodic.

Therefore, the partial sums $\sum_{n=1}^{N} A_n e^{-kn^2 t} \sin(nx)$ are a Cauchy sequence w.r.t. to uniform convergence, and hence, the uniform limit

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-ktn^2} \sin(nx)$$

exists and $u(x,t)$ is a continuous function on $[0,\pi]\times[0,\infty)$ and $u(x,t)$ satisfies $u(x,0)=f(x)$, $u(0,t)=u(\pi,t)=0$. In particular, u is continuous on $[0,\pi]\times[0,\infty)$.

Moreover, each term $A_n e^{-ktn^2}\sin(nx)$, $n\in\mathbb{N}$, has partial derivatives w.r.t. t and x:

$$-A_n n^2 k e^{-ktn^2}\sin(nx) \text{ and } A_n n e^{-ktn^2}\cos(nx)$$

and second derivatives w.r.t. x: $-A_n n^2 e^{-ktn^2}\sin(nx)$ and solves the heat equation by the separation of variable method. Recall the serie $\sum_{n=1}^{\infty} n^{\alpha} e^{-ktn^2}$ converges absolutely whenever $t>0$ for all $\alpha\in\mathbb{N}$. Hence, it follows that the series

$$\sum_{n=1}^{\infty} A_n n e^{-ktn^2}\sin(nx), \quad \sum_{n=1}^{\infty} A_n n^2 e^{-ktn^2}\sin(nx)$$

converge uniformly on $[0,\pi]\times[t_0,\infty)$ for $t_0>0$. This follows because, for instance,

$$\max_{(x,t)\in[0,\pi]\times[t_0,\infty)}\left|\sum_{n=N+1}^{\infty} A_n n e^{-ktn^2}\sin(nx)\right| \le \sum_{n=N+1}^{\infty}|A_n| n e^{-kt_0n^2}$$

$$\text{(Cauchy-Schwartz inequality)} \le \sqrt{\left(\sum_{n=N+1}^{\infty} A_n^2\right)\left(\sum_{n=N+1}^{\infty} n^2 e^{-2kt_0n^2}\right)}$$

It follows that we can compute the first and second order derivatives of $u(x,t)$ w.r.t. x and t by computing the partial derivatives of the partial sums:

$$u_x(x,t)=\sum_{n=1}^{\infty} A_n e^{-ktn^2} n\cos(nx),\ \ u_t(x,t)=\sum_{n=1}^{\infty} A_n k n^2 e^{-ktn^2}\sin(nx)$$

on $[0,\pi]\times[t_0,\infty)$ for $t_0>0$ and

$$ku_{x,x}(x,t)=-k\sum_{n=1}^{\infty} A_n n^2 e^{-ktn^2}\sin(nx)=u_t(x,t) \text{ for } (x,t)\in[0,\pi]\times(0,\infty).$$

In particular, $u\in C^2([0,\pi]\times(0,\infty))$ and solves the heat equation. □

7.8. **Complex form of the full Fourier serie.** Recall that

$$\sin(nx)=\frac{e^{inx}-e^{-inx}}{2i}, \quad \cos(nx)=\frac{e^{inx}+e^{-inx}}{2}.$$

Let $f\in C^0(\mathbb{R})$ be periodic. The full Fourier serie can be written in complex form

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta}=\mathcal{F}(f)$$

where

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

To see this we introduce a *Hermitien inner product.*

$$(f,g)=\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx \text{ for } f,g\in C^0(\mathbb{R},\mathbb{C}) \text{ periodic}$$

where $g(x)=\mathrm{Re}g(x)+\mathrm{Im}g(x)$ and $\overline{g(x)}=\mathrm{Re}g(x)-\mathrm{Im}g(x)$ is the complex conjugate of the complex number $g(x)$.

Note that $C^0(\mathbb{R},\mathbb{C})$ is complex vector space. $X_n=e^{inx}$, $n\in\mathbb{Z}$, are orthogonal w.r.t. $(\cdot,\cdot)$ and

$$\left\|e^{inx}\right\|_2^2=(e^{inx},e^{inx})=\int_{-\pi}^{\pi} e^{inx}e^{-inx}dx=2\pi.$$

In particular, the general Fourier coefficients take the form

$$c_n = \frac{1}{\|X_n\|}(f, X_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

We can interpret a 2π periodic function f on $\mathbb{R}$ as a function $\hat{f}$ on the $1D$ circle $\mathbb{S}^1 \subset \mathbb{R}^2 = \mathbb{C}$:

$$f(x) = \hat{f}\left(e^{ix}\right).$$

We interpret the heat equation

$$\begin{aligned} u_t &= k u_{x,x} \quad \text{on } \mathbb{R} \\ u(x,0) &= f(x) \quad \text{on } \mathbb{R} \\ u(x,t) &= u(x+2\pi,t)\ \forall x \in \mathbb{R} \end{aligned}$$

with $f \in C^1(\mathbb{R})$ that is 2π periodic as heat equation on $\mathbb{S}^1$.

The solution is given by

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n e^{-ktn^2} e^{inx} = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} f(y) \sum_{n=-\infty}^{\infty} e^{-ktn^2} e^{in(x-y)} dy$$

where c_n are the complex Fourier coefficients of f.

We can write this formula as

$$u(x,t) = \int_{-\pi}^{\pi} f(y) K^{\mathbb{S}^1}(x-y,t)dy$$

with

$$K^{\mathbb{S}^1}(\theta,t) = \sum_{n=-\infty}^{\infty} e^{-ktn^2} e^{in\theta}$$

$K^{\mathbb{S}^1}(\theta,t)$ is called the fundamental solution for the heat equation or heat kernel on $\mathbb{S}^1$.

Lecture 20

8. Laplace and Poisson equation, Harmonic functions

Some Preliminaries Given $u \in C^2(U)$ for an open subset $U \subset \mathbb{R}^n$.

- Recall that $U \subset \mathbb{R}^n$ is open if and only if for all $x \in U$ we can find $\epsilon_x > 0$ such that
$$\{y \in \mathbb{R}^n : |x-y|_2 < \epsilon_x\} =: B_{\epsilon_x}(x) \ \subset U$$
Also recall that $\overline{U}$ denote the closure of U, that is
$$\overline{U} = \left\{x \in \mathbb{R}^n : \exists (x_n)_{n\in\mathbb{N}} \subset U \text{ s.t. } \lim_{n\to n} x_n = x\right\}$$
Then, we define $\overline{U}\backslash U = \partial U$.
- A set $W \subset \mathbb{R}^n$ is called connected if there don't exist sets U_1, U_2 open and disjoint such that $U_1 \cap W, U_2 \cap W \neq \emptyset$ and $W \subset U_1 \cup U_2$.

  Or in other words, W is connected if for any pair of open and disjoint sets U_1, U_2 such that $W \subset U_1 \cup U_2$ it follows that either $W \cap U_1 = \emptyset$ or $W \cap U_2 = \emptyset$.
- $u \in C^2(U)$ if and only if all partial derivatives u_{x_i,x_j}, $i,j = 1\dots n$, exists and are continuous.

  Given $u \in C^2(U)$ the Laplace operator is defined by
$$\sum_{i=1}^n u_{x_i,x_i} =: \Delta u$$
Δ is a map between $C^2(U)$ and $C^0(U)$.

8.1. Laplace and Poisson Equation.

Remark 8.1. Laplace Equation $u \in C^2(U)$ satisfies the **Laplace equation** in U if
$$\Delta u = 0 \text{ in } U.$$

A function $u \in C^2(U)$ for some open **connected** set U with $\Delta u = 0$ is called *harmonic*.

In one dimension the Laplace equation becomes
$$\frac{d^2}{dx^2}u(x) = 0$$
and open, connected sets are intervals of the form (a,b) with $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$.

Hence, harmonic functions are linear functions
$$u(x) = Ax + B, \ A, B \in \mathbb{R}.$$

Remark 8.2. Poisson Equation Given $f \in C^0(U)$ the inhomogeneous version of the Laplace equation
$$\Delta u = f \text{ on } U$$
is called the **Poisson equation**.

If U is a domain with smooth boundary $\partial U \neq \emptyset$, then we usually require suplementary boundary conditions. For the Laplace equation this leads to the following boundary value problems.

Remark 8.3. Dirichlet Problem Let $g \in C^0(\partial U)$, Does there exist $u \in C^2(U) \cap C^0(\overline{U})$ such that
$$\begin{aligned} \Delta u &= 0 && \text{in } U \\ u(x) &= g(x) && \text{for } x \in \partial U. \end{aligned}$$

Smooth boundary: ∂U is a smooth $(n-1)$-dimensional submanifold in $\mathbb{R}^n$.

In this case there is a unique tangent plan $T_x\partial U$ for every $x \in \partial U$ and a unique smooth normal vector field $N : \partial U \to \mathbb{R}^n$: $\langle N(x), v\rangle = 0$ for all $v \in T_x\partial U$ and for all $x \in \partial U$.

Recall $u \in C^1(\overline{U})$ can be defined by saying there exists an open set $\tilde{U}$ such that $U \subset \tilde{U}$ and there exists $\tilde{u} \in C^1(\tilde{U})$ such that $\tilde{u}|_{\overline{U}} = u$.

Remark 8.4. Neumann Problem Let $g \in C^0(\partial U)$, Does there exist $u \in C^2(U) \cap C^1(\overline{U})$ such that

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } U \\ \frac{\partial}{\partial N} u(x) &= g(x) \quad \text{for } x \in \partial U. \end{aligned}$$

8.2. **Physical interpretion.** Physically, $u \in C^2(U)$ with $\Delta u = 0$ describes a distribution in equilibrium in U.

We can think of u as steady state solution of the diffusion equation for higher dimensions:

$$u_t = k\Delta u.$$

To see this imagine $u(x,t)$ as the distribution of some quantity in equilibrium in $U \subset \mathbb{R}^3$, that is there is no change over time. That is, for any subdomain $V \subset U$ we have

$$\int_V u_t(\mathbf{x}, t)d\mathbf{x} = \frac{d}{dt}\int_V u(\mathbf{x}, t)d\mathbf{x} = 0.$$

On the other, $\frac{d}{dt}\int_V u(\mathbf{x}, t)d\mathbf{x}$ is equal to the total flux through the boundary of V

$$\int_{\partial V} \langle \mathbf{F}(\mathbf{x}, t), N(\mathbf{x})\rangle d\mathbf{x}$$

where $\mathbf{F}(\mathbf{x}, t) \in \mathbb{R}^3$, $x \in U$, is the flux density. As in a previous lecture we assume Fick's law.

Remark 8.5. The the flux – or directional change of $u(x,t)$ in a point x at time t is proportional to the gradient $\nabla u(\mathbf{x}, t)$

$$\mathbf{F}(\mathbf{x}, t) = -k\nabla u(\mathbf{x}, t).$$

Hence, by the divergence theorem

$$0 = \frac{d}{dt}\int_V u(\mathbf{x}, t)d\mathbf{x} = \int u_t(\mathbf{x}, t)d\mathbf{x} = k\int_V \Delta_{\mathbf{x}} u(\mathbf{x}, t)d\mathbf{x}$$

implying that $0 = u_t(\mathbf{x}, t) = k\Delta_x u(\mathbf{x}, t)$. In particular $u(\mathbf{x}, t) = u(\mathbf{x})$ does not depend on t.

Other Interpretations

- **Electrostatics.**

  Electric current is described by a vector field $\mathbf{E}$ in a domain $U \subset \mathbb{R}^3$ that satisfies

  Maxwell's equations $\quad \mathrm{Curl}(\mathbf{E}) = \left(\frac{\partial \mathbf{E}}{\partial x_3} - \frac{\partial \mathbf{E}}{\partial x_2}, \frac{\partial \mathbf{E}}{\partial x_2} - \frac{\partial \mathbf{E}}{\partial x_1}, \frac{\partial \mathbf{E}}{\partial x_1} - \frac{\partial \mathbf{E}}{\partial x_3}\right) = 0, \quad \mathrm{Div}\mathbf{E} = 4\pi\rho$

  where ρ is the charge density of U.

  Curl$\mathbf{E} = 0$ in $\mathbb{R}^n$ is equivalent to

  $$\int_a^b \langle \mathbf{E} \circ \gamma(t), \gamma'(t)\rangle dt = 0$$

  for any closed curve $\gamma \in C^1([a,b], \mathbb{R}^n)$ $(\gamma(a) = \gamma(b)$ and $\gamma'(a) = \gamma'(b))$.

  We know that in this case there exists a potential $\phi \in C^2(\mathbb{R}^n)$ such that $\mathbf{E} = \nabla\phi$.

  Hence, the vector field $\mathbf{E}$ is the gradient of a potential $-\phi$ that satisfies the Poisson equation:

  $$\mathbf{E}(\mathbf{x}) = -\nabla\phi(\mathbf{x}) \quad \Rightarrow \quad \Delta\phi = -4\pi\phi(x).$$

- **Classical Newtonian Gravity.**

  Let $\mathbf{g}$ be the gravitational force vector field in $\mathbb{R}^3$ according to a mass distribution ρ. Again one has the following laws
  $$\mathrm{Curl}\mathbf{g} = 0 \text{ in } \mathbb{R}^3 \text{ and } \mathrm{Div}\mathbf{g} = -4\pi G\rho$$
  G is the gravitational constant.

  Hence, there exists a potential function ϕ such that $\nabla\phi = -\mathbf{g}$ and $\Delta\phi = -4\pi G\rho$.
- **Fluid dynamics.**

  Recall the transport equation
  $$u_t + \langle V, \nabla u\rangle$$
  for a vector field $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. We studied this equation a some lectures ago and assume that
  $$\mathrm{Div}V = 0$$
  which means the flow of V is incompressible and there are no sources and sinks.

  Now we also ssume V describes an irrotational flow. That means again that
  $$\mathrm{Curl}V = 0 \text{ in } \mathbb{R}^n.$$
  We know that in this case there exists a potential $\phi \in C^2(\mathbb{R}^n)$ such that $V = \nabla\phi$.

  Hence, ϕ satisfies the Laplace equation:
  $$\mathrm{Div}V = \Delta\phi = 0.$$

8.3. **Polar Coordinates in $\mathbb{R}^2$.** In $\mathbb{R}^2$ the Laplace operator is $\Delta u = u_{x,x} + u_{y,y} = 0$.

Let us express Δ in polar coordinates
$$(x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta) \text{ for } (r,\theta) \in (0,\infty) \times [0, 2\pi).$$
The differential of the map $(r,\theta) \in (0,\infty) \times (0,2\pi) \mapsto (r\cos\theta, r\sin\theta)$ and its inverse is
$$D(x,y) = \begin{pmatrix} x_r & y_r \\ x_\theta & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \text{ and } [D(x,y)]^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix}.$$
Consider u in the new coordinates, that is $\tilde{u} := u \circ (x,y)$. We compute
$$\begin{pmatrix} \tilde{u}_r \\ \tilde{u}_\theta \end{pmatrix} = \begin{pmatrix} -u_x\sin\theta + u_y\cos\theta \\ u_x\cos\theta + u_y\sin\theta \end{pmatrix} = D(x,y)\begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$
Here u_x and u_y is short for $u_x \circ (x,y)$ and $u_y \circ (x,y)$ respectively. Hence
$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = [D(x,y)]^{-1}\begin{pmatrix} \tilde{u}_r \\ \tilde{u}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\tilde{u}_r - \frac{1}{r}\sin\theta\tilde{u}_\theta \\ \sin\theta\tilde{u}_r + \frac{1}{r}\cos\theta\tilde{u}_\theta \end{pmatrix} = \begin{pmatrix} (\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta})\tilde{u} \\ (\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta})\tilde{u} \end{pmatrix}.$$
Hence, the operator $\frac{\partial}{\partial x}$ transform under the coordinate change $x = r\cos\theta$, $y = r\sin\theta$ into
$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$$
and similar for $\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}$. Therefore, it follows
$$\begin{aligned} u_{x,x} \circ (x,y) &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\tilde{u} \\ &= \Big(\cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial\theta\partial r} + \frac{1}{r}\sin^2\theta\frac{\partial}{\partial r} \\ &\qquad - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial\theta\partial r} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2}\Big)\tilde{u}. \end{aligned}$$

and also

$$u_{y,y}\circ(x,y) = \left(\sin\theta\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right)\tilde u$$
$$= \Big(\sin^2\theta\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial\theta\partial r}+\frac{1}{r}\cos^2\theta\frac{\partial}{\partial r}$$
$$+\frac{1}{r}\cos\theta\sin\theta\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\cos\theta\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2}\cos^2\theta\frac{\partial^2}{\partial\theta^2}\Big)\tilde u.$$

It follows that

$$u_{x,x}\circ(x,y)+u_{y,y}\circ(x,y) = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\tilde u = \tilde u_{r,r}+\frac{1}{r}\tilde u_r+\frac{1}{r^2}\tilde u_{\theta,\theta}.$$

Corollary 8.6. *The Laplace operator is invariant w.r.t. rotations of $\mathbb{R}^2$ at the center.*

Beweis. A rotation is linear transformation w.r.t. θ in polar coordinates. □

8.4. **Spherical Coordinates in $\mathbb{R}^3$.** Let us compute the Laplace operator $u_{x,x}+u_{y,y}+u_{z,z}=\Delta u$ in $\mathbb{R}^3$ in spherical coordintates

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta.$$

For that we first consider cylindrical coordinates

$$x = s\cos\phi,\ y = s\sin\phi,\ z = z.$$

We set $\tilde u = u\circ(x,y,z)$. Similar as for polar coordinates we obtain

$$\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} = \begin{pmatrix}\cos\phi\tilde u_s-\frac{1}{s}\sin\phi\tilde u_\phi\\ \sin\phi\tilde u_s+\frac{1}{s}\cos\phi\tilde u_\phi\\ \tilde u_z\end{pmatrix}\quad\text{and}\quad u_{x,x}+u_{y,y} = \tilde u_{s,s}+\frac{1}{s}\tilde u_s+\frac{1}{s^2}\tilde u_{\phi,\phi}.$$

In particular, we see $u_z=\tilde u_z$ and $u_{z,z}=\tilde u_{z,z}$.

Then, we apply cylindrical coordinates a second time setting

$$z = r\cos\theta,\ s = r\sin\theta,\ \phi=\phi,$$

and setting $\hat u = \tilde u\circ(s,\phi,z)$. As before we compute

$$\tilde u_s = \sin\theta\hat u_r+\frac{1}{r}\cos\theta\hat u_\theta$$

as well as

$$\tilde u_{s,s}+\tilde u_{z,z} = \hat u_{r,r}+\frac{1}{r}\hat u_r+\frac{1}{r^2}\hat u_{\theta,\theta}\quad\Rightarrow\quad u_{z,z} = \hat u_{r,r}+\frac{1}{r}\hat u_r+\frac{1}{r^2}\hat u_{\theta,\theta}-\tilde u_{s,s}$$

and in particular $\tilde u_\phi=\hat u_\phi$ and $\tilde u_{\phi,\phi}=\hat u_{\phi,\phi}$. It follows

$$u_{x,x}+u_{y,y}+u_{z,z} = \hat u_{r,r}+\frac{1}{r}\hat u_r++\frac{1}{r^2}\hat u_{\theta,\theta}+\frac{1}{s}\tilde u_s+\frac{1}{s^2}\tilde u_{\phi,\phi}$$
$$= \hat u_{r,r}+\frac{1}{r}\hat u_r+\frac{1}{r^2}\hat u_{\theta,\theta}+\frac{1}{r\sin\theta}\left(\sin\theta\hat u_r+\frac{1}{r}\cos\theta\hat u_\theta\right)+\frac{1}{r^2\sin^2\theta}\hat u_{\phi,\phi}$$
$$= \hat u_{r,r}+\frac{2}{r}\hat u_r+\frac{1}{r^2}\left(\hat u_{\theta,\theta}+\frac{\cos\theta}{\sin\theta}\hat u_\theta+\frac{1}{\sin^2\theta}\hat u_{\phi,\phi}\right).$$

We can now look for special solutions of the Laplace equation in $\mathbb{R}^2$ or in $\mathbb{R}^3$ that only depend on $\sqrt{x^2+y^2}$ or $\sqrt{x^2+y^2+z^2}$, that is $r>0$ in polar or spherical coordinates.

In $\mathbb{R}^2$ the Laplace equation for such functions reduces to

$$0 = \tilde u_{r,r}+\frac{1}{r}\tilde u_r\ \Rightarrow\ 0 = r\tilde u_{r,r}+\tilde u_r = (r\tilde u_r)_r\ \Rightarrow\ c_1 = r\tilde u_r.$$

Hence the solutions are $\tilde{u}(r) = c_1 \log r + c_2$. In Euclidean coordinates

$$u(x,y) = c_1 \log\left(\sqrt{x^2+y^2}\right) + c_2.$$

In $\mathbb{R}^3$ the Laplace equation for such functions reduces to

$$0 = \hat{u}_{r,r} + \frac{2}{r}\hat{u}_r \;\Rightarrow\; 0 = r^2\hat{u}_{r,r} + 2r\hat{u}_r \;\Rightarrow\; c_1 = r^2\hat{u}_r.$$

Hence, the solutions are $\hat{u}(r) = -\frac{c_1}{r} + c_2$. In Euclidean coordinates

$$u(x,y,z) = -\frac{c_1}{\sqrt{x^2+y^2+z^2}} + c_2.$$

8.4.1. *Application to Newtonian Gravity.* Imagine a star in an otherwise empty universe. We model the star as the point $0 \in \mathbb{R}^3$ with all its mass concentrated in 0.

What does **Classical Newtonian Gravity** tell us about the gravitational forces in the space around the star? We can write Newton's law as the following Poisson equation

$$\Delta\phi = 4\pi G\rho \text{ in } \mathbb{R}^3\backslash\{0\} \;\Rightarrow\; \frac{1}{4\pi G}\Delta\phi = \rho \text{ in } \mathbb{R}^3\backslash\{0\}$$

where ρ is the mass density in $\mathbb{R}^3\backslash\{0\}$.

By assumption the universe is empty in $\mathbb{R}^3\backslash\{0\}$. Hence, the mass density ρ is 0.

Moreover, we assume the Universe is fully isotrop and homogeneous. Hence, the gravitational forces are the same independent of the direction and hend only depend on the distance $r = \sqrt{x^2+y^2+z^2}$ to the star at 0. Hence, the Poisson equation becomes

$$\frac{1}{4\pi G}\left[\tilde{\phi}_{r,r} + \frac{2}{r}\tilde{\phi}_r\right] = 0.$$

A solution is

$$\tilde{\phi}(r) = -\frac{4\pi Gm}{r} + c_2 \quad \text{for constants } m, c_2 > 0.$$

We also assume that very far away from the star there is almost no gravitational pull. Hence $c_2 = 0$ and therefore

$$\phi(x,y,z) = -\frac{4\pi Gm}{\sqrt{x^2+y^2+z^2}} \quad \text{and} \quad \mathbf{g}(x,y,z) = -\nabla\phi(x,y,z)$$

where m describes the mass of the star.

Lecture 21

8.5. **Some Preliminaries about connected sets.** A subset $U \subset \mathbb{R}^n$ is open if and only if for all $x \in U$ we can find $\epsilon_x > 0$ such that

$$\{y \in \mathbb{R}^n : |x - y|_2 < \epsilon_x\} =: B_{\epsilon_x}(x) \ \subset U.$$

where $B_\epsilon(x) = \{y \in \mathbb{R}^n : |y - x|_2 < \epsilon\}$.

Let $U \subset \mathbb{R}^n$ be arbitrary, not necessarily open. The set

$$U^\circ = \{x \in U : \exists \epsilon_x > 0 \text{ s.t. } B_\epsilon(x) \subset U\}$$

is called *open interior* of U.

A subset $A \subset \mathbb{R}^n$ is closed if

$$(x_n)_{n\in\mathbb{N}} \subset A \text{ and } \lim_{n\to\infty} x_n = x \ \Rightarrow \ x \in A.$$

If $U \subset \mathbb{R}^n$ is again arbitrary,

$$\overline{U} = \left\{x \in \mathbb{R}^n : \exists (x_n)_{n\in\mathbb{N}} \subset U \text{ s.t. } \lim_{n\to n} x_n = x\right\}$$

denotes the *closure* of U. The closure $\overline{U}$ of U is closed.

Fact 8.7. *A subset $U \subset \mathbb{R}^n$ is open if and only if $\mathbb{R}^n \backslash U$ is closed.*

For U open we define $\overline{U}\backslash U = \partial U$, the boundary of U.

Definition 8.8 (Connected Sets)**.** A set $W \subset \mathbb{R}^n$ is called connected if there don't exist sets U_1, U_2 open and disjoint such that $U_1 \cap W, U_2 \cap W \neq \emptyset$ and $W \subset U_1 \cup U_2$.

Or equivalently, W is connected if for any pair of open and disjoint sets $U_1, U_2 \subset \mathbb{R}^n$ such that $W \subset U_1 \cup U_2$ it follows that either $W \cap U_1 = \emptyset$ or $W \cap U_2 = \emptyset$.

Recall that $u \in C^2(U)$ if and only if all partial derivatives u_{x_i,x_j}, $i, j = 1 \dots n$, exists and are continuous.

Given $u \in C^2(U)$ the Laplace operator is defined by

$$\sum_{i=1}^n u_{x_i,x_i} =: \Delta u$$

Δ is a map from $C^2(U)$ to $C^0(U)$.

Definition 8.9 (Laplace Equation)**.** $u \in C^2(U)$ satisfies the **Laplace equation** in U if

$$\Delta u = 0 \text{ in } U.$$

A function $u \in C^2(U)$ for some open set U with $\Delta u = 0$ is called *harmonic*.

Remark 8.10. In 1 dimension the Laplace equation becomes $\frac{d^2}{dx^2}u(x) = 0$ and open, connected sets are intervals of the form (a, b) with $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$. Hence, harmonic functions on open intervals are linear functions

$$u(x) = Ax + B, \ A, B \in \mathbb{R}.$$

8.6. **Maximum Principle for harmonic functions.**

Theorem 8.11. *Let $U \subset \mathbb{R}^n$ be open and let $u : \overline{U} \to \mathbb{R}$ be a function such that $u \in C^2(U) \cap C^0(\overline{U})$ be harmonic. Precisely, $u \in C^0(\overline{U})$ and $u|_U \in C^2(U)$.*

- **Weak Maximum Principle:** *The maximum and the minimum value of u are attained on ∂U:*

$$\max_{x\in\overline{U}} u(x) = \max_{x\in\partial U} u(x) \quad \text{and} \quad \min_{x\in\overline{U}} u(x) = \min_{x\in\partial U} u(x)$$

- **Strong Maximum Principle:** *If U is connected and there exists $x_0 \in U$ such that*

$$u(x_0) = \max_{x\in\overline{U}} u(x) \quad or \quad u(x_0) = \min_{x\in\overline{U}} u(x)$$

  then $u \equiv const \equiv u(x_0)$.

Remark 8.12. The strong maximum principle implies the weak one. But we will prove both principles separately.

Proof of the weak maximum principle.

The proof is similar to the proof of the weak maximum principle for solutions of the diffusion equation.

Define $v(\mathbf{x}) = u(\mathbf{x}) + \epsilon\frac{1}{2n}|\mathbf{x}|^2$ for some $\epsilon > 0$. We have $v \in C^2(U) \cap C^0(\overline{U})$.

First, let us assume v attains its maximum value in $x_0 \in U$.

Then, by the second derivative test in calculus the matrix $D^2v(\mathbf{x}_0)$ is negative semi-definite. Equivalently, all eigenvalues of $A = D^2v(\mathbf{x}_0)$ are non-positive.

It follows that the trace of the matrix A is also non-positive. Hence

$$\operatorname{tr}(A) = \operatorname{tr}(D^2v(x_0)) = \sum_{i=1}^n v_{x_i,x_i}(\mathbf{x}) = \Delta v(\mathbf{x}_0) \le 0.$$

On the other hand

$$\Delta v = \Delta u + \Delta(\epsilon\frac{1}{2n}|\mathbf{x}|^2) = 0 + \frac{\epsilon}{2n}\sum_{i=1}^n (x_i^2)_{x_i,x_i} = \epsilon > 0$$

This is a contradiction.

Hence $v \in C^0(\overline{U})$ attains its maximum value on ∂U. We obtain the following chain of inequalities

$$u(\mathbf{x}) \le u(\mathbf{x}) + \epsilon\frac{1}{2n}|\mathbf{x}|^2 \le v(\mathbf{x}) \le \max_{x\in\partial U} v(\mathbf{x}) \le \max_{x\in\partial U} u(\mathbf{x}) + \epsilon\frac{1}{2n}\max_{x\in\partial U}|\mathbf{x}|_2^2 \;\; \forall x \in \overline{U}.$$

Since $\epsilon > 0$ was arbitrary, let $\epsilon \to 0$ and it follows

$$u(\mathbf{x}) \le \max_{x\in\partial U} u(\mathbf{x}) \;\forall \mathbf{x} \in \overline{U}.$$

□

8.6.1. *Mean Value property.* The *mean value property* for harmonic functions states that the value of a harmonic function at any point equals its average on a ball or on sphere (spherical mean) centered at the given point. More precisely:

Theorem 8.13 (Mean Value Property)**.** *Let $u \in C^2(U)$ be harmonic for $U \subset \mathbb{R}^n$ open. Then*

$$u(\mathbf{x}_0) = \frac{1}{Vol(B_r(\mathbf{x}_0))}\int_{B_r(x_0)} u(\mathbf{x})d\mathbf{x} \quad \forall \mathbf{x}_0 \in U \text{ and } \forall B_r(\mathbf{x}_0) \subset U.$$

and

$$u(\mathbf{x}_0) = \frac{1}{\omega_{n-1}r^{n-1}}\int_{\partial B_r(\mathbf{x}_0)} u(\mathbf{x})d\mathcal{S}^{n-1}_{\partial B_r(\mathbf{x}_0)} \quad \forall x_0 \in U \text{ and } \forall B_r(\mathbf{x}_0) \subset U.$$

where ω_{n-1} is the $(n-1)$-dimensional surface of $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$. $\mathcal{S}^{n-1}_{\partial B_r(x_0)}$ is the surface measure of $\partial B_r(\mathbf{x}_0)$.

8.6.2. *Proof of the mean value property.* Let $x_0 \in U$ and $r > 0$ such that $B_r(x_0) \subset U$. W.l.o.g. we can assume $x_0 = 0$ by replacing $u(\mathbf{x})$ with $u(\mathbf{x} - \mathbf{x}_0)$. We set $B_r = B_r(0)$. Consider

$$\phi(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r(0)} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r}$$

We will show that $\phi'(r) = 0$. Then $\phi(r)$ is constant. On the other hand

$$\lim_{r\to 0} \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r} = u(0) \text{ by continuity of } u.$$

Indeed, for $\forall \epsilon > 0$ there is $\delta > 0$ such that $\forall r \in (0, \delta)$ it holds $|u(0) - u(\mathbf{x})| \leq \epsilon$ if $|\mathbf{x}| = r$. Hence

$$\left| \tfrac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r} - u(0) \right| \leq \lim_{r\to 0} \tfrac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r} |u(\mathbf{x}) - u(0)| d\mathcal{S}^{n-1}_{\partial B_r} \leq \tfrac{\epsilon}{\omega_{n-1} r^{n-1}} \int d\mathcal{S}^{n-1}_{\partial B_r}.$$

Let us show that $\phi'(r) = 0$. For that we first observe that

$$\frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r}(\mathbf{x}) = \frac{1}{\omega_{n-1}} \int_{\partial B_1} u(r\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_1}(\mathbf{x})$$

Then

$$\frac{d}{dr} \int_{\partial B_1} u(r\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_1}(\mathbf{x}) = \int_{\partial B_1} \langle \nabla u(r\mathbf{x}), \mathbf{x} \rangle d\mathcal{S}^{n-1}_{\partial B_1}(\mathbf{x}) = \frac{1}{r^{n-1}} \int_{\partial B_r} \langle \nabla u(r\mathbf{x}), \frac{\mathbf{x}}{r} \rangle d\mathcal{S}^{n-1}_{\partial B_r}(\mathbf{x})$$

We could exchange integration w.r.t. x and differentiation w.r.t. r, because $r \in [0, R) \mapsto u(rx) \in C^1([0, R))$ (because $u \in C^1(U)$) as long as $B_R(0) \subset U$.

By the divergence theorem the right hand side is equal to $\int_{B_r} \Delta u(\mathbf{x}) d\mathbf{x} = 0$.

Hence $\phi'(r) = 0$ for $r \in [0, R)$ and $R > 0$ as before.

It follows that $\phi(r) = \mathit{const} = c$ on $[0, R)$ and since $\phi(r) \to u(0)$ for $r \to 0$ we have $c = u(0)$. □

Proof of the strong maximum principle. see next lecture.

Lecture 22.

Remark 8.14. Laplace Equation $u \in C^2(U)$ satisfies the **Laplace equation** in U if

$$\Delta u = 0 \text{ in } U.$$

A function $u \in C^2(U)$ for some open set U with $\Delta u = 0$ is called *harmonic*.

Maximum Principle for harmonic functions

Theorem 8.15. *Let $U \subset \mathbb{R}^n$ be open and let $u : \overline{U} \to \mathbb{R}$ be a function such that $u \in C^2(U) \cap C^0(\overline{U})$ be harmonic.*

- **Weak Maximum Principle:** *The maximum and the minimum value of u are attained on ∂U:*

$$\max_{x \in \overline{U}} u(x) = \max_{x \in \partial U} u(x) \quad \textit{and} \quad \min_{x \in \overline{U}} u(x) = \min_{x \in \partial U} u(x)$$

- **Strong Maximum Principle:** *If U is connected and there exists $x_0 \in U$ such that*

$$u(x_0) = \max_{x \in \overline{U}} u(x) \quad \textit{or} \quad u(x_0) = \min_{x \in \overline{U}} u(x)$$

  then $u \equiv const \equiv u(x_0)$.

Remark 8.16. Connected sets A subset $W \subset \mathbb{R}^n$ is connected if for any pair of open and disjoint sets $U_1, U_2 \subset \mathbb{R}^n$ such that $W \subset U_1 \cup U_2$ it follows that either $W \cap U_1 = \emptyset$ or $W \cap U_2 = \emptyset$.

Theorem 8.17 (Mean Value Property)**.** *Let $u \in C^2(U)$ be harmonic for $U \subset \mathbb{R}^n$ open. Then, it holds*

$$u(\mathbf{x}_0) = \frac{1}{Vol(B_r(\mathbf{x}_0))} \int_{B_r(x_0)} u(\mathbf{x}) d\mathbf{x} \quad \forall \mathbf{x}_0 \in U \textit{ and } \forall B_r(\mathbf{x}_0) \subset U.$$

and

$$u(\mathbf{x}_0) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r(\mathbf{x}_0)} u(\mathbf{x}) d\mathcal{S}^{n-1}_{\partial B_r(\mathbf{x}_0)} \quad \forall x_0 \in U \textit{ and } \forall B_r(\mathbf{x}_0) \subset U.$$

where ω_{n-1} is the $(n-1)$-dimensional surface of $\partial B_1(0) = \mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$.

8.7. **Proof of the strong maximum principle.** Let $u \in C^2(U) \cap C^0(\overline{U})$ be harmonic for a connected and open subset $U \subset \mathbb{R}^n$.

Assume there exists $\mathbf{x}_0 \in U$ such that $u(\mathbf{x}_0) = \max_{\mathbf{x} \in \overline{U}} u(\mathbf{x}) =: M$.

We define

$$V = \{\mathbf{x} \in U : u(\mathbf{x}) = M\} \neq \emptyset.$$

as well $W = U \backslash V$.

Claim: V is open.

Proof of the Claim. Pick $\mathbf{x} \in V$ and $r > 0$ such that $B_r(\mathbf{x}) \subset U$. By the mean value property

$$M = u(\mathbf{x}) = \frac{1}{v_n(r)} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y} \le M.$$

Hence

$$0 = \frac{1}{v_n(r)} \int_{\partial B_r(\mathbf{x})} (u(\mathbf{y}) - M) d\mathbf{y} = 0$$

Since $u(\mathbf{y}) - M \le 0$, it follows $u(\mathbf{y}) = M$ on $B_r(\mathbf{x})$. Hence $B_r(\mathbf{x}) \subset V$.

Claim: $W = U \backslash V$ is open.

Proof of the Claim. If $\mathbf{x} \in W$, then $|u(\mathbf{x}) - M| > 0$. Since u is continuous there exists δ such that $|\mathbf{y} - \mathbf{x}| < \delta$ implies $|u(\mathbf{y}) - u(\mathbf{x})| \leq \frac{|M-u(\mathbf{x})|}{2}$. Then it follows that $|u(\mathbf{y}) - M| \geq |u(\mathbf{x}) - M| - |u(\mathbf{y}) - u(\mathbf{x})| \geq \frac{|M-u(\mathbf{x})|}{2} > 0$. Hence $B_\delta(\mathbf{x}) \subset W$ and therefore W is open.

Finally, since U is connected and since V and W are open, either $V = \emptyset$ or $W = \emptyset$. But since $\mathbf{x}_0 \in V$ and therefore $V \neq \emptyset$, it follows $W = \emptyset$ and $U = V$. □

8.8. **Poisson Formula.** Our goal is to solve the Dirichlet problem on a disk $\overline{B_a(0)} = \{x \in \mathbb{R}^2 : |x|_2 \leq a\}$ in $\mathbb{R}^2$.

Let $B_a(0) = \{x \in \mathbb{R}^2 : |x|_2 < a\}$, and $\partial B_a(0) = \overline{B_a(0)} \backslash B_a(0)$.

Definition 8.18. Dirichlet Problem for the Laplace equation on $B_a(0)$ Let $h \in C^0(\partial B_a(0))$.

Find $u \in C^2(B_a(0)) \cap C^0(\overline{B_a(0)})$ such that

$$\begin{aligned} u_{x,x} + u_{y,y} &= 0 \quad \text{in } B_a(0) \\ u &= h \quad \text{in } \partial B_a(0) \end{aligned}$$

For that we first consider $u_{x,x} + u_{y,y} = 0$ in polar cooardinates:

$$0 = \tilde{u}_{r,r} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta,\theta}$$

where $\tilde{u}(r,\theta) = u(r\cos\theta, r\sin\theta)$ and $r > 0$ and $\theta \in \mathbb{R}$.

Similar, we can rewrite the boundary data h as $\tilde{h}(\theta) = h(a\cos\theta, a\sin\theta)$.

Note that $\tilde{h}(\theta)$, $\theta \in \mathbb{R}$, is 2π-periodic. We apply the method of separation of variables to the Laplace equation in polar coordinates (compare with exercise): Assume $\tilde{u}(r,\theta) = R(r)\Theta(\theta)$. Then

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) = -\frac{1}{r^2}R(r)\Theta''(\theta)$$

Hence

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda = const.$$

The general solution for Θ is

$$\Theta(\theta) = \begin{cases} A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta), & \lambda > 0, \\ A + Bx, & \lambda = 0, \\ A\cosh(\sqrt{-\lambda}\theta) + B\sinh(\sqrt{-\lambda}), & \lambda < 0 \end{cases}$$

Since Θ is periodic, we only have to consider the cases $\lambda > 0$ and $\lambda = 0$ for $B = 0$.

Moreover, by evaluation of the function for the points 0 and 2π we see that $\lambda = n^2$, $n \in \mathbb{N} \cup \{0\}$.

The equation for R becomes

$$r^2R''(r) + rR'(r) - n^2R(r) = 0.$$

Solutions for $n \in \mathbb{N}$ are r^n, r^{-n} and $Cr^n + Dr^{-n}$ for $C, D \in \mathbb{R}$, and $\log r$, C and $C + D\log r$ for $n = 0$.

Since we are looking for smooth solutions u on $B_a(0)$ that are continuous we can assume that $D = 0$.

Now we consider infinite sums of the form

$$\tilde{u}(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n \left(A_n\cos(n\theta) + B_n\sin(n\theta)\right). \tag{44}$$

Finally, let us bring the boundary condition into play. At $r = a$ we require

$$\tilde{h}(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right). \tag{45}$$

So, assuming that $\tilde{h} \in C^1(\mathbb{R})$ (and 2π-periodic) this is the full Fourier series that converges uniformily and the Fourier coefficients are uniquely determined by the formulas

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{h}(\phi)d\phi, \; A_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} \tilde{h}(\theta)\sin(n\phi)d\phi, \; B_n = \frac{1}{a^n\pi}\int_{-\pi}^{\pi} \tilde{h}(\phi)\cos(n\phi)d\phi.$$

Uniform convergence of (45) implies uniform convergence of (44). By replacing A_n and B_n with the Fourier coefficients of h we can rewrite the formula for u as

$$\begin{aligned}\tilde{u}(r,\theta) &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left(\sin(n\theta)\sin(n\phi) + \cos(n\phi)\cos(n\theta)\right)\right] h(\theta)d\theta \\ &= \int_{-\pi}^{\pi}\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi))\right]\frac{d\pi}{2\pi}\end{aligned}$$

Recall $\cos\theta = \frac{e^{i\theta}+e^{i\theta}}{2}$ and the formula $\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$ for $z \in \mathbb{C}$ with $|z| < 1$. Hence

$$\begin{aligned}1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi)) &= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)} \\ &= 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}} \\ &= 1 + \frac{re^{i(\theta-\phi)}(a - re^{-i(\theta-\phi)}) - re^{-i(\theta-\phi)}(a - re^{i(\theta-\phi)})}{(a - re^{i(\theta-\phi)})(a - re^{-i(\theta-\phi)})} \\ &= \frac{a^2 - r^2}{a^2 - ar2\cos(\theta-\phi) + r^2}.\end{aligned}$$

We get

Remark 8.19. Poisson solution formula for the Laplace equation on the disk

$$\tilde{u}(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{h(\phi)(a^2 - r^2)}{a^2 - 2ar\cos(\theta-\phi) + r^2}d\phi$$

We can also write this formula again in Euclidean coordinates.

For that note that an infinitesimal length segement of the boundary $\partial B_a(0)$ is given by $ds = ad\phi$ where $d\phi$ is the infinitesimal angle of the segment ds.

Also note that for $\mathbf{x} = (r,\theta)$ and $\mathbf{y} = (s,\phi)$ we have

$$|\mathbf{x} - \mathbf{y}|^2 = r^2 + s^2 - 2rs\cos(\theta - \phi)$$

by the cosine rule. It follows that

Proposition 8.20. *Poisson formula, second version*

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a}\int_{\partial B_a(0)}\frac{u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2}ds(\mathbf{y}).$$

Theorem 8.21. *Let $h \in C^0(\partial B_a(0))$ be given in polar coordinates by $h(a\cos\theta, a\sin\theta) = \tilde{h}(\theta)$ for $\tilde{h} \in C^0(\mathbb{R})$ that it 2π periodic. Then the Poisson formula provides the unique harmonic function on $B_a(0)$ for which*

$$\lim_{x\to x_0} u(\mathbf{x}) = h(\mathbf{x}_0) \;\forall \mathbf{x}_0 \in \partial B_a(0).$$

Proof of the Theorem Uniqueness follows by the weak maximum principle.

Given the $\tilde{h}$ as in the theorem the Poisson formula yields

$$\tilde{u}(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta-\phi)\tilde{h}(\phi)\frac{d\phi}{2\pi} = \int_{-\pi}^{\pi} P(r,\phi)\tilde{h}(\theta-\phi)\frac{d\phi}{2\pi} \tag{46}$$

where $P(r,\theta) = \frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}$ is the Poisson kernel.

We have 3 important facts

- $P(r,\theta) > 0$ because $0 < r < a$ and $a^2 - 2ar\cos\theta + r^1 \geq a^2 - 2ar + r^2 = (a-r)^2$.
- $\int_{-\pi}^{\pi} P(r,\theta)\frac{d\theta}{2\pi} = 1$ by piecewise integration of the previous series.
- $P(r,\theta)$ solve the Laplace equation on $B_a(0)$. Moreover $P(r,\theta) \in C^2([0,a)\times\mathbb{R})$.

The last fact allows us to differentiate under the integral in (46) and we can check that

$$\tilde{u}_{r,r} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta,\theta} = \int_{-\pi}^{\pi} \underbrace{\left[\frac{\partial^2}{\partial r^2}P(r,\theta-\phi) + \frac{1}{r}\frac{\partial}{\partial r}P(r,\theta-\phi) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}P(r,\theta-\phi)\right]}_{=0}\tilde{h}(\phi)\frac{d\phi}{2\pi}.$$

So $\tilde{u}$ is harmonic on $B_a(0)$. It remains to prove that $\tilde{u}(r,\theta) \to h(\theta_0)$ if $(r,\theta) \to (a,\theta)$.

For that let us consider $r \in [0,a)$ such that $a - r < \delta$. We have

$$u(r,\theta) - h(\theta_0) = \int_{-\pi}^{\pi} P(r,\theta-\phi)[h(\phi) - h(\theta_0)]\frac{d\phi}{2\pi}$$

by the second fact.

But $P(r,\theta)$ is concentrated in $\theta = 0$ in the sense that for $\theta \in (\delta/2, 2\pi - \delta/2)$ we have

$$|P(r,\theta)| = \frac{a^2-r^2}{a^2-2ar\cos\theta+r^2} = \frac{a^2-r^2}{(a-r)^2 + 4ar\sin^2(\theta/2)} < \epsilon. \tag{47}$$

for some $\delta > 0$ and $a - r$ small. (We used $1 - \cos\theta = \cos(\frac{\theta}{2} - \frac{\theta}{2}) - \cos(\frac{\theta}{2} + \frac{\theta}{2}) = -2\sin^2(\frac{\theta}{2})$)

Now we break the integral into two pieces:

$$|u(r,\theta) - h(\theta_0)| \leq \int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta-\phi)|h(\phi) - h(\theta_0)|\frac{d\phi}{2\pi} + \int_{|\phi-\theta_0|>\delta} P(r,\theta-\phi)|h(\phi) - h(\theta_0)|\frac{d\phi}{2\pi}$$

Given $\epsilon > 0$ we can choose $\delta > 0$ small such that $|h(\phi) - h(\theta_0)| < \epsilon$ for $|\phi - \theta_0| < \delta$.

Hence, the first integral can be estimated by

$$\int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta-\phi)\epsilon\frac{d\phi}{2\pi} \leq \int_{-\pi}^{\pi} P(r,\theta-\phi)\frac{d\phi}{2\pi} = \epsilon.$$

For the second integral we use (47) and that h is bounded on $\partial B_a(0)$ by a constant M:

$$\int_{|\phi-\theta_0|>\delta} P(r,\theta-\phi)2M\frac{d\phi}{2\pi} \leq \epsilon 2M$$

provided $|\theta - \theta_0| < \frac{\delta}{2}$.

Application: Mean Value Property, 2n Proof. Let u be harmonic on U and let $B_r(\mathbf{x}_0) \subset U$.

We replace $u(\mathbf{x})$ with $u(\mathbf{x}-\mathbf{x}_0)$ and $B_r(\mathbf{x}_0)$ and U with $B_r(0)$ with $U - \mathbf{x}_0$. By Poisson's formula

$$u(0) = \frac{r^2-0^2}{2\pi r}\int_{\partial B_r(0)} \frac{u(\mathbf{y})}{|\mathbf{y}-0|^2}ds(\mathbf{y}) = \frac{r^2}{2\pi r}\int_{\partial B_r(0)} \frac{u(\mathbf{y})}{r^2}ds(\mathbf{y}) = \frac{1}{\omega_1 r}\int_{\partial B_r(0)} f(\mathbf{y})ds(\mathbf{y}).$$

Lecture 23
Recall the notation

$$B_a(0) = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |\mathbf{x}| < a\} = B_a, \quad \overline{B_a(0)} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |\mathbf{x}| \le a\} = \overline{B_a}.$$

Moreover $\overline{B_a} \backslash B_a =: \partial B_a$.

Definition 8.22 (**Dirichlet Problem for the Laplace equation on** B_a**).** Let $h \in C^0(\partial B_a)$. Find $u \in C^2(B_a) \cap C^0(\overline{B_a})$ such that

$$\Delta u = u_{x_1,x_1} + u_{x_2,x_2} = 0 \ \text{ in } \ B_a \ \ \& \ \ u = h \ \text{ in } \ \partial B_a. \tag{48}$$

Polar coordinates: $(x_1, x_2) = (r\cos\theta, r\sin\theta), \ r \in (0,\infty), \theta \in \mathbb{R}$.

Definition 8.23 (Dirichlet Problem for the Laplace equation on B_a in polar coordinates)**.** Let $\tilde{h}(\theta) \in C^0(\mathbb{R})$ with $\tilde{h}(\theta + 2\pi) = \tilde{h}(\theta)$ (for isnstance we choose $\tilde{h}(\theta) = h(a\cos\theta, a\sin\theta)$).

Find $\tilde{u} \in C^2((0,a) \times \mathbb{R}) \cap C^0((0,a] \times \mathbb{R})$ such that

$$\tilde{\Delta}\tilde{u} = \tilde{u}_{r,r} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta,\theta} = 0 \ \text{ in } \ (0,a) \times \mathbb{R} \ \ \& \ \ \tilde{u}(a,\theta) = \tilde{h}(\theta) \ \text{ for } \ \theta \in \mathbb{R}. \tag{49}$$

and

$$\tilde{u}(r,\theta) = \tilde{u}(r, \theta + 2\pi) \ \text{ for } \ r \in (0,\infty), \theta \in \mathbb{R} \ \ \& \ \ \limsup_{r \to 0} |\tilde{u}(r,\theta)| < \infty.$$

Theorem 8.24 (Poisson Formula in Polar coordinates)**.** *The unique solution of the Dirichlet Problem* (49) *is given by the Poisson formula*

$$\tilde{u}(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\tilde{h}(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} d\phi = \int_0^{2\pi} P(r, \theta - \phi)\tilde{h}(\phi)\frac{d\phi}{2\pi}.$$

where

$$P(r,\theta) = \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta)}$$

is called the **Poisson kernel**.

Theorem 8.25 (Poisson Formula in Euclidean coordinates)**.** *The unique solution* $u \in C^2(B_a) \cap C^0(\overline{B_a})$ *of* (48) *is given by*

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{\partial B_a} \frac{h(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} ds(\mathbf{y}).$$

In particular, we showed that

$$\tilde{u}(r,\theta) \to \tilde{h}(\theta_0) \ \text{ if } \ (r,\theta) \to (a, \theta_0).$$

In fact, from the proof we can see that this convergence is uniform w.r.t. θ:

$$\sup_{\theta_0} |\tilde{u}(r,\theta_0) - \tilde{h}(\theta_0)| \to 0 \ \text{ if } r \to a.$$

8.9. **Consequences of the Poisson formula in polar coordinates.** Recall the L^2 or **mean square norm**

$$\left\|\tilde{h}\right\|_2 = \sqrt{\int_0^{2\pi} (\tilde{h}(\theta))^2 d\theta}.$$

for $\tilde{h} \in C^0(\mathbb{R})$ that is 2π-periodic.

Theorem 8.26 (Mean square convergence of the Full Fourier series)**.** *Let $\tilde{h} \in C^0(\mathbb{R})$ that is 2π-periodic. Then, the full Fourier serie of $\tilde{h}$ converges in L^2 or mean square sense to $\tilde{h}$. More precisely*

$$\left\|\tilde{h} - \mathcal{S}^N\right\|_2 \to 0 \text{ for } N \to \infty.$$

where $\mathcal{S}^N$ are the partial sums of the **full Fourier series** *of $\tilde{h}$.*

Proof. In the proof of the Poisson formula we saw that

$$P(r, \theta - \phi) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (\sin(n\theta)\sin(n\phi) + \cos(n\theta)\cos(n\phi))$$

uniformily in $\phi, \theta \in \mathbb{R}$ and $r \in [c, d] \subset (0, a)$.

We insert this back into Poisson's formula. By uniform convergence we can exchange integration w.r.t. ϕ and summation w.r.t. $n \in \mathbb{N}$. It follows

$$\tilde{u}(r,\theta) = \underbrace{\int_0^{2\pi} h(\phi)\frac{d\phi}{2\pi}}_{=A_0(r)} + \sum_{n=1}^{\infty} \underbrace{\left(\left(\frac{r}{a}\right)^n \int_0^{2\pi} \tilde{h}(\phi)\cos(n\phi)\frac{d\phi}{\pi}\right)}_{=:A_n(r)} \cos(n\theta)$$

$$+ \sum_{n=1}^{\infty} \underbrace{\left(\left(\frac{r}{a}\right)^n \int_0^{\pi} \tilde{h}(\phi)\sin(n\phi)\frac{d\phi}{2\pi}\right)}_{=:B_n(r)} \sin(n\theta). \tag{50}$$

The right hand side is still a uniformily converging series in $\theta \in \mathbb{R}$. Hence, after multiplying with $\sin(k\theta)$, $k \in \mathbb{N}$, $\cos(k\theta)$, $k \in \mathbb{N}$ or 1, and then integrating w.r.t. θ over $[0, 2\pi]$ we get that $A_n(r)$ and $B_n(r)$ are the Fourier coefficients of $\theta \mapsto \tilde{u}(r, \theta)$ and (50).

We denote the Fourier partial sums $\mathcal{S}^N(r,\theta) = A_0(r) + \sum_{n=1}^N (A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta))$.

$\mathcal{S}^N(r,\cdot)$ converges uniformily to $\tilde{u}(r,\cdot)$ if $0 < r < a$.
Note that $A_n(a)$, $B_n(a)$ become the Fourier coefficients of $\tilde{h}$ and that

$$A_n(r) = \left(\tfrac{r}{a}\right)^n A_n(a), \ B_n(r) = \left(\tfrac{r}{a}\right)^n B_n(a), \ n \in \mathbb{N}.$$

Hence $\mathcal{S}^N(a,\cdot) = \mathcal{S}^N(\cdot)$ are the partial sums of the Fourier serie of $\tilde{h}$.

But we don't know if $\mathcal{S}^N$ converges uniformily to $\tilde{h}$.

Recall

$$\left\|\tilde{h}\right\|_2 = \sqrt{\int_0^{2\pi} (\tilde{h}(\theta))^2 d\theta} \le \sup_{\theta\in\mathbb{R}} |\tilde{h}(\theta)|.$$

It follows

$$0 < \left\|\tilde{h} - \mathcal{S}^N(\cdot)\right\|_2 \le \underbrace{\left\|\tilde{h} - \tilde{u}(r,\cdot)\right\|_2}_{\le \sup_{\theta\in\mathbb{R}}|\tilde{h}(\theta) - \tilde{u}(r,\theta)|} + \underbrace{\left\|\tilde{u}(r,\cdot) - \mathcal{S}^N(r,\cdot)\right\|_2}_{\le \sup_{\theta\in\mathbb{R}}|\tilde{u}(r,\theta) - \mathcal{S}^N(r,\theta)|} + \underbrace{\left\|\mathcal{S}^N(r,\cdot) - \mathcal{S}^N(\cdot)\right\|_2}_{\le |(\frac{r}{a})^n - 1| \|\mathcal{S}^N(\cdot)\|_2}.$$

Given $\eta > 0$ we pick $r \in (0, a)$ close to a such that

$$\sup_\theta \left|\tilde{h}(\theta) - \tilde{u}(r,\theta)\right| \le \eta \quad \& \quad \left|\left(\frac{r}{a}\right)^n - 1\right| \le \eta$$

Moreover, by Bessel's inequality, we have

$$\left\|\mathcal{S}^N(\cdot)\right\|_2^2 = A_0^2 + \sum_{n=1}^N (A_n^2 + B_n^2) \le \left\|\tilde{h}\right\|_2^2.$$

Hence

$$\left\|\tilde{h} - \mathcal{S}^N(\cdot)\right\|_2 \le \eta + \sup_{\theta\in\mathbb{R}} |\tilde{u}(r,\theta) - \mathcal{S}^N(r,\theta)| + \eta\left\|\tilde{h}\right\|_2$$

Therefore

$$\limsup_{N\to\infty}\left\|\tilde{h} - \mathcal{S}^N(\cdot)\right\|_2 \le \limsup_{N\to\infty}\left(\eta + \sup_{\theta\in\mathbb{R}} |\tilde{u}(r,\theta) - \mathcal{S}^N(r,\theta)| + \eta\left\|\tilde{h}\right\|_2\right) \le \eta(1+\left\|\tilde{h}\right\|_2).$$

Since $\eta > 0$ was arbitrary, $\limsup_{N\to\infty}\left\|\tilde{h} - \mathcal{S}^N(\cdot)\right\|_2 = \lim_{N\to\infty}\left\|\tilde{h} - \mathcal{S}^N(\cdot)\right\|_2 = 0.$ □

More Consequences. Let $\Omega \subset \mathbb{R}^2$ be open. $u \in C^2(\Omega)$ harmonic if and only if $\Delta u = 0$ on Ω.

Theorem 8.27. *Mean value property Let $u \in C^2(\Omega)$ be harmonic. Then*

$$u(\mathbf{x}) = \frac{1}{2\pi a}\int_{\partial B_a(\mathbf{x})} u(\mathbf{y})ds(\mathbf{y}) = \frac{1}{\mathrm{vol}(B_a(\mathbf{x}))}\int_{B_a(\mathbf{x})} u(\mathbf{y})d\mathbf{y}$$

for any $a > 0$ such that $B_a(\mathbf{x}) \subset \Omega$.

Theorem 8.28. *Liouville theorem Let $u \in C^2(\mathbb{R}^2)$ be harmonic and $\sup|u| \le C < \infty$.*
Then $u(\mathbf{x}) \equiv const$.

Theorem 8.29. *Let $\Omega \subset \mathbb{R}^2$ open and $u \in C^0(\Omega)$ such that the mean value property holds:*

$$u(\mathbf{x}) = \frac{1}{2\pi a}\int_{\partial B_a(\mathbf{x})} u(\mathbf{y})ds(\mathbf{y}) \quad \text{for } a > 0 \text{ whenever } B_a(\mathbf{x}) \subset \Omega.$$

It follows $u \in C^\infty(\Omega)$ and u is harmonic.

Lecture 24

Let $h \in C^0(\mathbb{R})$ 2π-periodic. We showed that the full Fourier series of h converges in L^2-sense to h.

Remark 8.30. More generally, the following holds.

If h is a 2π-periodic function that is integrable on $[0, 2\pi]$ (or on any interval of length 2π) s.t. $\int_0^{2\pi}(h(x))^2dx < \infty$ then the full Fourier series of h converges in L^2-sense to h.

This follows because we can approximate any 2π-periodic function that is integrable on $[0, 2\pi]$ with $\int_0^{2\pi} h(x)^2dx < \infty$ in L^2-sense by a sequence of 2π periodic, continuous functions. I will omit details.

Corollary 8.31 (Parseval Identity)**.**

$$(\frac{1}{2}A_0)^2\int_0^{2\pi} 1dx + \sum_{n=1}^{\infty} A_n^2 \int_0^{2\pi} \cos(nx)^2dx + \sum_{n=1}^{\infty} B_n^2 \int_0^{2\pi} \sin(nx)^2dx = \int_0^{2\pi} [h(x)]^2dx.$$

Proof. Follows directly from the *Least Square Approximation Theorem.* □

Application. Let $h(x)$ be the periodic extension of x on $[-\pi, \pi]$. Then

$$\int_0^{2\pi} h(x)^2dx = \int_{-\pi}^{\pi} x^2dx = \frac{2}{3}\pi^3 < \infty$$

Moreover, since x is an odd function on $[-\pi, \pi]$, we compute that $A_n = 0$ and

$$B_n = \int_0^{2\pi} h(x)\sin(nx)dx = (-1)^{n+1}\frac{2}{n}, \quad B_n^2 = \frac{4}{n^2}.$$

Hence, by Parseval's inequality it follows $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Theorem 8.32 (Liouville Theorem)**.** *Let $u \in C^2(\mathbb{R}^2)$ be harmonic s.t. $|u(x)| \leq C < \infty$ $\forall x \in \mathbb{R}^2$. Then $u \equiv const.$*

Proof. We pick $\mathbf{x} = x, \mathbf{y} = x \in \mathbb{R}^2$, $r > 0$ and set $R = r + |x-y|_2$ where $|x-y|_2 = \sqrt{\sum_{i=1}^2 (x_i - y)^2}$. It follows $B_r(x) \subset B_R(y)$.

Since u is harmonic and $-C \leq u \leq C$, $\hat{u} = u + C > 0$ is harmonic as well.

Applying the mean value property for $\hat{u}$ yields

$$\begin{aligned} \hat{u}(x) &= \frac{1}{\mathrm{vol}(B_r(x))}\int_{B_r(x)} \hat{u}(z)dz \\ (\hat{u} \geq 0) \quad &\leq \frac{1}{\mathrm{vol}(B_r(x))}\int_{B_R(y)} \hat{u}(z)dz = \frac{\mathrm{vol}(B_R(y))}{\mathrm{vol}(B_r(x))}\hat{u}(y) = \frac{(r+|x-y|_2)^2}{r^2}\hat{u}(y) \to \hat{u}(y). \end{aligned}$$

Hence $u(x) \leq u(y)$. Exchanging the role of x and y yields $u(x) = u(y)$, and since x and y were arbitrary we obtain the result. □

Theorem 8.33. *Let $\Omega \subset \mathbb{R}^2$ be open and $u \in C^0(\Omega)$ satisfy the mean value property*

$$u(x) = \frac{1}{\mathrm{vol}(B_r(x))}\int_{B_r(x)} u(z)dz \ \forall x \in \Omega \text{ and } \forall r > 0 \text{ s.t. } B_r(x) \subset \Omega.$$

Then $u \in C^\infty(\Omega)$ and u is harmonic.

Proof. Pick $\phi \in C_0^\infty(B_1)$ s.t. $\int_{\mathbb{R}^2} \phi(x)dx = \int_{B_1} \phi(x)dx = 1$ where $B_1 = B_1(0) = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ and radial $\phi(x) = \psi(|x|)$.

Applying the transformation formula for polar coordinates we see

$$1 = \int \phi(x)dx = \int_0^{2\pi}\int_0^1 \phi(r\cos\theta, r\sin\theta)rdrd\theta = 2\pi\int_0^1 r\psi(r)dr.$$

We also define $\phi_\epsilon(x) = \frac{1}{\epsilon^2}\phi(x/\epsilon)$. In particular $\phi_\epsilon \in C_0^\infty(B_\epsilon)$.

Let us pick $y \in \Omega$ and $\epsilon > 0$ s. t. $B_\epsilon(y) \subset \Omega$. Then

$$\begin{aligned}\int_{\mathbb{R}^2} u(z)\phi_\epsilon(z-y)dy &= \int_{B_\epsilon(y)} u(z)\phi_\epsilon(z-y)dz = \int_{B_\epsilon} \underbrace{u(z+y)}_{=:f(z)}\phi_\epsilon(z)dz \\ \left[\tilde{f}(r,\theta) := f(r\cos\theta, r\sin\theta)\right] \qquad &= \int_0^{2\pi}\int_0^\epsilon \tilde{(}r,\theta)\frac{1}{\epsilon^2}\psi(r/\epsilon)rdrd\theta \\ &= \int_0^\epsilon \underbrace{\int_0^{2\pi}\tilde{f}(r,\theta)rd\theta}_{\int_{\partial B_r(0)} f(z)ds(z) = 2\pi r f(0)} \frac{1}{\epsilon^2}\psi(r/\epsilon)dr = u(y). \qquad (51)\end{aligned}$$

Since $\phi_\epsilon \in C_0^\infty(B_\epsilon)$, for the left hand side we compute

$$\frac{\partial}{\partial y_i}\int_{\mathbb{R}^2} u(z)\phi_\epsilon(z-y)dy = \int_{\mathbb{R}^2} u(z)\frac{\partial}{\partial y_i}[\phi_\epsilon(z-y)]dy, \ i = 1, 2.$$

Hence, all partial derivatives of u exist in y. The right hand side of (51) is continuous w.r.t. y because $y \mapsto \frac{\partial}{\partial y_i}[\phi_\epsilon(\cdot - y)]$ is continuous w.r.t. uniform convergence. Hence $u \in C^1(\Omega)$.

Similarly, we can compute all higher order partial derivatives and hence $u \in C^\infty(\Omega)$.

u is harmonic. $x \in \Omega$. By the mean value property

$$u(x) = \frac{1}{2\pi r}\int_{\partial B_r} u(y)ds(y) = \frac{1}{2\pi r}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)rd\theta \quad \forall r > 0 \text{ s.t. } B_r(x) \subset \Omega.$$

We set $u(y+x) = f(y)$. we compute

$$\begin{aligned}0 &= \frac{d}{dr}\int_0^{2\pi} f(r\cos\theta, r\sin\theta)d\theta = \int_0^{2\pi}\langle \nabla f(r\cos\theta, r\sin\theta), \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\rangle d\theta \\ &= \frac{1}{r}\int_0^{2\pi}\langle \nabla f(r\cos\theta, r\sin\theta), N(r\cos\theta, r\sin\theta)\rangle rd\theta = \frac{1}{r}\int_{\partial B_r(x)}\langle \nabla u, N\rangle ds(x) = \frac{1}{r}\int_{B_r(x)} \Delta u dx.\end{aligned}$$

where $N(r\cos\theta, r\sin\theta) = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ is the unite normal for $\partial B_r(0)$ in $(r\cos\theta, r\sin\theta)$. ☐

Remark 8.34. The previous results hold in any dimension $n \geq 1$.

9. Green Identities and Green function

9.1. **Green Identities.** Let $\Omega \subset \mathbb{R}^n$ be open with smooth boundary. Let $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$. One computes the following

$$\nabla \cdot (v\nabla u) = \text{Div}(v\nabla u) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(v\frac{\partial u}{\partial x_i}\right) = \langle \nabla v, \nabla u\rangle + v\Delta u \ \text{ in } \Omega.$$

(1) **First Green Identity.** For $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$ it holds

$$\int_\Omega \langle \nabla v, \nabla u\rangle dx + \int_\Omega v\Delta u = \int_\Omega \text{Div}(v\nabla u)dx = \int_{\partial\Omega}\langle N, v\nabla u\rangle ds = \int_{\partial\Omega} v\langle N, \nabla u\rangle ds$$

where N is the unite normal vector field along the boundary of Ω.

We used the divergence theorem for the second equality.

(2) **Second Green Identity.** If $u, v \in C^2(\overline{\Omega})$, then

$$\int_U v\Delta u dx - \int u\Delta v dx = \int_{\partial U} v\frac{\partial u}{\partial N}ds - \int_{\partial U} u\frac{\partial v}{\partial N}ds$$

where $\frac{\partial u}{\partial N}(x) = \langle N(x), \nabla u(x)\rangle$ and the same for v.

9.2. **Dirichlet Principle.** Physical Idea: Among all functions u on Ω that describe a possible state of physical system, the prefered state is the one with minimal *kinetic energy*.

States of the system: $u \in C^1(\overline{\Omega})$.

Boundary condition: $u|_{\partial\Omega} = h$, $h \in C^1(\Omega)$.

Kinetic Energy: $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$ (*Dirichlet energy*).

Theorem 9.1. *Let $\Omega \subset \mathbb{R}^n$ be open with smooth boundary $\partial\Omega$. A function $u \in C^2(\overline{\Omega})$ is the unique harmonic function with $u|_{\partial\Omega} = h$ if and only if it minimizes E w.r.t. all $w \in C^1(\overline{\Omega})$ s.t. $u|_{\partial\Omega} = h$. More precisely*

$$E(u) \le E(w)\ \forall w \in \{w \in C^1(\overline{\Omega}) : w|_{\partial\Omega} = h\} =: \mathcal{E}.$$

Proof. Let $u \in \mathcal{E}$ be harmonic and $w \in \mathcal{E}$. Set $v := u - w$. Then $v|_{\partial\Omega} = 0$. With Green's first identity we compute

$$\begin{aligned} E(w) &= \frac{1}{2}\int_\Omega |\nabla(u-v)|^2 dx = \frac{1}{2}\int_\Omega \left[|\nabla u|^2 - 2\langle \nabla u, \nabla v\rangle + |\nabla v|^2\right] dx \\ &= E(u) + \underbrace{E(v)}_{\ge 0} + \int_\Omega v\Delta u dx - \int_{\partial\Omega} v\frac{\partial u}{\partial N} dx \ge E(u). \end{aligned}$$

Now, assume $u \in \mathcal{E}$ minimizes the Dirichlet energy. Let $\phi \in C^1_c(\Omega)$. Then $u + t\phi \in \mathcal{E}\ \forall t \in \mathbb{R}$ and

$$E(u + t\phi) = E(u) + \frac{1}{2}t\int_\Omega \langle \nabla u, \nabla\phi\rangle dx + t^2 E(\phi).$$

Hence $t \mapsto E(u + t\phi)$ is a Polynomial and continuously differentiable. Since u minimizes E, it follows

$$0 = \frac{d}{dt}\Big|_{t=0} E(u + t\phi) = \frac{1}{2}\int \langle \nabla u, \nabla\phi\rangle dx = -\frac{1}{2}\int_\Omega \phi\Delta u dx + \underbrace{\int_{\partial\Omega} \phi\frac{\partial u}{\partial N} ds}_{=0}\quad \forall \phi \in C^2_c(\Omega).$$

The fundamental theorem of Calculus of Variations yields $\Delta u = 0$ on Ω. □

Remark 9.2.

$$C^k_0(\Omega) = \left\{\phi \in C^k(\Omega) : \phi|_{\partial\Omega} = 0\right\},\quad C^k_c(\Omega) = \left\{\phi \in C^k(\Omega) : \overline{\{x \in \Omega : \phi(x) \neq 0\}} \subset \Omega \text{ compact}\right\}.$$

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9.3. **Representation formula.**

Theorem 9.3 (Representation formula)**.** *Let $n = 3$ and $\Omega \subset \mathbb{R}^3$ with smooth boundary. Let $u \in C^2(\overline{\Omega})$ and $x \in \Omega$. Then*
$$u(x) = \int_{\partial\Omega} \left[-u(y) \frac{\partial}{\partial N} \left(\frac{1}{|y-x|} \right) + \frac{1}{|y-x|} \frac{\partial u}{\partial N}(y) \right] \frac{ds}{4\pi}.$$

Proof. Recall $\Phi(x) := -\frac{1}{4\pi}\frac{1}{|x|}$ is harmonic on $\mathbb{R}^3 \backslash \{0\}$. Especially $\Phi(y-x) =: v(y)$ is harmonic on $\mathbb{R}^3 \backslash \{x\}$.

Pick $\epsilon > 0$ and such that $B_\epsilon(x) \subset \Omega$ and define $\Omega_\epsilon = \Omega \backslash \overline{B_\epsilon(x)}$. Hence Ω_ϵ is open and has smooth boundary.

By Green's second identity
$$\begin{aligned} 0 = \int_{\Omega_\epsilon} (v\Delta u - u \Delta v)\, dx &= \int_{\partial\Omega_\epsilon} \left(v\frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds \\ &= \int_{\partial\Omega} \left(v\frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds + \int_{\partial\Omega_\epsilon} \left(v\frac{\partial u}{\partial N} - u \frac{\partial v}{\partial N} \right) ds. \end{aligned}$$

The first integral on the RHS is minus the RHS of the displayed equation in the theorem. For the second term on the RHS we compute
$$\text{2n term, RHS} = \int_{\partial B_\epsilon(0)} \Big[-\frac{1}{4\pi|y|} \underbrace{\frac{\partial u}{\partial N}(y+x)}_{=:f(y)} + \underbrace{u(y+x)}_{=:g(y)} \frac{\partial}{\partial N} \frac{1}{4\pi|y|} \Big] ds(y) = (*)$$

Let us introduce spherical coordinates $(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$ and define $\tilde{f}(\epsilon,\theta,\phi) = f(\epsilon\cos\theta\sin\phi, \epsilon\sin\theta\sin\phi, \epsilon\cos\phi)$ and similarly for g. Note that for inward point normal field N on $\partial B_\epsilon(0)$ one has $\frac{\partial}{\partial N} = -\frac{d}{dr}$. Then
$$\begin{aligned} (*) &= \int_0^{2\pi} \int_0^\pi \Big[-\frac{1}{4\pi\epsilon} \tilde{f}(\theta,\phi) - \tilde{g}(\epsilon,\theta,\phi) \frac{\partial}{\partial r}\Big|_{r=\epsilon} \left(\frac{1}{4\pi r} \right) \Big] \epsilon^2 \sin\phi d\phi d\theta \\ &= \int_0^{2\pi} \int_0^\pi \Big[-\frac{1}{4\pi\epsilon} \tilde{f}(\epsilon,\theta,\phi) + \tilde{g}(\epsilon,\theta,\phi) \frac{1}{4\pi\epsilon^2} \Big] \epsilon^2 \sin\phi d\phi d\theta \\ &= \frac{\epsilon}{4\pi} \int_0^{2\pi} \int_0^\pi -\tilde{f}(\epsilon,\theta,\phi) \sin\phi d\phi d\theta + \underbrace{\frac{1}{4\pi\epsilon^2} \int_0^{2\pi} \int_0^\pi \tilde{g}(\epsilon,\theta,\phi)\epsilon^2 \sin\phi d\phi d\theta}_{\to \frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi u(x)\sin\phi d\phi d\theta = u(x)} \end{aligned}$$

We will show that the first integral tends to 0. First, we observe that
$$|f(y)| = |\frac{\partial u}{\partial N}(y+x)| = |\langle N(y+x), \nabla u(y+x) \rangle| \le |\nabla u(y+x)|.$$

Since $u \in C^2(\overline{\Omega})$, $|\nabla u|$ is continuous. Hence, given $C > 0$ we can pick $\epsilon_0 > 0$ such that
$$|\nabla u|(y+x) \le |\nabla|(x) + C \ \forall y \in \partial B_\epsilon(0) \text{ and } \epsilon \in (0, \epsilon_0).$$

Hence, for the first integral on the RHS we estimate
$$\begin{aligned} \frac{\epsilon}{4\pi} \left| \int_0^{2\pi} \int_0^\pi \tilde{f}(\epsilon,\theta,\phi) \sin\phi d\phi d\theta \right| &\le \frac{\epsilon}{4\pi} \int_0^{2\pi} \int_0^\pi |\tilde{f}(\epsilon,\theta,\phi)| \sin\phi d\phi d\theta \\ &\le \frac{\epsilon}{4\pi} \int_0^{2\pi} \int_0^\pi (|\nabla(x)| + C) \sin\phi d\phi d\theta = \epsilon(|\nabla u|(x) + C). \end{aligned}$$

So when ϵ tends to 0, this proves the claim. □

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Representation formula in 2D. Let $\Omega \subset \mathbb{R}^2$ with smooth boundary and let $u \in C^2(\overline{\Omega})$ be harmonic. Then

$$u(x) = \int_{\partial\Omega} \Big[u(y) \frac{\partial}{\partial N} \log|y-x| - \log|y-x| \frac{\partial u}{\partial N}(y) \Big] \frac{ds(y)}{2\pi}.$$

Definition 9.4 (Fundamental solution)**.** The harmonic function

$$x \in \mathbb{R}^n \backslash \{0\} \mapsto \Phi_n(x) = \begin{cases} \frac{1}{2\pi} \log|x|, \\ -\frac{1}{n(n-2)\operatorname{vol}(B_1(0))} \frac{1}{|x|^{n-2}} \end{cases}$$

is called the fundamental solution for the Laplace equation.

Remark 9.5. Using $\Phi_n(x)$ one derives a Representation formula for any $n \in \mathbb{N}$.

9.4. **Green's function. Goaol:** Given $\Omega \subset \mathbb{R}^n$ with smooth boundary, we want to modify Φ_n s.t. the second term in the Repres. Formula vanishes.

Definition 9.6 (Green's function)**.** Let $\Omega \subset \mathbb{R}^n$ be open with smooth boundary.

The Green's function $G(x)$ for the operator Δ and the domain Ω at a point $x_0 \in \Omega$ is a function $G : \overline{\Omega} \backslash \{x_0\} \to \mathbb{R}$ s.t.

(i) $G \in C^2(\overline{\Omega} \backslash \{x_0\})$ and $\Delta G = 0$ on $\Omega \backslash \{x_0\}$
(ii) $G|_{\partial\Omega} = 0$
(iii) $G(x) - \Phi_n(x - x_0) =: H(x)$ is finite at x_0 and $H \in C^2(\overline{\Omega})$ with $\Delta H = 0$ on Ω.

Remark 9.7. It can be shown that the Green's function always exists.

The idea is to solve the problem

$$\begin{aligned} \Delta H^{x_0} &= 0 \text{ on } \Omega, \\ H^{x_0}|_{\partial\Omega} &= -\Phi_n(\cdot - x_0) \text{ on } \partial\Omega \end{aligned}$$

and define $G^{x_0} = H^{x_0}(x) + \Phi_n(x - x_0)$.

Notation: $G^{x_0}(x) = G(x, x_0)$.

Theorem 9.8. *Let $\Omega \subset \mathbb{R}^n$ open with smooth boundary and let $h \in C^2(\partial\Omega)$. Let $u \in C^2(\overline{\Omega})$ be a solution of the Dirichlet problem*

$$\begin{aligned} \Delta u &= 0 \text{ on } \Omega, \\ u|_{\partial\Omega} &= h \text{ on } \partial\Omega. \end{aligned}$$

Then

$$u(x_0) = \int_{\partial\Omega} h(x) \frac{\partial}{\partial N} G(x, x_0) dx.$$

Proof. We write $G(x, x_0) - H^{x_0}(x) = \Phi_n(x - x_0)$. Then, by the Repres. formula and Green's second identity applied for u and H^{x_0} we obtain

$$\begin{aligned} u(x_0) &= \int_{\partial\Omega} \Big[u(y) \frac{\partial}{\partial N} \Phi_n(y - x_0) - \Phi_n(y - x_0) \frac{\partial u}{\partial N}(y) \Big] ds(y) \\ &= \int_{\partial\Omega} h(y) \frac{\partial}{\partial N} G(x, x_0) ds(y) - \int_{\partial\Omega} \Big[u(y) \frac{\partial}{\partial N} H^{x_0}(y) - H^{x_0}(y) \frac{\partial u}{\partial N}(y) \Big] ds(y) \\ &= \int_{\partial\Omega} h(y) \frac{\partial}{\partial N} G(x, x_0) ds(y) - \underbrace{\int_{\Omega} \big[u(y) \Delta H^{x_0}(y) - H^{x_0}(y) \Delta u(y) \big] dy}_{=0}. \end{aligned}$$

□

Theorem 9.9. *Let $\Omega \subset \mathbb{R}^3$ open with smooth boundary, let $f \in C^0(\overline{\Omega})$. Let $u \in C^2(\overline{\Omega})$ be a solution of the Dirichlet problem*

$$\Delta u = f \text{ on } \Omega,$$
$$u|_{\partial\Omega} = 0 \text{ on } \partial\Omega.$$

Then

$$u(x_0) = \int_\Omega G(x, x_0) f(x) dx.$$

Proof. Let $\epsilon > 0$ s.t. $B_\epsilon(x_0) \subset \Omega$ and set $\Omega_\epsilon \backslash \overline{B_\epsilon(x_0)}$. Green's second identity yields

$$\begin{aligned}\int_\Omega G(x,x_0)f(x)dx - \int_{B_\epsilon(x_0)} G(x,x_0)f(x)dx &= \int_{\Omega_\epsilon} G(x,x_0)\Delta u(x)dx \\ &= \int_{\partial\Omega}\Big[G(x,x_0)\frac{\partial u}{\partial N}(x) - u(x)\frac{\partial}{\partial N}\Big|_x G(\cdot,x_0)\Big]dx \\ &+ \int_{\partial B_\epsilon(x_0)}\Big[G(x,x_0)\frac{\partial u}{\partial N}(x) - u(x)\frac{\partial}{\partial N}\Big|_x G(\cdot,x_0)\Big]dx. \end{aligned} \tag{52}$$

The first term on the RHS of the previous equality is 0. Since $G(x,x_0) = H^{x_0}(x) + \Phi(x - x_0)$, the second term on the RHS rewrites as

$$\int_{\partial B_\epsilon(x_0)}\Big[\Phi(x-x_0)\frac{\partial u}{\partial N}(x) - u(x)\frac{\partial}{\partial N}\Big|_x \Phi(\cdot,x_0)\Big]dx + \int_{\partial B_\epsilon(x_0)}\Big[H^{x_0}(x)\frac{\partial u}{\partial N}(x) - u(x)\frac{\partial}{\partial N}\Big|_x H^{x_0}(\cdot)\Big]dx.$$

The second term is 0 since H^{x_0} and u are harmonic on $B_\epsilon(x_0)$ and because of Green's 2nd identity. The first term converges to $u(x_0)$ as $\epsilon \to 0$. This is exactly what we showed in the proof of the Representation formula for the Laplace equation.

Finally, the second term on the LHS in (52) is

$$\int_{B_\epsilon(x_0)} G(x,x_0)f(x)dx = \int_{B_\epsilon(x_0)} (\Phi(x-x_0) + H^{x_0}(x))f(x)dx.$$

Since $H^{x_0}(x)$ and f are continuous in x_0, there exists $\epsilon_0 > 0$ and $C > 0$ such that

$$|H^{x_0}(x)f(x)| \le C \text{ if } |x - x_0| \le \epsilon \text{ and } \epsilon \in (0, \epsilon_0).$$

Hence, for $\epsilon \in (0, \epsilon_0)$

$$\left|\int_{B_\epsilon(x_0)} H^{x_0}(x)f(x)dx\right| \le \int_{B_\epsilon(x_0)} Cdx \le C\,\mathrm{vol}(B_\epsilon(x_0)) \to 0.$$

Moreover

$$\left|\int_{B_\epsilon(x_0)} \Phi(x-x_0)f(x)dx\right| \le \underbrace{\sup_{x\in B_\epsilon(x_0)} |f(x)|}_{\le M} \int_{B_\epsilon(0)} |\Phi(x)|dx \le \underbrace{M\int_0^{2\pi}\int_0^\pi\int_0^\epsilon \frac{1}{4\pi}\frac{1}{r}r^2 dr \sin\phi d\phi d\theta}_{=M\frac{1}{2}\epsilon^2 \to 0}.$$

We used that $n = 3$ to compute the integral.

In particular, it follows that the inetral $\int_\Omega G(x,x_0)f(x)dx$ is welldefined. Hence, when $\epsilon \to 0$, (52) yields the result. □

Theorem 9.10 (Symmetry of Green's function)**.** *Let Ω as before. Then, the corresponding Green's function satisfies $G(a,b) = G(b,a)$ $\forall a, b \in \Omega$.*

Proof. We assume again $n = 3$.

Set $v(x) = G(x,b)$ and $u(x) = G(x,a)$. Let $\epsilon > 0$ s.t. $B_\epsilon(a), B_\epsilon(b) \subset \Omega$ and $\overline{B_\epsilon(a)} \cap \overline{B_\epsilon(b)} = \emptyset$.

Set $\Omega_\epsilon \backslash \overline{B_\epsilon(a)} \cup \overline{B_\epsilon(b)}$. With Green's 2nd identity we compute

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} (v\Delta u - u\Delta v)\, dx \\ &= \underbrace{\int_{\partial\Omega} \left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right) ds}_{=0} + \int_{\partial B_\epsilon(a)} (...)ds + \int_{\partial B_\epsilon(b)} (...)ds \end{aligned}$$

Consider

$$\int_{\partial B_\epsilon(a)} \left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\right) ds = \int_{\partial B_\epsilon(a)} \left(v\frac{\partial \Phi(\cdot - a)}{\partial N} - \Phi(\cdot - a)\frac{\partial v}{\partial N}\right) ds + \underbrace{\int_{\partial B_\epsilon(a)} \left(v\frac{\partial H^a}{\partial N} - H^a\frac{\partial v}{\partial N}\right) ds}_{=\int_{B_\epsilon(a)}(v\Delta H^a - H^a\Delta v)dx = 0}$$

The first term on the RHS in the last equation converges to $v(a)$, exactly as in the proof of the Representation formula.

In the same way, one proves that $\int_{\partial B_\epsilon(b)}(...)ds \to -u(b)$. It folllows

$$G(a,b) = v(a) = u(b) = G(b,a).$$

□

9.5. **Green's function for the upper half space.** Let $\mathbb{H} = \{x \in \mathbb{R}^x : x_3 > 0\}$. For a point $y = (y_1, y_2, y_3) \in \mathbb{H}$ one define its refrection at $\partial\mathbb{H}$ as $y^* = (y_1, y_2, -y_3)$.

Theorem 9.11. *The function $H^y(x) = \frac{1}{4\pi|x-y^*|}$ is in $C^2(\overline{\mathbb{H}}$ and solves*

$$\Delta H^y = 0 \quad on\ \mathbb{H},$$
$$H^y|_{\partial\mathbb{H}} = -\Phi|_{\partial\mathbb{H}}.$$

Hence, the Green function of $\mathbb{H}$ is $G(x,y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi|x-y^*|}$.

Proof. (1) $H^y \in C^\infty(\overline{\mathbb{H}})$,
(2) $\Delta H^y = 0$ on $\mathbb{H}$.,
(3) If $x = (x_1, x_2, 0) \in \partial\mathbb{H}$ then $H^y(x) = \frac{1}{4\pi|x-y^*|} = \frac{1}{4\pi|x-y|}$.

Hence $G(x,y) := -\frac{1}{4\pi}\left(\frac{1}{|x-y|} - \frac{1}{|x-y^*|}\right)$ is the Green function of $\mathbb{H}$. □

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9.6. **Green function for the ball $B_a(0)$.** Recall

$$B_a = B_a(0) = \left\{ x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2 + x_3^3} < a \right\} \ \& \ \partial B_a = \left\{ x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2 + x_3^3} = a \right\}$$

and for $x \in B_a$ we define

$$x^* = \frac{a^2}{|x|^2} x.$$

It follows that $|x^*||x| = a^2$.

Theorem 9.12. *The function $H^y(x) = \frac{a}{|y|} \frac{1}{4\pi} \frac{1}{|x-y^*|}$ is in $C^2(\overline{B_a})$ and solves*

$$\Delta H^y = 0 \quad \text{on } \overline{B_a},$$
$$H^y|_{\partial B_a} = -\phi(x-y) = \frac{1}{4\pi} \frac{1}{|x-y|} \quad \text{on } \partial B_a.$$

Hence, the Green function of B_a is

$$G(x,y) = \phi(x-y) + H^y(x) = -\frac{1}{4\pi} \frac{1}{|x-y|} + \frac{1}{4\pi} \frac{a}{|y|} \frac{1}{|x-y^*|}.$$

Proof. (1) $H^y \in C^\infty(\overline{B_a})$,
(2) $\Delta H^y = 0$ on $\overline{B_a}$,
(3) Let $x \in \partial B_a$, that is $|x| = a$. *Claim.* $\frac{|y|}{a}|x-y^*| = |x-y|$.
It follows $H^y\Big|_{\partial B_a} = -\phi(x-y)$.

□

Corollary 9.13 (Poisson formual in 3D)**.** *Let $u \in C^2(\overline{B_a})$ be a solution of*

$$\Delta u = 0 \quad \text{on } \overline{B_a},$$
$$u|_{\partial B_a} = h \quad \text{on } \partial B_a$$

then

$$u(x) = \frac{a^2 - |x|^2}{4\pi a} \int_{\partial B_a} \frac{h(y)}{|x-y|^3} ds(y).$$

Proof. Recall

$$u(x) = \int_{\partial B_a} h(y) \frac{G(\cdot, x)}{\partial N}\Big|_y ds(y).$$

We compute for $x \in \partial B_a$

$$\begin{aligned}
\nabla G(\cdot, y)\Big|_x &= \nabla \left(\frac{1}{4\pi} \frac{1}{|x-y|} - \frac{1}{4\pi} \frac{a}{|y|} \frac{1}{|x-y^*|} \right)\Big|_x \\
&= \frac{1}{4\pi} \frac{1}{|x-y|^3}(x-y) - \frac{1}{4\pi} \frac{a^3}{|y|^3 |x-y^*|^3} \left(\frac{|y|^2}{a^2} x - \frac{|y|^2}{a^2} y^*\right) \\
&= \frac{1}{4\pi} \left(\frac{1}{|x-y|^3}(x-y) - \frac{1}{|x-y|^3}\left(\frac{|y|^2}{a^2} x - y\right) \right) \\
&= \frac{1}{4\pi} \frac{1}{|x-y|^3} \left(x - y - \frac{a^2}{|y|^2} x + y \right) = \frac{1}{4\pi} \frac{1}{|x-y|^3} \left(x - \frac{|y|^2}{a^2} x \right)
\end{aligned}$$

Now, we have that the unit normal vector in x is $\frac{x}{a}$. Hence

$$\langle N, \nabla G(\cdot, y) \rangle = \frac{1}{4\pi} \frac{1}{|x-y|^3} \left(\langle \frac{x}{a}, x \rangle - \frac{|y|^2}{a^2} \langle \frac{x}{a}, x \rangle\right) = \frac{a^2 - |y|^2}{4\pi a} \frac{1}{|x-y|^3}.$$

□

Remark 9.14. The Green function for $B_a \subset \mathbb{R}^n$ is

$$G(x,y) = \frac{-1}{n(n-2)\operatorname{vol}^n(B_1)}\left(\frac{1}{|x-y|^{n-2}} - \frac{a}{|y|}\frac{1}{|x-y^*|^{n-2}}\right)$$

The Poisson formula for $B_a \subset \mathbb{R}^n$, $n > 3$, is

$$u(x) = \frac{a^2-|x|^2}{n\operatorname{vol}^n(B_1)a}\int_{\partial B_a}\frac{h(y)}{|x-y|^n}ds(y).$$

Theorem 9.15. *Let $h \in C^0(\partial B_a)$ with $B_a \subset \mathbb{R}^n$. Let*

$$u(x) = \begin{cases} \frac{a^2-|x|^2}{n\operatorname{vol}^n(B_1)a}\int_{\partial B_a}\frac{h(y)}{|x-y|^n}ds(y) & x \in B_a \\ h(x) & x \in \partial B_a. \end{cases}$$

Then $u \in C^2(B_a)\cap C^0(\overline{B}_a)$ and u is the unique solution of $\Delta u = 0$ on B_a and $u|_{\partial B_a} = h$.

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9.7. **Consequences.**

Corollary 9.16 (Harnack inequality). *Let $\overline{B_R(x_0)} \subset \mathbb{R}^n$ and $x_0 \neq x \in B_R(x_0)$ with $|x - x_0| = r$. Let $u \in C^2(\overline{B_R(x_0)})$, $u \geq 0$ and $\Delta u = 0$ on $\overline{B_R(x_0)}$. Then*

$$\frac{1 - \frac{r}{R}}{\left(1 + \frac{r}{R}\right)^{n-1}} u(x_0) \leq u(x) \leq \frac{1 - \frac{r}{R}}{\left(1 + \frac{r}{R}\right)^{n-1}} u(x_0).$$

In particular, if $r \in (0, \frac{R}{2})$ then $\frac{1}{2^n} u(x_0) \leq u(x) \leq 2^n u(x_0)$.

Proof. Recall the Poisson formula $u(y) = \frac{R^2 - r^2}{n \operatorname{vol}(B_1) R} \int_{\partial B_R(x_0)} \frac{u(y)}{|y-x|^n} ds(y)$ and observe

$$R - r = |x_0 - y| - |x - x_0| \leq |y - x| \leq |x_0 - x| + |x - x_0| = R - r$$

Hence

$$\frac{(R+r)(R-r)}{(R+r)^n} \leq \frac{R^2 - r^2}{|y-x|} \leq \frac{(R+1)(R-1)}{(R-r)^n}$$

Then $u \geq 0$ together with the mean value property imply the statement. □

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Corollary 9.17. *Let $\Omega \subset \mathbb{R}^n$ open and connected, $u \in C^2(\Omega)$ and $\Delta u = 0$. Assume there exists $V \subset \mathbb{R}^n$ open s.t. $\overline{V} \subset \mathbb{R}^n$ and there exists $R > 0$ and $x_1, \dots, x_n \in V$ with $\overline{V} \subset \bigcup_{i=1}^N B_{R/2}(x_i)$ and $\overline{B_R(x_i)} \subset \Omega$ for $i = 1, \dots, N$. Then*

$$u(y) \leq 2^{2nN} u(x) \ \forall x \neq y \in \Omega.$$

In particular $\exists C > 0$ s.t. $\sup_V u \leq C \inf_V u$.

Proof. Given $x, y \in V$. Since V is connected, we can find $z_0, z_1, \dots, z_{l-1}, z_l$ with $l \leq N$ such that $z_0 = x$ and $z_l = y$ and $\{z_{k-1}, z_k\} \subset B_{R/2}(x_{i_k})$ for all $k \in \{1, \dots, l\}$. This can be seen by induction as follows. Pick i_1 s.t. $x \in B_{R/2}(x_{i_1})$. Since V is connected there exists $i_2 \in \{1, \dots, N\} \backslash \{i_1\}$ such that $B_{R/2}(x_{i_2}) \cap B_{R/2}(x_{i_1}) \neq \emptyset$. Hence we pick $z_1 \in B_{R/2}(x_{i_1}) \cap B_{R/2}(x_{i_2})$. Similar, there exists $i_3 \in \{1, \dots, N\}$ such that $B_{R/2}(x_{i_1}) \cap B_{R/2}(x_{i_2}) \cap B_{R/2}(x_{i_3}) \neq \emptyset$ and we can pick $z_2 \in B_{R/2}(x_{i_1}) \cap B_{R/2}(x_{i_3})$ or $z_2 \in B_{R/2}(x_{i_2}) \cap B_{R/2}(x_{i_3})$. We continue inductively til we find i_l for $l \leq N$ such that $y \in B_{R/2}(x_{i_l})$.

With the Harnack inequality, it follows

$$u(x) \leq 2^n u(x_{i_1}) \leq 2^{2n} u(z_{i_1}) \leq 2^{4n} u(x_{i_2}) \leq \cdots \leq 2^{2ln} u(y).$$

□

Some more Remarks on the Green functions. Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth $\partial\Omega$. The Green function was given by $G(x, y) = \Phi_n(x - y) + H^y(x)$ where $H^y \in C^2(\overline{\Omega})$ and H^y solves

$$\Delta H^y = 0 \ \text{ on } \ \Omega \subset \mathbb{R}^n,$$
$$H^y\Big|_{\partial\Omega} = -\Phi_n(\cdot - y)\Big|_{\partial\Omega}.$$

- If Ω is not connected, that is $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$ and $y \in \Omega_1$, then $G(\cdot, y) \equiv 0$ on Ω_2.

  This follows, since $-\Phi(\cdot - y)$ solves $\Delta H^y = 0$ on Ω_2 with $H^y \in C^2(\overline{\Omega})$ and $H^y|_{\partial\Omega_2} = -\Phi_n(\cdot - y)|_{\partial\Omega_2}$.
- If Ω is connected, it follows by the strong Maximum principle that $G(x, y) < 0$ for $x, y \in \Omega$ (Exercise).

- $\frac{\partial G(\cdot,y)}{\partial N}\Big|_x = K_y(x)$ for $x \in \partial\Omega$ is called Poisson kernel of Ω.

  It follows $K_y(x) \geq 0$, since $G(x,y) = 0$, $x \in \partial\Omega$ and $y \in \Omega$, and since $G(x,y) \leq 0$ for all $x, y \in \Omega$.

Theorem 9.18. *If Ω open, bounded with smooth $\partial\Omega$ and connected, then $K_y < 0$ everywhere on $\partial\Omega$.*

Proof. Let $x \in \partial\Omega$ and let N be the unit normal vector in x. Since $\partial\Omega$ is smooth there exists $h_0 < 0$ such that for $x_0 = x + h_0 N$ we have that $B_{|h_0|}(x_0) \subset \Omega\backslash\{y\}$ and $\partial B_{|h_0|}(x_0) \cap \partial\Omega = \{x\}$.

If $-h \in (-h_0, 0)$, then $z = x + hN$ and $|x_0 - z| = |h_0 - h| = -h_0 + h = |h_0| - |h|$. From the Harnack inequality, it follows

$$\frac{G(x+hN,y)}{h} = \frac{-G(x+hN)}{|h|} \geq \frac{1 + \frac{|h_0|-|h|}{|h_0}}{\left(1 + \frac{|h_0|-|h|}{|h_0|}\right)^{n-1}} \frac{1}{|h|} \left(-G(x_0,y)\right) \geq \frac{1}{|h_0|} \frac{1}{2^{n-1}} \left(-G(x_0,y)\right) > 0.$$

Hence $\frac{\partial G(\cdot,y)}{\partial N}\Big|_x = \lim_{h\uparrow 0} \frac{G(x+hN,y)}{h} > 0.$ □

Lecture 29

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\partial\Omega$ smooth.

Theorem 9.19 (Boundary Maximum Principle)**.** *Assume Ω is connected. Let $u \in C^2(\overline{\Omega})$ be harmonic. If $x \in \partial\Omega$ s.t. $u(x) = \max_{\overline{\Omega}} u = M$, then $\frac{\partial u}{\partial N}(x) > 0$ or $u \equiv M$. (N is the unit normal vectorfield of $\partial\Omega$.)*

Proof. Exercise □

Theorem 9.20 (Local estimates)**.** *Let Ω be open and let $u \in C^2(\Omega)$ be harmonic. If $x_0 \in \Omega$ such that $\overline{B_r(x_0)} \subset \Omega$. Then*

$$|\nabla u|(x_0) \leq \frac{C}{r^{n+1}} \int_{B_r(x_0)} |u(x)| dx$$

for some constant $C = C(n) > 0$.

Remark 9.21.
- The local gradient estimate generalizes the mean value property. Indeed

$$\text{MVP: } u(x_0) = \underbrace{\frac{1}{\text{vol}(B_r(x_0))}}_{\sim \frac{C(n)}{r^n}} \int_{B_r(x_0)} u(x) dx \Rightarrow |u(x_0)| \leq \frac{C(n)}{r^n} \int_{B_r(x_0)} |u(x)| dx.$$

- Similarly, one can derive higher estimates of the form

$$|\nabla^k u|(x_0) \leq \frac{C(n)}{r^{n+k}} \int_{B_r(x_0)} |u(x)| dx$$

Proof. Since u is harmonic on Ω, $u \in C^\infty(\Omega)$. Hence $0 = (\Delta u)_{x_i} = \Delta(u_{x_i})$ and u_{x_i} is harmonic $\forall i = 1, \dots, n$.

By the mean value property it follows

$$u_{x_i}(x_0) = \frac{1}{\text{vol}(B_{r/2}(x_0))} \int_{B_{r/2}(x_0)} u_{x_i}(x) dx$$

Note that $u_{x_i} = \nabla \cdot V$ where $V = (0, \dots, 0, \underbrace{u}_{i}, 0, \dots, 0)$. Hence

$$\implies \int_{B_{r/2}(x_0)} u_{x_i}(x) dx = \int_{\partial B_{r/2}(x_0)} \langle V, N \rangle ds = \int_{\partial B_{r/2}(x_0)} u N_i ds$$

where N_i ith component of the unit normal vector field N along $\partial B_{r/2}(x_0)$. In particular $|uN_i| \leq |u|$. It follows

$$|u_{x_i}(x_0)| = \left| \frac{1}{\text{vol}(B_{r/2}(x_0))} \int_{B_{r/2}(x_0)} u_{x_i}(x) dx \right| \leq \frac{1}{\text{vol}(B_{r/2}(x_0))} \int_{\partial B_{r/2}(x_0)} |u(x)| dx \leq \underbrace{\frac{vol(\partial B_{r/2}(x_0))}{\text{vol}(B_{r/2}(x_0)}}_{\frac{2n}{r}} \sup_{\partial B_{r/2}(x_0)} |u|$$

At the same time we know that $\partial B_{r/2}(x) \subset \overline{B_r(x_0)}$ and therefore by the mean value property again

$$|u(x)| \leq \frac{1}{\text{vol}(B_{r/2}(x))} \int_{B_r(x)} |u(y)| dy \leq \frac{(2/r)^n}{\text{vol}(B_1)} \int_{B_r(x)} |u(y)| dy \quad \Rightarrow \sup_{\partial B_{r/2}(x_0)} |u| \leq \frac{(2/r)^n}{\text{vol}(B_1)} \int_{B_r(x)} |u(y)| dy.$$

The previous estimates together yield

$$|u_{x_i}(x_0)| \leq \frac{n}{\text{vol}(B_1(x_0))} \left(\frac{2}{r}\right)^{n+1} \int_{B_r(x_0)} |u(y)| dy \ \forall i = 1, \dots, n.$$

Since $|\nabla u| = \sqrt{\sum_{i=1}^n (u_{x_i})^2}$. □

Corollary 9.22 (Liouville Theorem)**.** *Let $u \in C^2(\mathbb{R}^n)$ s.t. $\Delta u = 0$ with $|u| \leq C \ \Rightarrow \ u = const.$*

Proof. For any $x \in \mathbb{R}^n$ and any $R > 0$, it holds $B_R(x) \subset \mathbb{R}^n$. $\implies |\nabla u|(x) \leq \frac{C}{R} \to 0$. □

9.8. **Direct Method of Calculus of Variations.** Let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected with smooth boundary. Let $w \in C^1(\overline{\Omega})$ and let $h = w|_{\partial\Omega}$ (the restriction of w to $\partial\Omega$). To find a solution u for the equation

$$\Delta u = 0 \text{ on } \Omega,$$
$$u|_{\partial\Omega} = h \text{ on } \partial\Omega$$

one can apply the Dirichlet principle: Find a minimizer of the energy

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx, \quad v \in \{v \in C^1(\overline{\Omega}) : v|_{\partial\Omega} = h\} =: \mathcal{E}_h$$

$E \geq 0$ on $\mathcal{E}_h$.

Hence, an infinimum $I = \inf_{v \in \mathcal{E}} E(v)$ exists and a sequence $v_n \in \mathcal{E}$ such that $E(v_n) \to I$.

Claim. $\tilde{v}_n = v_n - w$ is a Cauchy sequence w.r.t. the norm $\|v\|_{1,2} = \sqrt{\int_\Omega v^2 dx + E(v)}$.

Indeed

$$\begin{aligned} E(\tilde{v}_n - \tilde{v}_m) = E(v_n - v_m) &= E(v_n) + E(v_m) - \int_\Omega \langle \nabla v_m, \nabla v_n \rangle dx \\ &\leq 2E(v_n) + 2E(v_m) - 4E(\frac{1}{2}(v_n + v_m)) \\ &\leq 2E(v_n) + 2E(v_m) - 4I \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

With the Poincaré inequality $\int_\Omega v^2 dx \leq C_\Omega \int_\Omega |\nabla v|^2 dx$ for $v \in \mathcal{E}_0 = \{v \in C^1(\overline{\Omega}) : v|_{\partial\Omega} = 0\}$ it follows $\|\tilde{v}_n - \tilde{v}_m\|_{1,2} \to 0$. Hence $\tilde{v}_n$ is a Cauchy sequence w.r.t. $\|\cdot\|_{1,2}$ in $\mathcal{E}_0$.

Let $\overline{\mathcal{E}} = H_0^{1,2}(\Omega)$ be the completion of $\mathcal{E}_0$ w.r.t. $\|\cdot\|_{1,2}$. Then there exists $v \in \overline{\mathcal{E}}$ such that $v_n \to v$ w.r.t. $\|\cdot\|_{1,2}$.

Questions: What is the space $\overline{\mathcal{E}}$? Can we define E on $\overline{\mathcal{E}}$? Is it true that $E(v_n) \to E(v)$? Is v harmonic?

we write $u \in L^2(\Omega)$ if $|u|^2$ is integrable on Ω.

Definition 9.23. A function $f : \Omega \to \mathbb{R}$ in $L^2(\Omega)$ has a weak derivative $\hat{f}_{x_i} : \Omega \to \mathbb{R}$ w.r.t. x_i if

$$\int_\Omega f\phi_{x_i} dx = -\int_\Omega \hat{f}_{x_i}\phi dx \ \forall \phi \in C_c^1(\Omega).$$

If f has a weak derivative w.r.t. x_i for all $i = 1, \dots, n$ and f_{x_i} is in $L^2(\Omega)$, then we write $f \in W^{1,2}(\Omega)$.

In particular, if $f \in W^{1,2}(\Omega)$, the weak gradient $\hat{\nabla} f = (\hat{f}_{x_1}, \dots, \hat{f}_{x_n})$ satisfies $|\hat{\nabla} f| \in L^2(\Omega)$, and we can define $E(f) = \frac{1}{2}\int_\Omega |\hat{\nabla} f|^2 dx$.

Example 9.24.
- For $f \in C^1(\overline{\Omega})$ the weak derivatives exists and coincide with the classical derivatives. This follows directly by the product rule for partial derivatives. In particular $f \in W^{1,2}(\Omega)$. Also the energy E w.r.t. $\hat{\nabla}$ is the same as w.r.t. ∇.
- Define

$$f(x) = \begin{cases} x & x < 0 \\ 0 & x \geq 0 \end{cases} \implies f \neq C^1([-1,1]) \text{ but } f \in W^{1,2}(\Omega) \text{ (Check!)}$$

We cite the following result.

Theorem 9.25. *$H_0^{1,2}(\Omega) \subset W^{1,2}(\Omega)$ is a closed subset w.r.t. $\|\cdot\|_{1,2}$.*

Since $E \geq 0$ on $H_0^{1,2}(\Omega)$, we can define $\hat{I} = \inf E(v) \geq 0$ where $v \in W^{1,2}(\Omega)$ s.t. $v - w \in H_0^{1,2}(\Omega)$ where $w \in C^1(\overline{\Omega})$ is as before.

Dirichlet principle (revisited). Let $v \in W^{1,2}(\Omega)$ s.t. $E(v) = \hat{I}$, and let $\phi \in C^1_0(\Omega)$. Then

$$0 = \frac{d}{dt}\Big|_{t=0} E(v + t\phi) = \int_\Omega \langle \hat{\nabla} v, \nabla \phi \rangle dx.$$

Definition 9.26. $v \in W^{1,2}(\Omega)$ is a weak solution of the Laplace equation with boundary value $h = w|_{\partial\Omega}$ for $w \in C^1(\overline{\Omega})$ if

$$\int_\Omega \langle \nabla v, \nabla \phi \rangle dx = 0 \ \forall \phi \in C^1_0(\overline{\Omega})$$
$$v - w \in H^{1,2}_0(\Omega).$$

Theorem 9.27 (Weyl Lemma)**.** *If u is a weak solution of the Laplace equation then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.*

More precisely, for every ball $B \subset \Omega$ that is sufficiently small, there exists $v \in C^\infty(B)$ such that $\Delta v = 0$ and $\int_B |v - u| dx = 0$.

Proof. Let $\phi \in C^\infty_c(B_1(0))$ with $1 = \int_{B_1(0)} \phi(x) dx$ and $\phi(x) = \phi(-x)$, and set $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(x/\epsilon)$. Then $\phi_\epsilon \in C^\infty(B_\epsilon(0))$ and $1 = \int_{B_\epsilon(0)} \phi_\epsilon(x) dx$.

We define $\Omega_r = \{x \in \Omega : |x - y| > r \ \forall y \in \partial\Omega\}$. Then $\overline{B_\epsilon(x)} \subset \Omega \ \forall x \in \Omega_r$ and $\epsilon \in (0, r/2)$.

We define $u_\epsilon(x) = \int_{B_\epsilon(x)} u(y)\phi_\epsilon(y - x) dy$, $x \in \Omega_r$.

Claim. $u_\epsilon \in C^\infty(\Omega_r)$.

We compute

$$\begin{aligned}\frac{\partial u_\epsilon}{\partial x_i}(x) &= \lim_{h\to 0} \frac{u_\epsilon(x+h) - u_\epsilon(x)}{h} \\ &= \lim_{h\to 0} \int_{B_\epsilon(x)} u(y) \underbrace{\frac{\phi_\epsilon(y - x - h) - \phi_\epsilon(y - x)}{h}}_{\overset{uniformily}{\rightarrow} u(y)\frac{\partial \phi_\epsilon}{\partial x_i}(y-x)} dy = \int_{B_\epsilon(x)} u(y) \frac{\partial \phi_\epsilon}{\partial x_i}(x - y) dy.\end{aligned}$$

Similarly, we can also compute all higher derivatives. Hence $u_\epsilon \in C^\infty(\Omega_r)$.

Morover, since $\phi_\epsilon(\cdot - x) \in C^1_0(\Omega)$, it follows by the definition of weak derivatives and since u is a weak solution of the Laplace equation

$$\Delta u_\epsilon(x) = \int_{B_\epsilon(x)} u(y)\Delta\phi_\epsilon(y - x) dy = -\int_{B_\epsilon(x)} \langle \hat{\nabla} u, \nabla \phi_\epsilon(y - x)\rangle dx = 0$$

This proves the claim.

The local gradient estimate yields

$$\begin{aligned}|\nabla u_\epsilon|(x) &\le \frac{C(n)}{r^{n+1}} \int_{B_r(x)} |u_\epsilon(y)| dy \le C(n, r) \int_\Omega |u_\epsilon(y)| dy \\ &\le C(n, r) \int_\Omega \int_\Omega |u(z)| \phi_\epsilon(z - y) dz dy \\ &= C(n, r) \int_\Omega |u(z)| \underbrace{\int_\Omega \phi_\epsilon(z - y) dy}_{=1} dz \le C(n, r) \int_\Omega |u(z)| dz =: C.\end{aligned}$$

By the mean value theorem for differentiable functions in n dimensions we see that u_ϵ is C-Lipschitz for all $\epsilon > 0$.

Now, we cite the following classical theorem.

Theorem 9.28 (Arzela-Ascoli)**.** *Let $K \subset \mathbb{R}^n$ be compact and let $(u_i)_{i\in\mathbb{N}} \subset C^0(K)$. Then (u_i) has uniformily converging subsequence if and only if*

(1) $\exists M > 0 : \ |u_i(x)| \leq M \ \forall x \in K, \forall i \in \mathbb{N}$
(2) $\forall \epsilon > 0, \exists \delta > 0$ *s.t.* $|u_i(x) - u_i(y)| < \epsilon \ \forall x, y \in K$ *with* $|x - y| < \delta$ *and* $\forall i \in \mathbb{N}$.

We apply the theorem to a sequence $u_{\epsilon_i}|_{\overline{B}} =: u_i$ for $\epsilon_i \to 0$ and an arbitrary ball B such that $\overline{B} \subset \Omega_r$.

Property 1. and 2. in the Arzela-Ascoli theorem for u_i follow since u_ϵ is C-Lipschitz for C independent of ϵ.

Hence, there exists $v \in C^0(\overline{B})$ such that $u_i \to v$ uniformily. At the same time u_i is harmonic on $B \subset \Omega_r$ and hence satisfies the mean value property

$$u_i(x) = \frac{1}{\text{vol}(B_\eta(x))} \int_{B_\eta(x)} u_i(y) dy, \quad B_\eta(x) \subset B.$$

Uniform convergenc implies that the right and the left hand side of this identity converges to

$$v(x) = \frac{1}{\text{vol}(B_\eta(x))} \int_{B_\eta(x)} v(y) dy.$$

Hence $v \in C^\infty(B)$ and v is harmonic.

Now, we claim without proof that $\int_B |u_{\epsilon_i} - u| dx \to 0$. This implies $\int_B |v - u| dx = 0$ on $B \subset \Omega_r$. Since $r > 0$ was arbitrary, we obtain the statement. □

lecture 30

10. Wave equation in $3D$ and higher dimensions

We consider the higer dimensional wave equation

$$u_{t,t} = c^2 \Delta u \text{ on } \mathbb{R}^n \times [0,\infty) \quad \text{where } c > 0. \tag{53}$$

Recall that $\Delta u = \sum_{i=1}^n u_{x_i,x_i} = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$.

Let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ be a solution of (53).

10.1. **Principle of Causality.** Let $u \in C^2(\mathbb{R}^n \times [0,\infty)$ be a solution of (53). Assume $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi$, $x \in \mathbb{R}^n$, for $\phi, \psi \in C^2(\mathbb{R}^n)$.

Theorem 10.1. *Let $(\bar{x},\bar{t}) \in \mathbb{R}^n \times (0,\infty)$ and u a before. Then $u(\bar{x},\bar{t})$ is completely determined by the values of ϕ and ψ in the ball $\overline{B_{c\bar{t}}(\bar{x})}$.*

Remark 10.2.
- The ball $B_{c\bar{t}}(\bar{x})$ in the 0 time slice is more precisely given by

$$B_{c\bar{t}}(\bar{x}) \times \{0\} = \underbrace{\{(x,t) \in \mathbb{R}^n \times [0,\infty) : |x-\bar{x}| \le c|t-\bar{t}|\}}_{\hat{C}(\bar{x},\bar{t})} \cap \mathbb{R}^n \cap \{0\}.$$

- Recall that for $n=1$ the solution u is represented by D'Alambert's formula

$$u(\bar{x},\bar{t}) = \frac{1}{2}\left[\phi(\bar{x}+c\bar{t}) + \phi(\bar{x}-c\bar{t})\right] + \underbrace{\frac{1}{2c}\int_{\bar{x}-c\bar{t}}^{\bar{x}+c\bar{t}} \psi(s)ds}_{=\frac{1}{\mathrm{vol}(B_{c\bar{t}}(\bar{x}))}\int_{B_{c\bar{t}}(\bar{x})}\psi(s)ds} .$$

Hence, for $n=1$ we have already observed that the theorem is correct.

Proof. First, we have

$$0 = (u_{t,t} - c^2\Delta u)u_t = \underbrace{u_{t,t}u_t}_{=\frac{1}{2}(u_t^2)_t} - c^2\nabla(u_t\nabla u) + \underbrace{\langle \nabla u_t, \nabla u\rangle}_{=\frac{1}{2}(|\nabla u|^2)_t}$$

$$= \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right)_t - c^2 \mathrm{Div}^n(u_t\nabla u) = \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right)_t + \sum_{i=1}^n c^2 \frac{\partial}{\partial x_i}\left(-u_t\frac{\partial u}{\partial x_i}\right).$$

Hence, the last equation takes the following form

$$0 = \mathrm{Div}^{n+1} V$$

for a vectore field V defined on $\mathbb{R}^{n+1}$ by $V = \begin{pmatrix} -u_t\frac{\partial u}{\partial x_1} \\ \dots \\ -u_t\frac{\partial u}{\partial x_n} \\ \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2 \end{pmatrix}$.

Now, we apply the divergence theorem in $\mathbb{R}^{n+1}$ to V on the frustum

$$F = \{(x,t) \in \mathbb{R}^n \times [0,\infty) : |x-\bar{x}| \le c|t-\bar{t}| \text{ and } t \in [0,s]\}$$

where $s \in (0,\bar{t})$.

Note that we need a version of the divergence theorem that allows regular corner and edges. The boundary ∂F of F has three parts:

$$T = \{(x,s) \in \mathbb{R}^n \times [0,\infty) : |x-\bar{x}| \le c|s-\bar{t}|\}$$
$$B = \{(x,0) \in \mathbb{R}^n \times [0,\infty) : |x-\bar{x}| \le c\bar{t}\}$$
$$K = \{(x,t) \in \mathbb{R}^n \times [0,\infty) : |x-\bar{x}| = c|t-\bar{t}| \ \& \ t \in [0,s]\}$$

The divergence theorem yields

$$0 = \int_{\partial F} \langle N, V\rangle dS = \int_B + \int_T + \int_K$$

where S denotes the n-dimensional surface measure on ∂F (that is B, T and K respectively) and N is the corresponding uni normal vector field.
For T one has $N = (0, \dots, 0, 1)$, and for B one has $N = (0, \dots, 0, -1)$. Hence, taking the formula for V into account, it follows

$$\int_T \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) dS + \int_K \langle N, V\rangle dS \leq \int_B \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) dS$$

Claim. $\int_K \langle N, V\rangle dS \geq 0$.

To prove the claim we compute the unit normal vectore field N along K. Consider the function $\Phi(x_1, \dots, x_n, t) = \sum_{i=1}^n (x_i - \bar{x}_i)^2 - c^2(t - \bar{t})^2$. Then, since

$$\{(x,t) \in \mathbb{R}^n \times [0, \infty) : |x - \bar{x}|^2 = c^2|t - \bar{t}|^2\} = \partial\hat{C}(\bar{x}, \bar{t}) =: C$$

and $C = \Phi^{-1}(0)$, it follows $K \subset \Phi^{-1}(0)$.

The unit normal vector field along the level set of smooth function Φ is given by $N = \frac{\nabla\Phi}{|\nabla\Phi|}$. We therefore compute

$$\frac{1}{|\nabla\Phi|}\nabla\Phi(x_1, \dots, x_n, t) = \frac{1}{2\sqrt{\sum_{i=1}^n (x_i - \bar{x}_i)^2 - c^4(t - \bar{t})^2}} 2 \begin{pmatrix} x_1 - \bar{x}_1 \\ \dots \\ x_n - \bar{x}_n \\ -c^2(t - \bar{t}) \end{pmatrix} = \frac{c}{\sqrt{1 - c}} \begin{pmatrix} \frac{x_1 - \bar{x}_1}{cr} \\ \dots \\ \frac{x_n - \bar{x}_n}{cr} \\ -\frac{t - \bar{t}}{|t - \bar{t}|} \end{pmatrix}$$

where we have set $\sum_{i=1}^n (x_i - \bar{x}_i)^2 =: r^2 = c^2(t - \bar{t})$. Since $t \in (0, \bar{t})$, $-(t - \bar{t}) = |t - \bar{t}|$.

For $x \in K$ it follows

$$\langle N, V\rangle(x) = \underbrace{-\sum_{i=1}^n \frac{x_i - \bar{x}_i}{cr} c^2 u_t u_{x_i}}_{c u_t \sum_{i=1}^n \frac{x_i - \bar{x}_i}{r} u_{x_i}} + \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2.$$

Setting $\hat{r} = \frac{1}{|x - \bar{x}|}(x - \bar{x}) = \frac{1}{r}\begin{pmatrix} x_1 - \bar{x}_1 \\ \dots \\ x_n - \bar{x}_n \end{pmatrix}$ we get that

$$\sum_{i=1}^n \frac{x_i - \bar{x}_i}{r} u_{x_i} = \langle \hat{r}, \nabla u\rangle = \frac{\partial u}{\partial \hat{r}} =: u_r.$$

Hence

$$\begin{aligned}\langle N, V\rangle(x) &= -c u_t u_r + \frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2 \\ &= \underbrace{\frac{1}{2}(c u_r - u_t)^2}_{\geq 0} + \left(\frac{c^2}{2}|\nabla u|^2 - \frac{c^2}{2}u_r^2\right)\end{aligned}$$

By the Cauchy-Schwarz inequality one has

$$u_r = \langle \hat{r}, \nabla u\rangle \leq |\hat{r}| \cdot |\nabla u| = |\nabla u|$$

Hence, $\frac{c^2}{2}\left(|\nabla u|^2 - u_r^2\right) \geq 0$ and therefore $\langle N, K\rangle(x) \geq 0$ for $x \in K$.

We can conclude that

$$\int_T \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) dS \leq \int_B \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) dS$$

Now, we assume there is a solution u of the wave equation such that ϕ and ψ vanish on B. Then

$$0 \le \int_T \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) dS \le \int_B \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) dS = \int_B \left(\frac{1}{2}\psi^2 + \frac{c^2}{2}|\nabla \phi|^2\right) dS = 0.$$

Hence, all the previous inequalities become equalities, and since $\frac{1}{2}(u_t)^2 + \frac{c^2}{2}|\nabla u|^2 \ge 0$, it follows that $\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2 = 0$.

Then, since $u_t^2 \ge 0$ and $|\nabla u|^2 \ge 0$, it follows $u_t = 0$ and $|\nabla u|^2 = 0$ ($\Rightarrow \nabla u = 0$) on $T = \hat{C}(\bar{x},\bar{t}) \cap \mathbb{R}^n \times \{s\}$ for $s \in (0, t_0)$.

Since $s \in (0, t_0)$ was arbitrary, $u_t = 0$ and $\nabla u = 0$ on $\hat{C}(\bar{x},\bar{t})$. It follows that $\nabla^{n+1}u = 0$ on $\hat{C}(\bar{x},\bar{t})\backslash\partial\hat{C}(\bar{x},\bar{t})$. So $u \equiv$ *constant* on $\hat{C}(\bar{x},\bar{t})$. Since $\phi = u(x,0) = 0$ and since u is continuous, it follows $u \equiv 0$ on $\hat{C}(\bar{x},\bar{t})$.

Hence $u(\bar{x},\bar{t}) = 0$.

If u and v are two solutions that coincide on B, we consider $u - v$ that is also a solution by linearity of the wave equation. Then $u - v$ vanishes on B, this implies $u(\bar{x},\bar{t}) = v(\bar{x},\bar{t})$. □

Remark 10.3.

- $\hat{C}(\bar{x},\bar{t})$ is also called the domain of dependence of $(\bar{x},\bar{t})$.
- One can see that the initial data ϕ and ψ in a point $(\hat{x}, 0)$ can influence the solution only in the light cone

  $$\{(x,t) \in \mathbb{R}^n \times [0,\infty) : |x - \hat{x}| \le ct\} =: \hat{C}^+(\hat{x}, 0).$$

  Indeed, assume there are solutions u and v such that $u(\hat{x},0) \ne v(\hat{x},0)$ but $u(\cdot,0) = v(\cdot,0)$ on $\hat{C}(\bar{x},\bar{t}) \cap \mathbb{R}^n \times \{0\} = B$ for a point $(\bar{x},\bar{t})$ such that $(\bar{x},\bar{t}) \notin \hat{C}^+(\hat{x},0)$. In particular $\hat{C}(\bar{x},\bar{t}) \cap \hat{C}^+(\hat{x},0) = \emptyset$.

  Assume the chance in $\hat{x}$ from u to v causes a change of the value of the solutions in $(\bar{x},\bar{t})$. This means $u(\bar{x},\bar{t}) \ne v(\bar{x},\bar{t})$. But this contradicts $u = v$ on B.

Conservation of Energy. Let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ be a solution of the wave equation. Assume $\phi = u(\cdot,0) \in C_c^2(\mathbb{R}^n)$, then $u(\cdot,t) \in C_c^2(\mathbb{R}^n)$ for all $t > 0$.

To see this assume $\phi(x) = 0$ for $x \ne B_R(0)$ for some $R > 0$. Then, for $y \notin B_{R+ct}(0)$ it follows that $u(y,t) = 0$ because of the causality principle.

Corollary 10.4. *Let $u \in C^2(\mathbb{R}^n \times \mathbb{R})$ be a solution of the wave equation with $u(\cdot,0) = \phi \in C_c^2(\mathbb{R}^n)$. Then, the total energy*

$$E(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) d(x_1,\dots,x_n)$$

is constant.

Proof. We saw before that

$$0 = \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right)_t - c^2\nabla\cdot(u_t\nabla u)$$

Integration over a ball $B_R(0)$ such that $u(\cdot,t) \equiv 0$ outside of $B_R(0)$ yields

$$0 = \int_{\mathbb{R}^n} \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right)_t d(x_1,\dots,x_n) - c^2 \int_{\partial B_R(0)} \langle N, \nabla u\rangle u_t ds$$

where we applied the divergence theorem for $V = u_t\nabla u$ on the ball $B_R(0)$. Since $u \equiv 0$ on $\partial B_R(0)$, the last integral vanishes.

Morover

$$0 = \int_{\mathbb{R}^n} \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right)_t d(x_1,\dots,x_n) = \frac{d}{dt}\int_{\mathbb{R}^n} \left(\frac{1}{2}u_t^2 + \frac{c^2}{2}|\nabla u|^2\right) d(x_1,\dots,x_n) = \frac{d}{dt}E(u(\cdot,t)$$

where we could pull the derivative w.r.t. t out of the integral because $u(\cdot,t)$ are compactly supported and smooth in t. □

Remark 10.5. The integral $\int_{\mathbb{R}^n} \frac{1}{2} u_t^2 dx$ is called the total kinetic energy of u at time t, and the integral $\int_{\mathbb{R}^n} \frac{c^2}{2} |\nabla u|^2 dx$ is the total potential energy at time t.

Kirchhoff's formula in $3D$ (see also subsecton 10.3)

Theorem 10.6. *Let $u \in C^2(\mathbb{R}^3 \times [0, \infty)$ be solution of $u_{t,t} = c^2 \Delta u$ with initial condition ϕ and ψ. Then, for $(\bar{x}, \bar{t}) \in \mathbb{R}^3 \times (0, \infty)$, it holds*

$$u(\bar{x}, \bar{t}) = \frac{1}{4\pi c^2 \bar{t}} \int_{\partial B_{c\bar{t}}(\bar{x})} \phi(x) ds(s) + \frac{\partial}{\partial \bar{t}} \left[\frac{1}{4\pi c^2 \bar{t}} \int_{\partial B_{c\bar{t}}(\bar{x})} \phi(x) ds(x) \right]$$

Remark 10.7. Hence, $u(\bar{x}, \bar{t})$ does not depend on $B_{c\bar{t}}(\bar{x})$ but only $\partial B_{c\bar{t}}(\bar{x})$.

This is also known as Huygen's principle. Any solution of the $3D$ wave equation propagates exactly at the speed (of light, or sound) c. At any time $\bar{t}$ a listener (or observer) hears (or sees) exactly what has occured at the time $t - d/c$ where d is the distance to the source.

Lecture 32

10.2. **Deriving the wave equation (for sound waves in $3D$).** We start with the equations of motions of a compressible fluid/gas.

The compressible Euler equations:

$$\frac{\partial}{\partial t}u + u \cdot \nabla u = -\frac{1}{\rho}\nabla(f \circ \rho) \tag{54}$$

$$\frac{\partial}{\partial t}\rho + \operatorname{Div}(\rho u) = 0 \tag{55}$$

where

- $u(x, y, z, t) \in \mathbb{R}^3$ the velocity of a particle in $(x, y, z) \in \mathbb{R}^3$ and at time $t \in \mathbb{R}$,
- $\rho(x, y, z, t) \in \mathbb{R}$ density of particle in (x, y, z) and at t.

The function $f : [0, \infty) \to \mathbb{R}$ models the internal pressure that is a function of ρ (for an ideal gas)

One also assumes that f is an increasing function w.r.t. ρ, that is $f'' \geq 0$.

Remark 10.8. For air one assume that $f(\rho) = p_0 \left(\frac{\rho}{\rho_0}\right)^\gamma$ where

- γ is the adeabatic intex ($\to 1.4$),
- p_0 sea level atmospheric pressure,
- ρ_0 density w.r.t. a reference temperature.

Assumptions 10.9. The absolute value of $\rho_0 - \rho$ and the absolute value of its derivatives are small ($\sim \epsilon$). Also the absolute value of u and its derivatives are small ($\sim \epsilon$).

Then, in the following, we neglect all terms of orrder $\sim \epsilon^2$.

(55):

$$\begin{aligned} 0 &= \rho_t + \operatorname{Div}(\rho u) \\ &= \rho_t + \overbrace{\underbrace{\langle \nabla(\rho - \rho_0), u\rangle}_{\sim \epsilon^2} + \underbrace{(\rho - \rho_0)\operatorname{Div} u}_{\sim \epsilon^2}}^{\operatorname{Div}((\rho-\rho_0)u)} + \rho_0 \operatorname{Div} u \end{aligned}$$

Hence, we replace (55) with

$$0 = \rho_t + \rho_0 \operatorname{Div} u$$

(54): $u \cdot \nabla u \sim \epsilon^2$ and

$$\nabla(f \circ \rho) = \overbrace{f'(\rho_0) + f''(\rho_0)\underbrace{(\rho - \rho_0)}_{\sim \epsilon^2} + \underbrace{o(|\rho - \rho_0|)}_{\sim \epsilon^2}}^{f'(\rho)} \overbrace{\underbrace{\nabla(\rho - \rho_0)}_{\sim \epsilon}}$$

$$\frac{1}{\rho} = \frac{1}{\rho_0} - \underbrace{\frac{1}{\rho_0^2}(\rho - \rho_0)}_{\sim \epsilon} + \underbrace{o(|\rho - \rho_0|)}_{\sim \epsilon}$$

Hence, we replace $\frac{1}{\rho}\nabla(f \circ \rho)$ with $\frac{f'(\rho_0)}{\rho_0}\nabla(\rho - \rho_0)$, and therefore replace (54) with

$$u_t = -\frac{f'(\rho_0)}{\rho_0}\nabla(\rho - \rho_0) = -\frac{f'(\rho_0)}{\rho_0}\nabla \rho$$

Now, we can compute

$$\rho_{t,t} = -\rho_0(\operatorname{Div} u)_t = -\rho_0 \operatorname{Div}(u_t) = \frac{f'(\rho_0)}{\rho_0}\operatorname{Div} \nabla \rho = \frac{f'(\rho_0)}{\rho_0}\Delta \rho.$$

Setting $c = \sqrt{\frac{f'(\rho_0)}{\rho_0}} \geq 0$ then ρ satisfies the wave equation.

10.3. **Kirchhoff's formula.**

Theorem 10.10. *Let $u \in C^2(\mathbb{R}^3 \times [0,\infty)$ be solution of $u_{t,t} = c^2 \Delta u$ with initial condition ϕ and ψ. Then, for $(\bar{x}, \bar{t}) \in \mathbb{R}^3 \times (0,\infty)$, it holds*

$$u(\bar{x},\bar{t}) = \frac{1}{4\pi c^2 \bar{t}} \int_{\partial B_{c\bar{t}}(\bar{x})} \phi(x) ds(s) + \frac{\partial}{\partial \bar{t}} \left[\frac{1}{4\pi c^2 \bar{t}} \int_{\partial B_{c\bar{t}}(\bar{x})} \phi(x) ds(x) \right]$$

Remark 10.11. Hence, $u(\bar{x},\bar{t})$ does not depend on $B_{c\bar{t}}(\bar{x})$ but only $\partial B_{c\bar{t}}(\bar{x})$.

We first prove the following Lemma

Lemma 10.12 (Euler-Poisson-Darboux equation)**.** *Let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ be a solution of $u_{t,t} = c^2 \Delta u$ on $\mathbb{R}^n \times [0,\infty)$ with initial conditions ϕ and ψ. Define $\bar{u}(r,t) = \frac{1}{\mathrm{vol}(\partial B_r(0))} \int_{\partial B_r(0)} u(x,t) ds(x)$.*

Then $\bar{u} \in C^2((0,\infty) \times [0,\infty))$ and $\bar{u}$ solves the Euler-Poisson-Darboux equation

$$\bar{u}_{t,t,} = c^2 \left(\bar{u}_{r,r} - \frac{n-1}{r} \bar{u}_r \right)$$
$$\bar{u}(r,0) = \bar{\phi}(r), \quad \bar{u}_t(r,0) = \bar{\psi}(r).$$

where $\bar{\phi}(r) = \frac{1}{\mathrm{vol}(\partial B_r(0))} \int_{\partial B_r(0)} \phi ds(x)$ and $\bar{\psi}(r) = \frac{1}{\mathrm{vol}(\partial B_r(0))} \int_{\partial B_r(0)} \psi(x) ds(x)$.

Remark 10.13. Recall that

$$\frac{d^2}{dr^2} - \frac{n-1}{r} \frac{d}{dr}$$

is the radial part of the Laplace operator in polar coordinates.

Proof of the lemma. We observe

$$\bar{u}(r,t) = \frac{1}{\mathrm{vol}(\partial B_r(0))} \int_{\partial B_r(0)} u(x,t) ds(x) = \underbrace{\frac{r^{n-1}}{\mathrm{vol}(\partial B_r(0))}}_{=:C(n)} \int_{\partial B_1(0)} u(rx,t) ds(x)$$

where the constant $C(n) > 0$ only depends on n.

We also have $u(rx,t) \leq \max_{B_{2r_0}(0)\times\{t\}} u(x,t)$ and $\frac{d}{dr} u(rx,t) = \langle \nabla u(rx,t), x \rangle \leq \max_{B_{2r_0}(0)\times\{t\}} |\nabla u|(x,t)$ for all $r \in (0,r_0)$. Hence

$$\Rightarrow \frac{d}{dr} \bar{u}(r,t) = C(n) \int_{\partial B_1(0)} \frac{\partial}{\partial r} u(rx,t) ds(x)$$

Similarly for higher derivatives w.r.t. r and derivatives w.r.t. t. Hence $\bar{u} \in C^2((0,\infty) \times [0,\infty))$.

Now

$$\begin{aligned}
\frac{d}{dr} \bar{u}(r,t) &= C(n) \int_{\partial B_1(0)} \langle \nabla u(rx), x \rangle ds(x) \\
&= \frac{1}{\mathrm{vol}(\partial B_r(0))} \int_{\partial B_r(0)} \langle \nabla u(x), \underbrace{\frac{x}{r}}_{=:N} \rangle ds(x) \\
&= \frac{r}{n} \underbrace{\frac{1}{\mathrm{vol}(B_r(0))}}_{\alpha(n) r^n} \int_{\partial B_r(0)} \underbrace{\Delta u(x)}_{=\frac{1}{c^2} u_{t,t}} ds(x)
\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dr}\left(r^{n-1}\frac{d}{dr}\bar{u}(r,t)\right) &= \frac{1}{n\alpha(n)}\frac{d}{dr}\int_{\partial B_r(0)} u_{t,t}(x,t)ds(x)\\ &= \frac{1}{n\alpha(n)}\int_{\partial B_r(0)} u_{t,t}(x,t)ds(x)\\ &= r^{n-1}\frac{1}{\mathrm{vol}(\partial B_r(0))}\int_{\partial B_r(0)} u_{t,t}(x,t)ds(x) = r^{n-1}(\bar{u})_{t,t}(r,t)\end{aligned}$$

It follows

$$(n-1)r^{n-2}(\bar{u})_r + r^{n-1}(\bar{u})_{r,r} = r^{n-1}\bar{u}_{t,t}$$

Deviding by r^{n-1} gives the desired equation. □

Proof. Proof of the theorem Recall we have $n = 3$. The previous lemma implies

$$\bar{u}_{t,t} = c^2\left(\bar{u}_{r,r} - \frac{2}{r}\bar{u}_r\right)$$

Let us define $v(r,t) = r\bar{u}(r,t)$. Then

$$\begin{aligned} v_r &= \bar{u} + r\bar{u}_r\\ \Rightarrow\ v_{r,r} &= \bar{u}_r + \bar{u}_r + r\bar{u}_{r,r} = \frac{1}{c^2}\bar{u}_{t,t} = \frac{1}{c^2}\bar{u}_{t,t}\end{aligned}$$

Since u is continuous in 0, it follows that $\bar{u}(r,t) = \frac{1}{4\pi r^2}\int_{\partial B_r(0)} u(x)ds(x) \to u(0)$ for $r \to 0$.

$$\Rightarrow\ v(r,t) \to 0 \text{ as } r \to 0$$

Hence v solves the half line problem

$$\begin{aligned} v_{t,t} &= c^2 v_{r,r} \text{ on } (0,\infty)\times[0,\infty)\\ \lim_{r\to 0} v(r,t) &= 0 \text{ for } t \geq 0\\ v(r;0) = r\bar{\phi}(r) &= \phi^*(r),\ v_t(r,0) = r\bar{\psi}(r) = \psi^*(r) \text{ for } r > 0\end{aligned}$$

The solution for this problem is given by the following formula

$$v(r,t) = \begin{cases} \frac{1}{2}\left[\phi^*(ct+r) - \phi^*(ct-r)\right] + \frac{1}{2c}\int_{ct-r}^{ct+r}\psi^*(s)ds & \text{for } 0 \leq r \leq c|t|,\\ \frac{1}{2}\left[\phi^*(r+ct) - \phi^*(r-ct)\right] + \frac{1}{2c}\int_{r-ct}^{r+ct}\psi^*(s)ds & \text{for } r \geq c|t|.\end{cases}$$

Therefore

$$\begin{aligned} u(0,t) &= \lim_{r\to 0}\bar{u}(r,t) = \lim_{r\to 0}\frac{1}{r}v(r,t)\\ &= \lim_{r\to 0}\left(\frac{1}{2r}\left[\phi^*(ct+r) - \phi^*(ct-r)\right] + \frac{1}{2cr}\int_{ct-r}^{ct+r}\psi^*(s)ds\right)\\ &= (\phi^*)'(ct) + \frac{1}{c}\psi^*(ct)\\ &= \frac{d}{dr}\left(r\bar{\phi}(r)\right)\Big|_{ct} + t\bar{\psi}(ct) = \frac{d}{dt}\left(t\bar{\phi}(ct)\right) + t\bar{\psi}(ct).\end{aligned}$$

That is the desired formula for $x = 0$. The case $x \neq 0$ is derived easily since a translation of u satisfies the wave equation as well. □

10.4. **Formula in $2D$ (Poisson formula).** Let $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ s.t.

$$u_{t,t} = c^2 \Delta u \text{ on } \mathbb{R}^2 \times [0,\infty)$$
$$u(x,y,0) = \phi(x,y) \text{ and } u_t(x,y,0) = \psi(x,y)$$

Define $v(x,y,z,t) = u(x,y,t)$. Then v solves the wave equation in $3D$ and we can apply Kirchhoff's formula:

$$v(\underbrace{x_0,y_0,0}_{=o},t) = \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(0)} \psi(x,y,t)ds(x,y,z) + \dots.$$

We use this formula to derive a formula that only involves integral in $\mathbb{R}^2$. This is also called *Hadamard's method of decent*.

Note that $\partial B_{ct}(o) = S^+ \cup S^-$ with $(x,y,z) \in S^\pm$ if and only if $z = \pm\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}$.

Hence S^+ (and S^-) respectively are parametrized by the graph of $z = z(x,y)$. Hence, we can compute a surface integral over S^+ using the transformation $ds(x,y,z) = \sqrt{1 + (z_x)^2(x,y) + (z_y)^2(x,y)}dxdy$.

$$\Rightarrow \quad v(o,t) = \frac{2}{4\pi c^2 t}\int_{\{(x,y):(x-x_0)^2+(y-y_0)^2 \le c^2t^2\}} \psi(x,y)\sqrt{1 + (z_x)^2 + (z_y)^2}dxdy.$$

Note that

$$1 + (z_x)^2 + (z_y)^2 = 1 + \left(-\frac{x}{z}\right)^2 + \left(-\frac{y}{z}\right)^2 = \frac{c^2t^2}{z^2} = \frac{c^2t^2}{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}$$

Hence

$$\begin{aligned} v(o,t) =& \frac{1}{2\pi c}\int_{\{(x,y):(x-x_0)^2+(y-y_0)^2 \le c^2t^2\}} \frac{\psi(x,y)}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}}dxdy \\ &+ \frac{d}{dt}\frac{1}{2\pi c}\int_{\{(x,y):(x-x_0)^2+(y-y_0)^2 \le c^2t^2\}} \frac{\phi(x,y)}{\sqrt{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}}dxdy. \end{aligned}$$

This is Poisson's formula for solutions of the wave equation in $2D$.

Remark 10.14. In particular, we see that Huygen's principle does not hold for the case of two space dimensions.

Lecture 33

Remark 10.15. An alternative form of Kirchhoff's formula is deduced as follows. Let $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ a solution of the wave equation with initial conditions ϕ and ϕ. Set

$$\bar{\phi}(x,t) = \fint_{\partial B_{ct}(x)} \phi(y)ds(y) \quad \text{and} \quad \bar{\psi}(x,t) = \fint_{\partial B_{ct}(x)} \psi(y)ds(y).$$

Then Kirchhoff's formula takes the form

$$u(x,t) = t\bar{\psi}(x,t) + \frac{\partial}{\partial t}(t\bar{\phi}(x,t))$$

We compute

$$\begin{aligned}
\frac{\partial}{\partial t}(t\bar{\phi}(x,t)) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x)} \phi(y)ds(y)\right) \\
&= \frac{\partial}{\partial t}\left(\frac{t}{4\pi c^2 t^2}\int_{\partial B_1(0)} \phi(x+cty)c^2t^2 ds(y)\right) \\
&= \frac{1}{4\pi}\int_{\partial B_1(0)} \phi(x+cty)ds(y) + \frac{t}{4\pi}\int_{\partial B_1(0)} \frac{\partial}{\partial t}\phi(x+cty)ds(y) \\
&= \frac{1}{4\pi c^2 t^2}\int_{\partial B_1(0)} \phi(x+cty)c^2t^2 ds(y) + \frac{tc}{4\pi}\int_{\partial B_1(0)} \langle \nabla\phi(x+cty), y\rangle ds(y) \\
&= \frac{1}{4\pi c^2 t^2}\int_{\partial B_{ct}(x)} \phi(y)ds(y) + \frac{tc}{4\pi c^2 t^2}\int_{\partial B_{ct}(x)} \langle \nabla\phi(y), \frac{y-x}{ct}\rangle ds(y) \\
&= \fint_{\partial B_{ct}(x)} \left(\phi(y) + \langle \nabla\phi(y), y-x\rangle\right) ds(y).
\end{aligned}$$

Hence

$$u(x,t) = \fint_{\partial B_{ct}(x)} \left(t\psi(y)\phi(y) + \langle \nabla\phi(y), y-x\rangle\right) ds(y).$$

Similar for $n = 2$ the formula can be written as

$$u(x,t) = \frac{1}{2}\fint_{B_{ct}(x)} \frac{t\psi(y) + \phi(y) + t\langle \nabla\phi(y), y-x\rangle}{\sqrt{t^2 - |y-x|^2}} ds(y).$$

10.5. **Solution of the wave equation in** $3D$**.**

Theorem 10.16 (Solution to the wave equation in $3D$)**.** *Let $n = 3$. $\phi, \psi \in C^2(\mathbb{R}^3)$. Define u by Kirchhoff's formula. Then $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ and u solves*

$$u_{t,t} = c^2\Delta u \quad \text{on } \mathbb{R}^3 \times (0,\infty)$$
$$\lim_{(x,t)\to(x_0,0)} u(x,t) = \phi(x), \quad \lim_{(x,t)\to(x_0,0)} u(x,t) = \psi(x) \quad \text{for } x \in \mathbb{R}^3.$$

Proof. For simplicity we assume $\phi = 0$. Then

$$u(x,t) = t\bar{\psi}(x,t) = t\fint_{\partial B_{ct}(x)} \psi(y)ds(y) = t\frac{1}{4\pi}\int_{\partial B_1(0)} \psi(x+tcy)ds(y).$$

By techniques we used before we immediately get that $C^2(\mathbb{R}^3 \times [0,\infty))$.

Now observe for $(x,t) \in \mathbb{R}^3 \times (0,\infty)$

$$u_t(x,t) = \bar{\psi}(x,t) + t\bar{\psi}(x,t)$$
$$u_{t,t}(x,t) = \bar{\psi}(x,t) + \bar{\psi}_t(x,t) + t\bar{\psi}_{t,t}(x,t) = \frac{1}{t}\frac{d}{dt}\left(t^2\bar{\psi}_t(x,t)\right)$$

Consder $\bar{\psi}_t(x,t)$:

$$\begin{aligned}\bar{\psi}_t(x,t) &= \frac{1}{4\pi}c\int_{\partial B_1(0)}\langle\nabla\psi(x+cty),y\rangle ds(y)\\ &= \frac{c}{4\pi c^2t^2}\int_{\partial B_{ct}(x)}\langle\nabla\psi(y),\underbrace{\frac{y-x}{ct}}_{=N}\rangle ds(y) = \frac{c}{4\pi c^2t^2}\int_{B_{ct}(x)}\Delta\psi(y)dy\end{aligned}$$

where N is the normal vectore along $\partial B_{ct}(x)$. Hence

$$\begin{aligned}\frac{d}{dt}\left(t^2\bar{\psi}_t(x,t)\right) &= \frac{d}{dt}\frac{1}{4\pi c}\int_0^{ct}\int_{\partial B_1(x)}\widetilde{\Delta\psi}(r,z)ds(z)r^2dr\\ &= \frac{1}{4\pi c}c\int_{\partial B_1(x)}\widetilde{\Delta u}(ct,z)c^2t^2ds(z) = \frac{1}{4\pi}\int_{\partial B_1(0)}\Delta\psi(x+y)c^2t^2ds(y)\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{t}\frac{d}{dt}\left(t^2\bar{\psi}_t(x,t)\right) &= \frac{1}{t4\pi}\int_{\partial B_1(0)}\Delta\psi(x+y)c^2t^2ds(y)\\ &= \frac{1}{t4\pi}\Delta\int_{\partial B_1(0)}\psi(x+y)c^2t^2ds(y)\\ &= c^2\Delta t\frac{1}{4\pi c^2t^2}\int_{\partial B_{ct}(x)}\psi(y)dy\\ &= c^2\Delta\left(t\bar{\psi}(x,t)\right).\end{aligned}$$

Let us check the initial conditions. We have $u_t(x,t) = \bar{\psi}(x,t) + t\bar{\psi}_t(x,t)$. Now we compute

$$\bar{\psi}(x,t) = \frac{1}{4\pi}\int_{\partial B_1(0)}\psi(x+tcy)ds(y) \to \psi(x_0) \text{ for } (x,t)\to(x_0,0).$$

Here we could move the limit inside the integral because $f(y) := \psi(x+cty) \to \psi(y)$ as $t\to 0$ uniformily on $\partial B_1(0)$. Because of the same reason we have

$$\begin{aligned}|t\bar{\psi}(x,t)| &= \left|\frac{t}{4\pi}\int_{\partial B_1(0)}\langle\nabla\psi(x+tcy),y\rangle ds(y)\right|\\ &\le t\underbrace{\frac{1}{4\pi}\int_{\partial B_1(0)}|\nabla\psi|(x+tcy)ds(y)}_{\to|\nabla\psi|(x)} \to 0 \text{ as } (x,t)\to(x_0,0).\end{aligned}$$

Moreover

$$t\bar{\psi}(x,t)\to 0 \quad\text{as } (x,t)\to(x_0,0).$$

Hence, the initial conditions are satisfied. □

Remark 10.17 (Kirchhoff's formula for $n = 2k+1 \ge 3$).

$$u(x,t) = \frac{1}{\gamma_n}\frac{\partial}{\partial t}\left[\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B_{ct}(x)}\phi(y)dy\right)\right] + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B_{ct}(x)}\psi(y)dy\right)$$

where $\gamma_n = 1\cdot 3\cdot 5\cdot\dots\cdot(n-2)$.

Remark 10.18. The solution of the wave equation involves derivatives of ϕ. This suggests that for $n\ge 3$ the solution need not to be as smooth as the initial condition, it may be less regular.

Lecture 34

10.6. **Wave equations with a source term.** We consider

$$u_{t,t} - c^2 \Delta u = f(x,t)$$
$$u(x,0) = 0{,}0 u_t(x,0)$$

For ϕ, ψ the solution of the homogeneous problem is

$$u(x,t) = \underbrace{t\bar{\psi}(x,t)}_{\mathcal{L}(t)\psi(x)} + \underbrace{\frac{d}{dt}\left(t\bar{\phi}(x,t)\right)}_{\frac{d}{dt}\mathcal{L}(t)\phi(x)}$$

By Dunhamel's principle the solution for the inhomogeneous problem is

$$v(x,t) = \int_0^t \mathcal{L}(t-s) f(x,s) ds = \int_0^t (t-s)\bar{f}(x, t-s, s) ds$$
$$= \int_0^t (t-s) \frac{1}{4\pi c^2 (t-s)} \int_{\partial B_{c(t-s)}} f(y,s) ds(y) ds$$

Since $|y-x| = c(t-s)$ for $x \in \partial B_{c(t-s)}(x)$, it follows that

$$v(x,t) = \frac{1}{4\pi c} \int_0^t \int_{\partial B_{c(t-s)}(x)} \frac{f(y, t-|y-x|/c)}{|y-x|} ds(y) ds = \frac{1}{4\pi c} \int_{B_{ct}(x)} \frac{f(\xi, t-|\xi-x|/c)}{|\xi - x|} d\xi$$

This formul has a very similar structure as the solution formula for the Laplace equation on a ball.

10.7. **Relativistic geometry.** A light ray is the path of a particle moving along $x(t) = x_0 + v_0 t$. Note that $x(t) \in \mathbb{R}^3$. The speed of $x(t)$ is

$$\frac{d}{dt} x(t) = v_0 \in \mathbb{R}^3 \ \text{ with } \ |v_0| = c$$

Characteristic surfaces. Let $S \subset \mathbb{R}^4$ be any $3D$ surface in $\mathbb{R}^4$ and set for $t \in \mathbb{R}$ fixed

$$S_t = S \cap \{(x,t) : x \in \mathbb{R}^3\} \subset \mathbb{R}^3 \ \ 2D \text{ surface.}$$

Definition 10.19. A surface $S \subset \mathbb{R}^4$ is called characteristic surface if it is a union of light rays of which each is orthogonal to S_t in $\mathbb{R}^3$.

Consider $g(x), x \in \mathbb{R}^3$ and $f(x,t) = t - g(x)$ and set $S = \{(x,t) \in \mathbb{R}^4 : f(x,t) = k\}$ for $k \in \mathbb{R}$.

Theorem 10.20. *A surface $S = \{(x,t) \in \mathbb{R}^4 : f(x,t) = k\} = S$ is a characteristic surface $\forall k \in \mathbb{R}$ if and only if g satisfies the eikonal equation $|\nabla g| = \frac{1}{c}$.*

Proof. Assume S is characteristic (fo all $k \in \mathbb{R}$). Let $x_0 \in \mathbb{R}^3$ and consider $(x_0, 0) \in \mathbb{R}^4$. Set $k_0 = f(x_0, 0) = g(x_0)$ and consider $S = \{(x,t) \in \mathbb{R}^4 : f(x,t) = k_0\}$.

By assumption $\exists (x(t), t) \in \mathbb{R}^4$ a light ray such that $x(0) = x_0$ and $(x(t), t) \in S \ \forall t$ and $\frac{d}{dt} x(t) = v_0 \perp S_t$ for all $t \in \mathbb{R}$.

Since $t = g(x(t)) = k_0$

$$\Rightarrow 0 = 1 - \langle \nabla g(x(t)), v_0 \rangle \Rightarrow \langle \nabla g(x_0), v_0 \rangle = 1.$$

The gradient of g is orthogonal to its level sets. Hence $\nabla g(x_0) \perp S_0 = \{x \in \mathbb{R}^3 : g(x) = k_0\}$.

On the other v_0 is orthogonal to S_0 by assumption. Therefore $\nabla g(x_0)$ and v_0 are parallel. It follows that

$$1 = \langle \nabla g(x_0), v_0 \rangle = |\nabla g(x_0)| |v_0| = c |\nabla g(x_0)|.$$

since x_0 was arbitrary, we have one direction.

The other direction is left as an exercise. □

Example 10.21. Consider $g(x) = \sum_{i=1}^{3} \frac{a_i}{c} x_i$ with $a_1^2 + a_2^2 + a_3^2 = 1$. Then $\nabla g(x) = \begin{pmatrix} a_1/c \\ a_2/c \\ a_3/c \end{pmatrix}$.

Hence $|\nabla g|^2 = \frac{1}{c}$ and $S = \{(x,t) \in \mathbb{R}^4 : t - g(x) = b/c\}$ is a characteristic surface with $S_t = \{x \in \mathbb{R}^3 : \sum_{i=1}^{3} a_i x_i = b + ct\}$.

Definition 10.22. (1) If $(v,t) \in \mathbb{R}^4$ satisfies $t > c|v|$, (v,t) is called timelike
(2) If $t < c|v|$, (v,t) is called spacelike, and
(3) If $t = c|v|$, (v,t) is called null.

$\mathbb{R}^4$ equipped with the bilinear form $\langle (v,t),(w,s)\rangle_1 = c\sum_{i=1}^{3} v_i w_i - ts$ is called Minkowski space time.

Lemma 10.23. *$S = \{(x,t) \in \mathbb{R}^4 : t - g(x) = k\} \subset \mathbb{R}^4$ is a characteristic surface if and only if a normal vector field N along S is a null vector field.*

Proof. The normal vectors along S are given by $\nabla f = \begin{pmatrix} -\nabla g \\ 1 \end{pmatrix}$. Then

$$\nabla f \text{ is null } \Leftrightarrow c|\nabla g| = 1$$

This proves the lemma by the previous theorem. □

Definition 10.24. A surface $F = \{(x,t) \in \mathbb{R}^4 : f(x,t) = t - g(x) = k\}$ is spacelike if $|\nabla g| < \frac{1}{c}$ on S. That is normal vectors along S are timelike.

For instance, if $g = const$ then $f(x,t) = t - const$ on S, then $|\nabla g| = 0 < \frac{1}{c}$.

Theorem 10.25 (without proof)**.** *Let $S = \{(x,t) : t - g(x) = k\} \subset \mathbb{R}^4$ be spacelike for a smooth function $g : \mathbb{R}^3 \to \mathbb{R}$. Then there exists a unique solution $u \in C^2(\mathbb{R}^4)$ of*

$$c^2 \Delta u = u_{t,t} \quad \text{with } u = \phi \text{ and } \frac{\partial u}{\partial N} = \psi \text{ on } S$$

where $\phi, \psi \in C^2(\mathbb{R}^3)$ and N is the unit normal vector field along S.

Since $N = \frac{1}{\sqrt{|\nabla g|^2+1}} \begin{pmatrix} -\nabla g \\ 1 \end{pmatrix}$ the second initial condition becomes

$$\frac{\partial u}{\partial N} = \frac{1}{\sqrt{|\nabla g|^2+1}} \langle \nabla u, \begin{pmatrix} -\nabla g \\ 1 \end{pmatrix} \rangle = \frac{-1}{\sqrt{|\nabla g|^2+1}} \langle \nabla^{\mathbb{R}^3} u, \nabla g\rangle + \frac{1}{\sqrt{|\nabla g|^2+1}} u_t.$$

Hence $u_t - \langle \nabla^{\mathbb{R}^3} u, \nabla g\rangle = \sqrt{|\nabla g|^2+1}\,\psi$.

Example 10.26. In $1D$ we have $u_{t,t} = c^2 u_{x,x}$. $S = \{(x,t) : \gamma(x) = t\}$ for $\gamma : \mathbb{R} \to \mathbb{R}$ smooth.

Lecture 35

Singularities.

Theorem 10.27 (without proof)**.** *Singularities of a a solution of $u_{t,t} = c^2\Delta u$ can only be carried on characteristic surfaces.*

Example 10.28. Recall the plucked string solution of the wave equation.

Example 10.29. Assume we have a solution in $\mathbb{R}^2$ of the form

$$u(x,t) = \begin{cases} \frac{1}{2}v(x,t)\,[t-g(x)]^2 & \text{if } g(x) \le t \\ 0 & \text{if } g(x) \ge t. \end{cases}$$

for $g \in C^2(\mathbb{R})$ and $v \in C^2(\mathbb{R}\times\mathbb{R})$.

Let us compute the partial derivatives:

$$u_t = v(t-g(x)) + \frac{1}{2}v_t(t-g(x))^2$$
$$u_x = -v(t-g(x)) + \frac{1}{2}v_x(t-g(x))^2$$

We see that $u \in C^1(\mathbb{R}^2)$. Moreover

$$u_{t,t} = v + 2v_t(t-g) + \frac{1}{2}v_{t,t}(t-g)^2$$
$$u_{x,x} = vg_x^2 - vg_{x,x}(t-g) - 2g_xv_x(t-g) + \frac{1}{2}v_{x,x}(t-g)^2$$

Since u solves the wave equation on $g(x) < t$, we have on $\{(x,t) : g(x) < t\}$

$$\begin{aligned} 0 &= u_{t,t} - c^2u_{x,x} \\ &= \frac{1}{2}(t-g)^2(v_{t,t} - c^2v_{x,x}) + v(1-c^2g_x^2) + (t-g)\left(2v_t + c^2vg_{x,x} + c^2 2g_xv_x\right) \end{aligned}$$

On $g(x) > t$ the wave equation holds trivially. Hence for u being a solution across the surface $\{g(x) = t\}$ the right hand side in the previous equation has to be 0 on $\{g(x) = t\}$.

$$\Rightarrow |g_x| = \frac{1}{c} \;\Rightarrow\; S = \{g(x) = t\} \text{ is a characteristic surface.}$$

Moreover, given g that satisfies the eikonal equation, deviding by $(t-g)$ it follows

$$-\frac{1}{2}(t-g)(v_{t,t} - c^2v_{x,x}) = 2(v_t + g_xv_x) + v\cdot g_{x,x} \quad \text{on } \{g(x) = t\}$$

Consequently v has to satisfy the following transport equation $v_t + c^2g_xv_x = -\frac{c^2}{2}vg_{x,x}$.

the information contained in v is transported along characteristic surfaces.

11. Schroedinger equations and stationary Schroedinger equations

A quantum mechanical system is described by the Schroedinger equation

$$\begin{aligned} -iu_t &= \tfrac{1}{2}k\Delta u + Vu \quad \text{on } \mathbb{R}^3\times\mathbb{R} \\ u(x,t) &\to 0 \text{ if } |x| \to \infty \\ u(x,0) &= \phi(x) \text{ on } \mathbb{R}^3 \end{aligned} \tag{56}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function that describes an external force. Instead of the boundary condition $u(x,t) \to 0$, $|x| \to \infty$, we also can could assume $\int_{\mathbb{R}^3} u(x,t)^2dx < \infty$.

We consider the following choices for V:

1. $V \equiv 0$: the free Schroedinger equation.
2. $V(x) = -|x|^2$: the harmonic oscilator.
3. $V(x) = \frac{1}{|x|}$: the hydrogen atom (on $\mathbb{R}^3\backslash\{0\}\times\mathbb{R}$).

Separation of variables for (56) yields ($u(x,t) = T(t)X(x)$):

$$\underbrace{-i\frac{T_t(t)}{T(t)}}_{(a)} = \underbrace{\frac{\frac{1}{2}k\Delta X(x) + V(x)X(x)}{X(x)}} = -\lambda.$$

We assume $\lambda \in \mathbb{R}$. Then

(a) $T(t) = e^{-\lambda i t} = \cos(\lambda t) + i\sin(\lambda t)$ that is T behaves like a wave.

(b)

$$\underbrace{-\frac{1}{2}k\Delta X - VX}_{=:LX} = \lambda X \text{ on } \mathbb{R}^3 \tag{57}$$

This is an eigenvalue equatin for the operator $L : C^2(\mathbb{R}^3) \to \{v : \mathbb{R}^3 \to \mathbb{R}\}$ with "boundary conditions at infinity"

$$X(x) \to 0 \text{ for } |x| \to 0 \text{ or } \int_{R^3} |X(x)|^2 dx < \infty.$$

If $V \equiv 0$, (57) has no solution. Hence, the method of separation of variables is not applicable.

Lecture 36

11.1. **Diffusion equation on $\mathbb{R}^n$.** Before we study the case $V = 0$, we consider the diffusion equation

$$u_t = k\Delta u \text{ on } \mathbb{R}^n \ \& \ u(x,t) \to 0 \text{ if } |x| \to \infty$$
$$u(x,0) = \phi(x), \quad \phi \in C^0_b(\mathbb{R}^n) \tag{58}$$

for higher space dimensions.

Theorem 11.1. *The solution of* (58) *is given by*

$$u(x,t) = \frac{1}{(4\pi kt)^{\frac{n}{2}}} \int_{R^n} e^{-\frac{|x-y|^2}{4tk}} dy.$$

Remark 11.2. Define

$$S_n(x,t) = \frac{1}{(4\pi tk)^{\frac{n}{2}}} e^{-|x|^2/4tk} = \Pi_{i=1}^n \underbrace{\frac{1}{\sqrt{4\pi tk}} e^{-(x_i)^2/4tk}}_{=S(x,t)}$$

where $S(x,t)$ is the fundamental solution of the heat equation in one space dimension. Then

$$u(x,t) = \int_{\mathbb{R}^n} S_n(x-y,t)\phi(y)dy = \Pi_{i=1}^n \int_{\mathbb{R}} S(x_i - y_i, t)\phi_i(y_i)dy_i.$$

if $\phi(x) = \Pi_{i=1}^n \phi_i(x_i)$.

Proof. One can check that

- $\frac{\partial}{\partial t} S_n(x,t) = k\Delta S_n(x,t)$ for $x \in \mathbb{R}^n$ and $t > 0$.
- $u \in C^2(\mathbb{R}^n \times (0,\infty))$ and $u_t = k\Delta u$ on $\mathbb{R}^n \times (0,\infty)$. (Because of the exponential decay of S_n for $|x| \to 0$, we can exchange differentiation and integration for $x \in \mathbb{R}^n$ and $t > 0$, exactly as in the $1D$ case.)
- $u \in C^0(\mathbb{R}^n \times [0,\infty))$ and $u(x,0) = \phi(x)$ (Exercise).

□

Remark 11.3. The solution formula still holds if $k = Re(k) + iIm(k) \in \mathbb{C}$ with $Re(k) > 0$. A hint to see this is that $Re(\frac{1}{k}) > 0$ and

$$\left| e^{-|x|^2/4kt} \right| = \left| e^{-Re(1/k)|x|^2/4t} e^{-iIm(1/k)|x|^2/4t} \right| \le e^{-Re(1/k)|x|^2/4t}.$$

But will not give a proof.

If we consider $k = i + \epsilon$ and let $\epsilon \to 0$ one gets

Theorem 11.4. *The solution of the free Schrödinger equation on $\mathbb{R}^3$ for $u(x,0) = \phi(x) \in C^0_c(\mathbb{R}^n)$ is given by*

$$u(x,t) = \frac{1}{(2\pi kit)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-|x-y|^2/2ikt} \phi(y)dy$$

11.2. **Time independent Schroedinger equation.** Separation of variables for the Schroedinger equation (56) yields $T_\lambda(t) \in \mathbb{C}$ that satisfies $i\lambda = \frac{T_\lambda(t)}{T_\lambda(t)}$ and

$$\begin{aligned} \Delta X + V \cdot X + \lambda X = 0 \ \text{ on } \ \mathbb{R}^3 \\ X(x) \to 0, \ |x| \to \infty \end{aligned} \tag{59}$$

This equation is also called the time independent Schroedinger equation.
If (59) is solvable for λ, λ is called an energy level of the QM system described by the operator $L = -\Delta - V$.

Goal: Find energy levels λ_k and corresponding eigenfunctions v_k that solve (59). If the system v_k is complete, then a general solution of the Schrödinger equation (56) is given by

$$u(x,t) = \sum_{k=0}^{\infty} T_{\lambda_k}(t) v_k(x).$$

Harmonic oscillator: $V(x) = -|x|^2$**.** Assume first $n = 1$ (space dimension). Equation (59) implies

$$v'' - x^2 v + \lambda v = 0.$$

For $\lambda = 1$ a solution is $e^{-x^2/2}$ and it satisfies the boundary condition at infinity $e^{-x^2/2} \to 0$ for $|x| \to \infty$.

For $\lambda \neq 1$, we assume $v(x) = w(x)e^{-x^2/2}$ and derive an equation for w as follows

$$v' = w' e^{-x^2/2} - x \cdot w e^{-x^2/2}$$
$$v'' = w'' e^{-x^2/2} - x w' e^{-x^2/2} - x w' e^{-x^2/2} - w e^{-x^2/2} + x^2 w e^{-x^2/2}$$

Then

$$(x^2 - \lambda) w e^{-x^2/2} = (w'' - 2xw' - w + x^2 w) e^{-x^2/2}$$
$$\Rightarrow \; 0 = w'' - 2xw' + (\lambda - 1) w$$

The last equation is known as *Hermite's differential equation.* To find solutions we apply **the power series method**.

Assume $w(x) = \sum_{k=0}^{\infty} a_k x^k$ and pluck it into Hermite's differential equation.

$$0 = \underbrace{\sum_{k=0}^{\infty} a_k k(k-1) x^{k-2}}_{=\sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}} - \sum_{k=0}^{\infty} a_k 2xkx^{k-1} + \sum_{k=0}^{\infty} (\lambda - 1) a_k x^k$$

Replacing k with $k+2$ in the first sum yields

$$0 = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k - \sum_{k=0}^{\infty} a_k 2k x^k + \sum_{k=0}^{\infty} (\lambda - 1) a_k x^k.$$

Since the coefficients of a power series determine the power series uniquely, it follows that

$$0 = a_{k+2}(k+2)(k+1) - a_k 2k + a_k(\lambda - 1) \;\Rightarrow\; a_{k+2}(k+2)(k+1) = (2k + 1 - \lambda) a_k$$

From this recursive formula for the coefficients we deduce first that

$$a_0 = 0 \;\Rightarrow\; a_k = 0 \;\forall k = 2, 4, \dots$$
$$a_1 = 0 \;\Rightarrow\; a_k = 0 \;\forall k = 3, 5, \dots$$

Moreover, if $\lambda = 2k + 1$ we get that $0 = a_{k+2i}$ for all $i \in \mathbb{N}$. Hence

If $a_0 \neq 0,\ a_1 = 0$ and $\lambda = 2k + 1$ for k even $\Rightarrow$ w is a even polynom of degree k.
If $a_0 = 0,\ a_1 \neq 0$ and $\lambda = 2k + 1$ for k odd $\Rightarrow$ w is an odd polynom of degree k.

In particular

$$\begin{array}{rllll} H_0(x) = 1 & \lambda = 1 & a_0 = 1 & a_1 = 0 & \\ H_1(x) = 2x & \lambda = 3 & a_0 = 0 & a_1 = 2 & \\ H_2(x) = 4x^2 - 2 & \lambda = 5 & a_0 = -2 & a_1 = 0 & \Rightarrow \; a_2 = \frac{-5+1}{2 \cdot 1}(-2) = 4 \end{array}$$

The set of polynomials H_k is called Hermite polynomials. It follows that solutions of (59) are

$$v_k(x) = H_k(x) e^{-x^2/2} \text{ for } \lambda_k = 2k + 1 \text{ and } \forall k \in \mathbb{N} \cup \{0\}$$

Remark 11.5.
(1) v_k satisfies the boundary conditon at infinity because $P(x)e^{-x^2/2} \to 0$ as $|x| \to \infty$ for any polynomial P.
(2) If $\lambda \neq 2k+1$, no power series solution satisfies the condition at infinity.
(3) the following formual holds
$$H_k(x) = (-1)^k e^{x^2} \frac{d}{d^k} e^{-x^2}$$
(4) $v_k(x) = H_k(x)e^{-x^2/2}$ are mutually orthogonal:
$$\int_R H_k(x)H_l(x)e^{-x^2} dx = 0 \text{ if } k \neq l.$$
(5) The set v_k is complete: $\forall f : \mathbb{R} \to \mathbb{R}$ such that $\int |f|^2 dx < \infty$ it holds $f = \sum_{k=0}^{\infty} A_k v_k$ in L^2 sense (mean square sense) where $A_k = (v_k, f)/(v_k, v_k)$ and $(u, v) = \int u \cdot v dx$.
$\longrightarrow$ "the spectrum of $-\Delta + |x|^2$ is λ_k, $k \in \mathbb{N} \cup \{0\}$.

Higher dimensions. The LHS of the equation
$$\Delta v - |x|^2 v = \lambda v \text{ on } \mathbb{R}^n$$
factorizes as
$$\sum_{i=1}^{n} (v_{x_i,x_i} - x_i^2 v) = \lambda v$$
This allows us to apply the separation of variables method w.r.t. to $x_1, \dots, x_n$ and solutions are given by
$$v_{\mathbf{k}}(x) = \prod_{j=1}^{n} H_{k_j}(x_j) e^{-x_j^2/2}$$
for $\lambda_{\mathbf{k}} = \sum_{j=1}^{n} \lambda_{k_j} = \sum_{j=1}^{n} (2k_j + 1)$ and $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$..

11.3. **Energy levels of the hydrogen atom:** $V(x) = \frac{1}{|x|}$**.** Assume $k = 1$:
$$-\Delta v - \frac{2}{|x|} v - \lambda v = 0 \text{ on } \mathbb{R}^3.$$
After a transformation with spherical coordinates the equation becmes
$$-\tilde{v}_{r,r} - \frac{2}{r}\tilde{v}_r - \frac{1}{r^2}(\text{ partial derivatives of } \phi \text{ and } \theta\ ) - \frac{2}{r}\tilde{v} - \lambda\tilde{v} = 0 \ r \in (0, \infty), \phi \in (0, 2\pi), \theta \in (0, \pi)$$
where $\tilde{v}(r, \phi, \theta) = v(r \cos\phi \cos\theta, r \sin\phi \cos\theta, r \sin\theta)$.

Assume that $\lambda < 0$. That is only study the negative part of the specturem of the operator $L = -\Delta - \frac{2}{|x|}$.

Assume also that v is spherical symmetric. Then partial derivatives w.r.t. ϕ and θ in the equation above vanish and it reduces to
$$\Rightarrow R'' + \frac{2}{r}R' + \lambda R + \frac{2}{r}R = 0$$
this equation is known as *Laguerre's differential equality.* We have the following boundary conditions
$$R(0) < \infty \ \text{ and } \ R(r) \to 0 \text{ as} r \to \infty.$$
The assymptotic behaviour of Laguerre's equations is $R'' + \lambda R = 0$ with solution $R(r) = e^{\pm\beta r}$ for $\beta = \sqrt{-\lambda}$. Therefore, taking into account the boundary condition at infinity, we make the following ansatz
$$R(r) = w(r)e^{-\beta r}$$

As for the harmonic oscillator we can deduce an ODE for w:

$$-w'' + 2(\beta - \frac{1}{r})w' + 2(\beta - 1)\frac{1}{r}w = 0$$

This yields

$$\frac{1}{2}rw'' - \beta rw' + w' + (\beta - 1)w = 0$$

The power series methode yields for this equation that

$$0 = \frac{1}{2}\sum_{k=0}^{\infty} a_k(k-1)kr^{k-1} - \beta\sum_{k=0}^{\infty} ka_kr^k + \sum_{k=0}^{\infty} ka_kr^{k-1} + (1-\beta)\sum_{k=0}^{\infty} a_kr^k$$

If relabel k to $k-1$ in the second and fourth series. we obtain

$$0 = \sum_{k=0}^{\infty} a_k(\frac{1}{2}(k-1)k + k)r^{k-1} + \sum_{k=1} a_{k-1}\left((1-\beta) - \beta(k-1)\right)r^{k-1}$$

Consequently

$$a_k\frac{k(k+1)}{2} = (\beta k - 1)a_{k-1} \ \forall k \in \mathbb{N}$$

If $\beta = \frac{1}{k}$ for some $k \in \mathbb{N}$ then $a_{k+i} = 0 \ \forall i \in \mathbb{N}$. Hence w_k is a polynomial of degree k. We also define

$$w_k(r)e^{-\beta_k r} = R_k(r) = \tilde{v}_k(r)$$

A special family of spherical symmetric solutions of the Schroedinger equation is therefore given by

$$u(x,t) = \sum_{k=0}^{\infty} e^{-2\lambda_k t}v_k(x) \text{ with } \lambda_k = \beta_k^2 = \frac{1}{k^2}$$

The set $\{-\frac{1}{k^2}\}_{k\in\mathbb{N}}$ are the energy levels of the hydrogen atom as experimentally observed by Bohr.

Remark 11.6.

(1) The functions v_k are not complet. There are two reasons for that. First, we assumed spherical symmetry (no angular momentum of the the atom). Second, Besides $\lambda_k < 0$ the specturm of the operator $-\Delta - \frac{1}{|x|}$ has a positive and continuous part that is $[0,\infty)$ (free electrons).

(2) If $\beta \neq \frac{1}{k}$, then w is an infinite power series. For large k the coefficients behave assymptotically as follows

$$\frac{k(k+1)}{2}a_k = (\beta k - 1)a_{k-1} \Rightarrow ka_k = \left(\beta\frac{2k}{k+1} - \frac{2}{k+1}\right)a_k \Rightarrow a_k = \beta\frac{2}{k}a_{k-1}$$

for large k. Hence $w \sim e^{2\beta r}$ that does not satisfy the boundary condition at infinity.

Lecture 37

12. Fourier method

12.1. **introducton.** We have considered different types of equations

(1) $-iu_t = \underbrace{\frac{1}{2}k\Delta u + Vu}_{-Lu}$ on $\mathbb{R}^n$

(2) $u_t = \underbrace{k\Delta}_{-Lu}$ on $\Omega \subset \mathbb{R}^n$ or $\mathbb{R}^n$

(3) $u_{t,t} = \underbrace{c^2\Delta u}_{-Lu}$ on $\Omega \subset \mathbb{R}^n$ or $\mathbb{R}^n$.

We assume the factor in front of Δ is always 1.

Seperation of Variables: Assume $u(x,t) = T(t)X(x)$. This yields equations for T (for instance $T' + \lambda T = 0$ in case of (2)) and

(60) for $X(x): \quad LX = \lambda X$ on $\Omega \subset \mathbb{R}^n$ or $\mathbb{R}^n$ together with a boundary condition.

for a constant λ and with solutions T_λ and X_λ.

Goal: If possible, find λ_k, $k \in \mathbb{N}$, and solutions v_k of (60) s.t. $\phi(x) = \sum_{k=0}^{\infty} A_k v_k(x)$ in L^2-sense or stronger for many initial conditions $\phi(x) = u(x,0)$.

$$\Longrightarrow \; u(x,t) = \sum_{k=0}^{\infty} T_{\lambda_k}(t) A_k V_k(x) \text{ solves problem (1) and (2)}$$

for the initial condition ϕ.

In case of (3) one has a pair of initial conditions $\phi = \sum A_k v_k$ and $\psi = \sum B_k v_k$ and $T_\lambda(t) = A\cos(\lambda t) + B\sin(\lambda t)$.

$$\Longrightarrow \; u(x,t) = \sum_{k=0}^{\infty} (A_k \cos(\lambda t) + B_k \sin(\lambda t)) v_k.$$

In general: The existence of solutions of (60) depends on V and Ω.

12.2. **Orthogonality of eigenfunctions.** Assume Ω is bounded and $\partial\Omega$ is smooth, and that $g, f : \Omega \subset \mathbb{R} \to \mathbb{C}$ continuous. An inner product between f and g is defined by

$$\int_\Omega f\overline{g}dx =: (f,g)$$

where $\overline{c}$ is complex conjugate of $c \in \mathbb{C}$. Hence $\overline{g} = \overline{Re(g) + iIm(g)} = Re(g) - iIm(g)$.

In particular $\|f\|^2 = (f,f) = \int_\Omega (Re(f)^2 + Im(f)^2)dx$.

Remark 12.1. The PDEs make sense for $\mathbb{C}$ value v: $\Delta v = \Delta Re(v) + i\Delta Im(v) = (Re(\lambda) + iIm(\lambda))(Re(v) + iIm(v))$.

Assume from now on that $V = 0$**.** If u, v are $\mathbb{R}$-valued Green's identity yields

$$\int_\Omega uLvdx - \int_\Omega vLudx = \int_\Omega u\Delta vdx - \int_\Omega v\Delta udx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial N} + v\frac{\partial u}{\partial N} \right) dx$$

One can check: homogeneous Dirichlet, Neumann or Robin conditions imply $(u, Lv) = (v, Lu)$.

If u, v are $\mathbb{C}$ valued, note that

$$\begin{aligned}\Delta u \cdot \overline{v} &= (\Delta Re(u) + i\Delta Im(u))(Re(v) - iIm(v)) \\ &= \Delta Re(u)Re(v) + \Delta Im(u)Im(v) + i\Delta Im(u)Re(v) - i\Delta Re(u)Im(u).\end{aligned}$$

Hence, it is straightforward to see that Green's identity again shows that $(Lu, v) = (u, Lu)$ provided one has Dirichlet, Neumann or Robin boundary conditions (for the Real and Imaginary part of u respectively).

We say $L = -\Delta$ on Ω with the corresponding boundary condition is a symmetric operator.

Corollary 12.2.
- *If u, v are eigenfcts for real eigenvalues $\lambda_1 \neq \lambda_2$ then $0 = (u, Lv) - (v, Lu) = \underbrace{(\overline{\lambda_2 - \lambda_1})}_{=\lambda_2-\lambda_1}(u, v)$. Hence u and v are orthogonal.*
- *If u is an eigenfunction for an eigenvalue $\lambda \in \mathbb{C}$, then $0 = (\overline{\lambda} - \lambda)\underbrace{(u, u)}_{>0}$.*

  Hence $\lambda = \overline{\lambda}$ and therefore $\lambda \in \mathbb{R}$.

  In particular, since $\Delta u = \Delta Re(u) + i\Delta Im(u)$, $Re(u)$ and $Im(u)$ are $\mathbb{R}$ valued eigenfunctions for the real EV λ.

Remark 12.3 (**Eigenvalues with multiplicity**). If there are 2 linear independent eigenfunctions u, v for the same eigenvalue λ, the Gram-Schmidt algorithm finds $\tilde{v}$ s.t. $(u, \tilde{v}) = 0$ and $u, \tilde{v}$ span the same linear space as u and v.

Theorem 12.4. *If $\phi(x) = \sum_{k=0}^{\infty} A_k v_k$ (in L^2 sense or stronger) for orthogonal functions v_k, then $A_k = \frac{(\phi, v_k)}{(v_k, v_k)}$.*

Proof. We have

$$(\phi, v_n) = \int_\Omega \phi v_n dx = \int_\Omega \left(\sum_{k=0}^{\infty} A_k v_k v_n\right) dx = \sum_{k=0}^{\infty} A_k \int_\Omega v_k v_n dx = A_k(v_n, v_n)$$

where L^2 converges allows to pull the sum outside of the integral. □

Completeness. Consider $L = -\Delta$ with hom. Dirichlet, Neumman or Robin bdy conditions on $\partial\Omega$. Then there exist infinitely many eigenfcts v_k, $k \in \mathbb{N}_0$ and

$$\phi = \sum_{k=0}^{\infty} A_k v_k \text{ in } L^2\text{-sense } \ \forall \phi \in C^0(\overline{\Omega}) \left(\text{ or } \forall \phi \text{ with } \int_\Omega \phi^2 dx < \infty \right).$$

Lecture 38

Let $\Omega \subset \mathbb{R}^n$ be bouned with $\partial\Omega$ smooth.

Theorem 12.5. *If $L = -\Delta$ (that is $V = 0$) on Ω with hom. Dirichlet bdy conditions, then all eigenvalues are positive.*

For hom. Neumann or hom. Robin bdy conditions all eigenvalues are non-negative.

Proof. For the Dirichlet case: Let $u \neq 0$ be an eigenfct. Green's first identity yields

$$\lambda(u,u) = \int_\Omega u\Delta u dx = \int_\Omega |\nabla u|^2 dx - \int_\Omega (\dots) dx = \int_\Omega |\nabla u|^2 dx \geq 0$$

If $\int_\Omega |\nabla u|^2 dx = 0$, then $\nabla u = 0$, Hence u is constant on every connected component of Ω and by the homogeneous Dirichlet bdy condition is must be 0. □

Example 12.6. Let $\Omega = Q = [0,\pi]^n$. We want to find a complete set of eigenfunctions for $-\Delta$ on Q with homog. Dirichlet BC. Note that

$$-\Delta v = -\sum_{i=1}^n u_{x_i,x_i} = -\sum_{i=1}^n \Delta^{1D} u(\dots, x_i, \dots).$$

For the $1D$ problem we found eigenfunctions $v_k = \sin(kx)$, $k \in \mathbb{N}$, with EV k^2. Then a complete set of eigenfunctions for the n-dimensional problem is given by

$$v_{\mathbf{k}}(x_1, ;x_n) = \prod_{i=1}^n v_{k_i}(x_i),\ \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n \text{ for the EV } \sum_{i=1}^n k_i^2.$$

12.3. **Vibration of $2D$ disk.** We study the following problem

$$(61)\qquad \begin{array}{rcccl} u_{tt} &=& c^2\Delta u & \text{on} & \Omega \\ u(\cdot,t) &=& 0 & \text{on} & \partial\Omega \\ u(x,0) &=& \phi(x) & u_t(x,0) & = \psi(x). \end{array}$$

where $\Omega = B_a(0)$.

Separation of Variables yields the following eigenvalue equation (compare with previous lecture):

$$(62)\qquad \underbrace{-c^2\Delta v}_{=:Lv} = \lambda \text{ on } \Omega \text{ for } \lambda > 0 \quad v(\cdot,t) = 0 \text{ on } \partial\Omega.$$

The symmetry of the domain Ω suggests to introduce polar coordinates: $\tilde{v}(r,\theta) = v(r\cos\theta, r\sin\theta)$. Then

$$\Delta\tilde{v} = v_{r,r} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta,\theta}$$

Another separation of variables equation yields for $\tilde{v}(r,\theta) = R(r)\Theta(\theta)$

$$\left(R'' + \frac{1}{r}R' + \lambda R\right)\Theta = \left(-\frac{1}{r^2}\Theta''\right)R$$

It follows

$$\frac{R'' + \frac{1}{r}R' + \lambda R}{\frac{1}{r^2}R} = -\frac{\Theta''}{\Theta} = \gamma \in \mathbb{R}.$$

The equation for Θ is

$$\Theta'' + \lambda\Theta = 0 \text{ together with periodic boundary condition } \Theta(\theta + 2\pi) = \Theta(\theta).$$

We saw before that the general solution is given by $\Theta(\theta) = A\cos(n\theta) + B\sin(n\theta)$ and necessarily $\gamma = n^2$ for $n \in \mathbb{N}$. Hence the equation for R is

$$R'' + \frac{1}{r}R + \left(\lambda - \frac{n^2}{r^2}\right)R = 0 \text{ on } [0,a] \text{ with } R(0) < \infty \text{ and } R(a) = 0.$$

Observe that because the eigenvalue equation comes with Dirichlet boundary condition, we have $\lambda > 0$ (see previous lecture).

Hence we can introduce a new variable $\rho = \sqrt{\lambda} r$ and $\tilde{R}(\rho) = R(\rho/\sqrt{\lambda})$. It follows

$$\lambda \tilde{R}'' + \frac{\sqrt{\lambda}}{r}\tilde{R}' + \left(\lambda - \frac{n^2}{r^2}\right)\tilde{R} = 0 \Rightarrow \tilde{R}'' + \frac{1}{\rho}\tilde{R}' + \left(1 - \frac{n^2}{\rho^2}\right)\tilde{R} = 0$$

The last equation is known as *Bessel's differential equation.*

Remark 12.7. The coefficients of this ODE are singular at $\rho = 0$. But $\rho = 0$ is a so-called *regular singular point.* This implies that $\tilde{R} \sim C\rho^\alpha$ for $\rho \to 0$ for some $\alpha \in \mathbb{R}$.

Therefore we make the following ansatz. We assume that

$$\tilde{R}(\rho) = w(\rho)\rho^\alpha \text{ where } w \text{ is given as a power series: } w(\rho) = \sum_{k=0}^\infty a_k \rho^k.$$

Here we will not derive an equation for w, but we will write

$$\tilde{R}(\rho) = \sum_{k=0}^\infty a_k \rho^{\alpha+k}$$

and plug it into the differential equation for $\tilde{R}$:

$$0 = \rho^\alpha \sum_{k=0}^\infty a_k(\alpha+k)(\alpha+k-1)a_k\rho^{k-2} + (\alpha+k)a_k\rho^{k-2} + a_k\rho^k - n^2 a_k \rho^{k-2}.$$

$$\begin{aligned} \Rightarrow\ & k=0: \ \left(\alpha(\alpha-1)+\alpha-n^2\right)a_0 = 0,\\ & k=1: \ \left(\alpha(\alpha+1)+\alpha+1-n^2\right)a_1 = 0,\\ & k\geq 2: \ \left((\alpha+k)(\alpha+k-1)+(\alpha+k)-n^2\right)a_k = -a_{k-2} \end{aligned}$$

Hence

$$\begin{aligned} \text{first equ. :} \ &\Rightarrow \ \alpha = \pm n, \ a_0 \in \mathbb{R}. \ \text{ Assume } n \geq 0,\\ \text{second equ. :} \ &\Rightarrow \ \left((\alpha+1)^2 - n^2\right)a_1 = 0 \ \Rightarrow \ a_1 = 0,\\ \text{third equ. :} \ &\Rightarrow \ a_k = -\frac{a_{k-2}}{(\alpha+k)^2-n^2} = -\frac{a_{k-2}}{(\alpha+k-n)(\alpha+k+n)} = -\frac{a_{k-2}}{k(2n+k)}. \end{aligned}$$

In particular, it follows that $a_k = 0$ for $k \in \mathbb{N}$ odd.

Hence, it follows that

$$\begin{aligned} a_{2j} &= -\frac{a_{2j-2}}{2j\cdot 2(n+j)}\\ \text{if } a_0 = 1 \ \text{ then } &(-1)^j \frac{1}{j!2^{2j}(n+1)\cdots(n+j)},\\ \text{if } a_0 = \frac{1}{2^n}\frac{1}{n!} \ \text{ then } &(-1)^j\frac{1}{j!2^{2j+n}(n+j)!}\\ \Rightarrow \ \tilde{R}(\rho) &= \sum_{j=0}^\infty (-1)^j \left(\frac{1}{2}\rho\right)^{n+2j}\frac{1}{j!(n+j)!} = J_n(\rho) \end{aligned}$$

$J_n(\rho)$ is called the Bessel function of order n.

Remark 12.8. (1) The solutions for $\alpha = -n$ look like ρ^{-n} for $\rho \to 0$ or they look like $\log \rho$. Hence, since we require $\tilde{R}(0) < \infty$ we dismiss them.

(2) The series above converges on $[0, \infty)$ and hence $J_n(\rho)$ solves the Bessel equation. $\Rightarrow R(r) = cJ_n(\sqrt{\lambda} r)$ for some $c \in \mathbb{R}$.

(3) It holds $J_n(\rho) \sim C\rho^\alpha$ for $C = const$ for $\rho \to 0$ and
$$J_n(\rho) \sim \sqrt{\frac{2}{\pi\rho}}\cos(\rho - \frac{\pi}{4} - \frac{n\pi}{2}) + o(\frac{1}{\rho^{3/2}}) \text{ for } \rho \to \infty.$$

For the homogeneous Dirichlet boundary condition we require $J_n(\sqrt{\lambda}a) = 0$. The function $\lambda \mapsto J_n(\sqrt{\lambda}a)$ has infinitely many, countably many roots $0 < \lambda_{n,1} < \lambda_{n,2} < \dots$.

Theorem 12.9. *Solutions for the eigenvalue equation (62) are given by*
$$\tilde{v}_{n,m}(r,\theta) = J_n(\sqrt{\lambda_{n,m}}r)\cdot\cos(n\theta),\ \ \tilde{w}_{n,m}(r,\theta) = J_n(\sqrt{\lambda_{n,m}}r)\cdot\sin(n\theta)$$
This is a complete orthonormal system of eigenfunctions for the operator $L = -\Delta$ on $B_a(0)$ ($\lambda_{n,m}$, $(n,m) \in \mathbb{N}^2$ are the eigenvalues of L with multiplicity 2)

Together with Fourier method from the previous lecture we can solve (61). Lets do this for a particular case.

Example 12.10. We want to find the solution $\tilde{u}$ with $\tilde{u}(r,\theta,0)$ and $\tilde{u}_t(r,\theta,0) = \tilde{\psi}(r)$ (ψ only depends on the radius variable r). If we expand $\tilde{\phi}$ with the eigenfunction, we found, then we get
$$\tilde{\psi}(r) = \sum_{m=1}^{\infty}\beta_{0,m}C_{0,m}J_0(\beta_{0,m}r)$$
where $\beta_{0,m} = \sqrt{\lambda_{0,\beta}}$. Hence
$$\tilde{u}(r,\theta,t) = \sum_{m=1}^{\infty}C_{0,m}J_{0,m}(\beta_{0,m}r)\sin(\beta_{0,m}t)$$
where
$$\beta_{0,m}C_{0,m} = \int_0^a \tilde{\psi}(r)J_0(\beta_{0,m}r)rdr/\int_0^a (J_0(\beta_{0,m}r))^2 rdr.$$

Remark 12.11. Note that $\tilde{v}_{n,m}, \tilde{w}_{n,m}$ is indeed an orthornormal set. For instance we can compute
$$\int_0^{2\pi}\int_0^a \tilde{v}_{n,m}\tilde{v}_{k,l}rdrd\theta = \int_0^{2\pi}\cos(n\theta)\cos(k\theta)d\theta\cdot\int_0^a J_n(\beta_{n,m}r)J_k(\beta_{k,l}r)rdr$$
where the first integral is 0 if and only if $m \neq k$ and the second integral is 0 if and only if $(n,m) \neq (k,l)$.

12.4. **Vibrations of a $3D$ ball.** Consider the eigenvalue problem

$$\begin{array}{rcll} -\Delta v &=& \lambda v & \text{on} \quad \Omega \\ v &=& 0 & \text{on} \quad \partial\Omega \end{array} \tag{63}$$

where $\Omega = B_a(0) \subset \mathbb{R}^3$.

By Spherical coordinates the PDE becomes
$$0 = \Delta v + \lambda v = \tilde{v}_{r,r} + \frac{2}{r}\tilde{v}_r + \frac{1}{r^2}\left[\frac{1}{(\sin\theta)^2}v_{\phi,\phi} + \frac{1}{\sin\theta}\left(\sin\theta\tilde{v}_\theta\right)_\theta\right] + \lambda\tilde{v}$$
where $\tilde{v}(r,\phi,\theta) = v(r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta)$, $r \in [0,a], \phi \in [0,2\pi], \theta \in [0,\pi]$.

With another separation of variables $\tilde{v}(r,\phi,\theta) = R(r)Y(\phi,\theta)$ we get
$$R'' + \frac{2}{r}R' + (\lambda - \frac{\gamma}{r^2})R = 0 \text{ for } \gamma \in \mathbb{R} \text{ and with } R(0) < \infty \text{ and } R(a) = 0.$$
and
$$\underbrace{\frac{1}{\sin^2\theta}Y_{\phi,\phi} + \frac{1}{\sin\theta}\left(\sin\theta Y_\theta\right)_\theta}_{=\Delta^{\partial B_1(0)}Y} + \gamma Y = 0$$
with boundary conditions $Y(\phi + 2\pi, \theta) = Y(\phi,\theta)$ and $Y(\phi,0), Y(\phi,\pi) < \infty$.

Remark 12.12. $\Delta^{\partial B_1(0)}$ is the Laplace operator on $\partial B_1(0)$ in spherical coordinates.

Defininng $\sqrt{r}R(r) = w(r)$, then w satisfies

$$w'' + \frac{1}{r}w' + \left(\lambda - \frac{\gamma + \frac{1}{4}}{r^2}\right) w = 0.$$

Then a change of variable $\tilde{R}(\rho) = w(\rho/\sqrt{\lambda})$ yields that $\tilde{R}$ satisfies again Bessel differential equality.

Hence $w(r) = J_{\sqrt{\gamma+\frac{1}{4}}}(\sqrt{\lambda}r)$ and $R(r) = \frac{1}{\sqrt{r}} J_{\sqrt{\gamma+\frac{1}{4}}}(\sqrt{\lambda}r)$ with boundary conditions $R(0)$ finite and $R(a) = 0$.

Angular funciton $Y(\phi, \theta)$. We assume one more time separated variables: $Y(\phi, \theta) = q(\phi)p(\theta)$. It follows

$$\frac{1}{(\sin\theta)^2} q''(\phi)p(\theta) + \frac{1}{\sin\theta}\left(\sin\theta q'(\theta)\right)' p(\phi) + \gamma q(\phi)q(\theta) = 0$$

$$\Rightarrow \frac{q''}{q} = -\frac{\sin\theta(\sin\theta p')'}{p} - \gamma(\sin\theta)^2 =: -\alpha \in \mathbb{R}.$$

Recall the periodic boundary condition for Y in the variable ϕ. Hence, we know that

$$\alpha = m^2 \text{ with } m \in \mathbb{N}_0 \text{ and } q(\phi) = A\cos(m\phi) + B\sin(m\phi).$$

$$\Rightarrow \frac{(\sin\theta p')'}{\sin\theta} + \left(\gamma - \frac{m^2}{(\sin\theta)^2}\right) p = 0$$

with boundary conditions $p(0), p(\pi) < \infty$.

Another variable change $s = \cos\theta$ ($\to \sin\theta = \sqrt{1-s^2}$) yields

$$\frac{d}{ds}\left((1-s^2)\frac{dp}{ds}\right) + \left(\gamma - \frac{m^2}{1-s^2}\right) p = 0$$

with $p(-1), p(1)$ finite. This equation is known as (asssociated) Legendre equation. Its eigenvalues are $l(l+1)$ for $l \in \mathbb{N}$, $l \geq m$ with eigenfunctions

$$P_l^m(s) = \frac{(-1)^m}{2^l l!}(1-s^2)^{\frac{m}{2}} \frac{d^{l+m}}{ds^{l+m}}\left[s^2 - 1\right]^l$$

This are the associated Legendre functions.

Conclusion: The function

$$\tilde{v}(r, \phi, \theta) = R(r)q(\phi)p(\theta) = J_{\sqrt{\gamma+\frac{1}{4}}}(\sqrt{\lambda}r)\frac{1}{\lambda}(A\cos(m\phi) + B\sin(m\phi))P_l^m(\cos\phi)$$

solves the eigenvalue equation (63) where $l \geq m, \gamma = l(l+1)$. It follows $\sqrt{l(l+1) + \frac{1}{4}} = l + \frac{1}{2}$.

One can also replace the sin and cos term with $e^{im\phi}$ for $m \in \mathbb{Z}$.

Moreover, let $\lambda_{l,i}$ be the roots of $J_{l+\frac{1}{2}}(\sqrt{\lambda}a)$. Then

$$\tilde{v}_{l,m,i}(r, \phi, \theta) = \frac{1}{\sqrt{\lambda}} J_{l+\frac{1}{2}}(\sqrt{\lambda}r) P_l^{|m|}(\cos\theta)e^{im\phi}, m \in \mathbb{Z}, l \geq |m|, i \in \mathbb{N}.$$

is a complete set of orthogonal eigenfunctions with EV $\lambda_{l,i}$ that has multiplicity $2l + 1$ because there $2l + 1$ m's such that $l \geq |m|$.

Lecture 39

12.5. **Bessel's differential equation.** Bessel's differential equation is

$$u'' + \frac{1}{z}u' + \left(1 + \frac{s^2}{z^2}\right) u = 0 \quad \text{for } z > 0.$$

For $s \in \mathbb{N}$ and $u(0) < \infty$ we derived by the power series method that

$$J_n(z) = \sum_{j=0}^{\infty} (-j)^j \frac{1}{j!(n+j)!} \left(\frac{z}{2}\right)^{2j+n}$$

where the coefficients a_k are defined recursively with $a_0 = \frac{1}{2^n n!}$.

Now let $s \in (0, \infty)$ and let $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ is the *Gamma function.* Γ satisfies

- $\Gamma(n+1) = n!\ \forall n \in \mathbb{N}$.
- $\Gamma(s+1) = s\Gamma(s) = s(s-1) \cdot \dots, \cdot (s-n)\Gamma(s-n) \quad \forall n \in \mathbb{N}$ with $n > s$.

Setting $a_0 = \frac{1}{2^s \Gamma(s+1)}$ the previous recursion formula for the coefficients yields

$$J_s(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(j+s+1)} \left(\frac{z}{2}\right)^{2j+1}$$

This is the Bessel function of order $s > 0$ and it solves Bessel's differential equation.

Further extension to $s \in (-1, 0)$ is possible. Then J_s and J_{-s} are linear independent solutions for Bessl's differential equation, but $J_{-s}(z) \to \infty$ for $z \to 0$.

- **Assymptotic behavior:**
$$J_s(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - \frac{s\pi}{2} - \frac{\pi}{4}) + o(z^{-\frac{3}{2}})$$
- **Recursion formula:**
$$J_{s\pm 1} = \frac{s}{z} J_s \mp J_s'(z)$$
In particular, for $s = \frac{1}{2} + n - 1$ one has
$$J_{\frac{1}{2}+n}(z) = (-1) z^{\frac{1}{2}+n} \frac{d}{dz}\left(z^{-\frac{1}{2}-n} J_{\frac{1}{2}+n-1}(z)\right) = (-1)^n z^{n+\frac{1}{2}} \left(z^{-1}\frac{d}{dz}\right)^n \left(z^{-\frac{1}{2}} J_{\frac{1}{2}}(z)\right)$$
- For $s = \frac{1}{2}$, $v(z) = \sqrt{z} J_{\frac{1}{2}}(z)$ solves $v'' + v = 0$ with $v(0) = 0$. Hence $v(z) = c \cdot \sin z$ and
$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin(z)$$
Similar $J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos(z)$. Therefore, we obtain the following nice formula
$$J_{n+\frac{1}{2}}(z) = (-1)^n \sqrt{\frac{2}{\pi}} z^{n+\frac{1}{2}} \left(z^{-1}\frac{d}{dz}\right)^n \left(\frac{\sin z}{z}\right).$$

12.6. **Legendre functions.** Eigenfunction of $-\Delta$ on $B_a(0) \subset \mathbb{R}^3$ with Dirichlet BC are

$$\tilde{v}_{l,m,j}(r,\phi,\theta) = \frac{1}{\sqrt{r}} J_{\frac{1}{2}+l}(\sqrt{\lambda_{l,j}} r) P_l^{|m|}(\cos\theta) e^{im\phi} \text{ with } m \in \mathbb{Z}, l \in \mathbb{N}_0 \text{ s.t. } l \geq |m| \text{ and } j \in \mathbb{N}.$$

The corresponding eigenvalue is $\lambda_{l,j}$ where $(\lambda_{l,j})_{j\in\mathbb{N}}$ are the roots of the function $\lambda \mapsto J_{\frac{1}{2}+l}(\sqrt{\lambda} a)$.

Remark 12.13. The functions $P_l^m(s)$ for $m, l \in \mathbb{N}$ with $l \geq m$ are the *associated Legndre functions* given by $P_l^m(s) = \frac{(-1)^m}{2^l l!} (1-s^2)^{\frac{m}{2}} \underbrace{\frac{d^m}{ds^m} \frac{d^l}{ds^l} (s^2-1)^l}_{=: q_l^m}$ where q_l^m are Polynomials of degree $l - m$.

- $P_l^m(s)$ solves the *associated Legendre equation* $((1-s^2))P')' + \left(l(l+1) - \frac{m^2}{1-s^2}\right) P = 0$.
- $\frac{1}{2^l l!}\frac{d^l}{ds^l}(1-s^2)^l$ are called *Legndre polynomials* and they solve the *Legendre differential equation*

$$((1+s^2)v')' + l(l+1)v = 0.$$

Orthogonality

$$(\tilde{v}_{l,m,j}, \tilde{v}_{k,n,t}) = \int_0^a \int_0^{2\pi} \int_0^{\pi} \tilde{v}_{l,m,j}(r,\phi,\theta)\tilde{v}_{k,n,t}(r,\phi,\theta)\sin\theta r^2 dr d\phi d\theta$$
$$= \underbrace{\int_0^a J_{\frac{1}{2}+l}(\sqrt{\lambda_{l,j}}r)J_{\frac{1}{2}+k}(\sqrt{\lambda_{k,j}}r)rdr}_{=0 \text{ if } (l,j)\neq(k,t)} \underbrace{\int_0^{2\pi} e^{i(m-n)\phi}d\phi}_{=0 \text{ if } m\neq n} \underbrace{\int_0^{2\pi} P_l^{|m|}(\cos\theta)P_k^{|n|}(\cos\theta)\sin\theta d\theta}_{=0 \text{ if } (m,l)\neq(n,k)}$$

Spherical Harmonics The set of functions $Y_l^m(\phi,\theta) = P_l^{|m|}(\cos\theta)e^{im\phi}$ $m \in \mathbb{Z}$, $l \in \mathbb{N}$ with $l \geq |m|$ is a complete orthogonal set of eigenfunctions for the Laplace operator on the sphere $\partial B_1(0) =: \mathbb{S}^2 \subset \mathbb{R}^3$. In spherical coordinates the Laplace operator on the sphere is

$$\frac{1}{\sin^2\theta}Y_{\phi,\phi} + \frac{1}{\sin\theta}\left(\sin\theta Y_\theta\right)_\theta.$$

Example: Laplace equation with non-homogeneous Dirichlet BC

$$\Delta u = 0 \text{ on } B_a(0)$$
$$u = g \text{ on } \partial B_a(0)$$

Expand $g = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l^m Y_l^m(\phi,\theta)$.

Assuming $\tilde{u}(r,\phi,\theta) = R(r)Y(\phi,\theta)$ the equation becomes

$$R'' + \frac{2}{r}R' + \frac{\gamma}{r^2}R = 0$$

with $\gamma = l(l+1)$ for $l \geq m$ and spherical harmonic $Y_l^m = Y$.

The equation for R is **not a Bessel equation of the form we saw before** but of Euler type:

$$uv' + \frac{\beta}{r}v' + \frac{\gamma}{r^2}v = 0.$$

Then: If s and t are the roots of $x(x-1)+\beta x + \gamma = 0$, solutions are given by $v(t) = Ct^r + Dt^s$.

In our case $\beta = 2$ and the quadratic equation $x^2 + x + l(l+1) = (x-l)(x+(l+1)) = 0$ has exactly 2 roots. The positive root is l and a solution for our Euler type Equation is $R(r) = r^l$ or $(r/a)^l$.

Hence $u(r,\phi,\theta) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l^m \left(\frac{r}{a}\right)^l Y_l^m(\phi,\theta)$ is a solution of the problem.

Lecture 40

12.7. **Hydrogen atom revisited.** Consider Schroedinger's equation

$$iu_t = -\frac{1}{2}\Delta u + V \cdot u$$

for a radial potential function $V = \tilde{V}(r)$ with $r^2 = x_1^2 + x_2^2 + x_3^2$ and $u = u(x, y, z, t) \in \mathbb{C}$. We assume the global boundary condition

$$\int_{\mathbb{R}^3} |u(x,t)|^2 dx < \infty.$$

A separation of variables yields solutions of the form $u(x_1, x_2, x_3, t) = v(x_1, x_2, x_3)e^{-i\lambda t/2}$ where

$$-\Delta v + 2V \cdot v = \lambda v$$

Let assume that $\lambda \in \mathbb{R}$ and $v(x_1, x_2, x_3) \in \mathbb{R}$.

After introducing spherical coordinates another separation of variables yields solutions of the eigenvalue equation of the form $\tilde{v}(r, \phi, \theta) = R(r) \cdot Y(\phi, \theta)$ where

$$\lambda r^2 - 2r^2\tilde{V}(r) + \frac{r^2R'' + 2rR'}{R} = \gamma = -\frac{1}{Y}\left\{\frac{1}{\sin^2\theta}Y_{\theta,\theta} + \frac{1}{\sin\theta}\left(\sin\theta Y_\theta\right)_\theta\right\}$$

for constant $\gamma \geq 0$. We observed that the expression inside $\{ \}$ on the RHS is the Laplace operator on $\partial B_1(0)$ in spherical coordinates.

Hence, the second equation corresponds to an eigenvalue equation of the Laplace operatore on $\partial B_1(0)$. The eigenvalues are given by $\gamma = l(l+1) > 0$ where $l \in \mathbb{N}$ with $l \geq |m|$ for $m \in \mathbb{Z}$. The corresponding eigenfunctions are the spherical harmonics given by

$$Y_l^m(\phi, \theta) = e^{im\phi}P_l^{|m|}(\cos\theta).$$

The equaton for R becomes

$$R'' + \frac{2}{r}R' + \left(\lambda - 2\tilde{V}(r) - \frac{l(l+1)}{r^2}\right)R = 0.$$

For the hydrogen atom where $\tilde{V}(r) = -\frac{1}{r}$ we obtain

$$R'' + \frac{2}{r}R' + \left(\lambda + \frac{2}{r} - \frac{l(l+1)}{r^2}\right)R = 0$$

where we assume the boundary conditions $R(0) < \infty$ and $R(r) \to 0$ for $r \to \infty$.

As in our previous treatment of the hydrogen atom we set $w(r) = e^{\beta r}R(r)$ with $\beta = \sqrt{-\lambda}$. Recall that we assumed that $\lambda < 0$.

$$\implies w'' + 2\left(\frac{1}{r} - \beta\right)w' + \left[\frac{2(1-\beta)}{r} - \frac{l(l+1)}{r^2}\right]w = 0.$$

Power series method: Assume $w(r) = \sum_{k=0}^\infty a_k r^k$. Then

$$\sum k(k-1)a_k r^{k-2} + 2\sum k a_k r^{k-2} - 2\beta \sum k a_k r^{k-1} + 2(1-\beta)\sum a_k r^{k-1} - l(l+1)\sum a_k r^{k-2} = 0$$

or

$$\sum_{k=0}^\infty \underbrace{[k(k-1) + 2k - l(l+1)]}_{=k(k+1)-l(l+1)} a_k r^{k-2} + \sum_{k=1}^\infty \underbrace{[-2\beta(k-1) + (2-2\beta)]}_{=2(1-k\beta)} a_{k-1} r^{k-2} = 0$$

We can conclude that

$$l(l+1)a_0 = 0 \quad \text{and} \quad [k(k+1) - l(l+1)]a_k = -2(1-k\beta)a_{k-1}.$$

We see that for $k < l$ it must follow that $a_k = 0$. Then a_l is completely arbitrary, the following coefficients are computed using the recursion formula.

Finally the series is polynomial if $\beta = \frac{1}{n}$ for some $n \in \mathbb{N}$ since the recursion formula implies that $a_k = 0$ for $k \geq n$.

Thus the eigenvalues are $\lambda = -\frac{1}{n^2}$ and the corresponding eigenfunctions are

$$v_{nlm}(r, \phi, \theta) = e^{-r/n} L_n^l(r) \cdot Y_l^m(\phi, \theta)$$

where L_n^l are the polynomials of the form $L_n^l(r) = \sum_{k=l}^{n-1} a_k r^l$.

Remark 12.14. The eigenfunctions v_{nlm} are still not complete because we cannot consider $\lambda \geq 0$ which corresponds to the case of a free electron.

Angular Momentum in QM. In Newton mechanic, given a particle $x(t) \in \mathbb{R}^3$, the angular momentum is defined as $x(t) \times \dot{x}(t)$ (here $\times$ is the cross product between vectors).

The angular momentum of QM system is the operator $\mathbf{L} = -ix \times \nabla = i \begin{pmatrix} x_2\partial_3 - x_3\partial_2 \\ x_3\partial_1 - x_1\partial_3 \\ x_1\partial_2 - x_2\partial_1 \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}$.

In spherical coordinates:

$$L_1 = i(\cot\theta \cos\phi \partial_\phi + \sin\phi \partial_\theta), \;\; L_2 = i(\cot\theta \sin\phi \partial_\theta - \cos\phi \partial_\theta), \;\; L_3 = -i\partial_\phi$$

$$\implies |\mathbf{L}|^2 = L_1^2 + L_2^2 + L_3^2 = -\frac{1}{\sin^2\theta}\partial_{\phi,\phi} - \frac{1}{\sin\phi}\partial_\theta\left(\sin\theta \partial_\theta\right) = -\text{“Laplace operator on } \partial B_1(0)\text{”}$$

Hence, the eigenfunctions of $|\mathbf{L}|^2$ are the spherical harmonics. Moreover, one can check that

$$L_3 Y_l^m = m Y_l^m, \quad L_\pm Y_l^m = (L_1 \pm iL_2) Y_l^m = ((l \mp m)(l \pm m))^{\frac{1}{2}}\, Y_l^m.$$

Hence the spherical harmonics is a (complete) set of eigen functions for the operator $\mathbf{L}$.

Remark 12.15. In QM one cannot have a pure rotation around a single axis (for instance x_3). This would correspond to the existence of eigenfunctions $Y_l^m(\phi, \theta)$ that depends on ϕ but not on θ. Hence $P_l^{|m|}(\cos\theta) = const$, $\Rightarrow l = 0 \Rightarrow |m| = 0$ and therefore $Y_l^m(\phi, \theta) = const$.

Lecture 41

13. General eigenvalue problems

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected with $\partial\Omega$ smooth ($n = 3$ or $n \in \mathbb{N}$).
We consider the general eigenvalue problem

$$-\Delta u = \lambda u \quad \text{on} \quad \Omega \tag{64}$$

with Dirichlet boundary condition $u|_{\partial\Omega} = 0$.
We saw before that for this problem $\lambda \geq 0$. How can we find the eigenvalues and the solutions of the corresponding eigenvalue problem?
Dirichlet Energy:

$$E(u) = \int_\Omega |\nabla u|^2 dx \quad \text{for } u \in C^1(\overline{\Omega}) \quad \text{(We drop the } \tfrac{1}{2} \text{ in front of the integral)}$$

The minimizers u of E on $C^1(\overline{\Omega})$ are harmonic (Dirichlet principle). Hence if $u|_{\partial\Omega} = 0$, the maximum principle implies $u = 0$. Therefore $\lambda = 0$ is not an eigenvalue.

New Minimization problem (with additional constraints):
Find the minimizer of E on $C^1(\overline{\Omega})$ on $\mathcal{E}_0 := \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ with $\int_\Omega u^2 dx = 1$.

$$\iff \quad \text{Find minimizer of} \quad \frac{E(u)}{\|u\|_2^2} = R(u) \quad \text{on } \mathcal{E}_0\backslash\{0\}. \tag{65}$$

$R(u)$ is called the *Rayleigh quotient* of u.

Theorem 13.1 (Minimum principle for the first EV)**.** *Assume $u \in C^2(\overline{\Omega}) \cap \mathcal{E}_0$ solves the minimization problem (65). Then u solves the EV problem (64) with $\lambda = R(u) =: \lambda_1$.*

Moreover, assuming that any solution of the EV problem (64) for some λ is in $C^2(\overline{\Omega}) \cap \mathcal{E}_0$, then λ_1 is the smallest eigenvalue of $-\Delta$ on Ω with homogeneous Dirichlet BC.

Proof. Let $\phi \in C^1(\overline{\Omega})$ with compact support in Ω. Then $u + \epsilon\phi = w(\epsilon) \in \mathcal{E}_0$ $\forall \epsilon \in (-\delta, \delta)$. The functions $w(\epsilon)$ are called *trial functions.*

We set

$$f(\epsilon) = E(w(\epsilon)) = \int_\Omega |\nabla u|^2 dx + 2\epsilon \int_\Omega \langle \nabla u, \nabla \phi\rangle dx + \epsilon^2 \int_\Omega |\nabla \phi|^2 dx,$$
$$g(\epsilon) = \|w(\epsilon)\|_2^2 = \int_\Omega u^2 dx + 2\epsilon \int_\Omega u\phi dx + \epsilon^2 \int_\Omega \phi^2 dx$$

$$\Longrightarrow f'(0) = 2\int_\Omega \langle \nabla u, \nabla\phi\rangle dx, \quad g'(0) = 2\int_\Omega u\phi dx$$

Since u is a minimizer of R on $\mathcal{E}_0\backslash\{0\}$, we have

$$\Longrightarrow 0 = \frac{d}{d\epsilon} R(w(\epsilon))\big|_{\epsilon=0} = \frac{f'(0)g(0) - f(0)g'(0)}{g^2(0)}.$$

Since $g(0) = \|w(0)\|_2^2 = \|u\|_2^2 \neq 0$, it follows

$$0 = \int_\Omega \langle \nabla u, \nabla \phi\rangle dx \, \|u\|_2^2 - \int_\Omega u \cdot \phi dx \cdot E(u).$$

$$\Longrightarrow \quad R(u) \int_\Omega u\phi dx = \frac{E(u)}{\|u\|_2^2} \int_\Omega u\phi dx = \int_\Omega \langle \nabla u, \nabla\phi\rangle dx \overset{\text{1st Green Id}}{=} -\int_\Omega \Delta u \phi dx.$$

Since this holds for any function $\phi \in C^1(\Omega)$ with compact support in Ω, the Fundamental theorem of Calculus of Variations implies

$$-\Delta u = \lambda u \text{ on } \Omega.$$

If λ_2 is any other eigenvalue with an eigenfunction $v \in C^2(\overline{\Omega}) \cap \mathcal{E}_0$, then

$$-\int_\Omega v\Delta v dx = \int_\Omega |\nabla v|^2 dx = \lambda_2 \int_\Omega v^2 dx.$$

Hence

$$\lambda_2 = R(v) \geq R(u) = \lambda_1.$$

□

Recall $(u, v) = \int_\Omega uvdx$.

Theorem 13.2 (Minimum principle for the nth EV)**.** *Assume we found the first $n-1$ EVs $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$ with the corresponding eigenfunctions $v_1, \dots, v_n \in C^2(\overline{\Omega}) \cap \mathcal{E}_0$.*

Then the nth EV of $-\Delta$ on Ω with homogeneous Dirichlet BC is given by the following minimum with additional constraints

$$\lambda_n = \min \frac{E(w)}{\|w\|_2^2} \quad \text{where } w \in \mathcal{E}_0 \backslash \{0\},\ 0 = (w, v_1) = \cdots = (w, v_{n-1}). \tag{66}$$

If $u \in C^2(\overline{\Omega}) \cap \mathcal{E}_0$ solves the minimization problem (66) then u solves (64) for the $\lambda = R(u) = \lambda_n$.

Proof. Let $u \in C^2(\overline{\Omega}) \cap \mathcal{E}_0$ be a minimizer of (66) and let $\phi \in C^1(\Omega)$ with compact support in Ω.

Note we can choose $v_1, \dots, v_{n-1}$ to be orthogonal. Then, we define

$$\tilde{\phi} = \phi - \sum_{k=1}^{n-1} c_k v_k \quad \text{where} \quad c_k = \frac{(\phi, v_k)}{(v_k, v_k)}$$

The definition of $\tilde{\phi}$ is such that $(\tilde{\phi}, v_k) = 0\ \forall k = 1, \dots, n-1$. Moreover, $\tilde{\phi} \in C^1(\overline{\Omega}) \cap \mathcal{E}_0$. We also set $\tilde{w}(\epsilon) = u + \epsilon\tilde{\phi}$. Then $w(\epsilon) \in \mathcal{E}_0 \cap C^1(\overline{\Omega})$ and

$$(\tilde{w}(\epsilon), v_k) = (u + \epsilon\tilde{\phi}, v_k) = (u, v_k) + \epsilon(\tilde{\phi}, v_k) = 0.$$

Hence $\tilde{w}(\epsilon)$ are trial functions for the minimization problem (66) and we can define f, g has before and compute

$$f'(0) = 2\int_\Omega \langle \nabla u, \nabla \tilde{\phi} \rangle dx, \quad g'(0) = 2\int_\Omega u\tilde{\phi} dx$$

and

$$0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{f(\epsilon)}{g(\epsilon)}.$$

$$\implies \int_\Omega \langle \nabla u, \nabla \tilde{\phi} \rangle dx = \lambda^* \int_\Omega u\tilde{\phi} dx$$

where $\lambda^* = R(u) = \frac{E(u)}{\|u\|_2^2}$. Hence

$$\implies -\int_\Omega \Delta u \tilde{\phi} dx = \lambda^* \int_\Omega u\tilde{\phi} dx$$

$$\implies -\int_\Omega (\Delta u - \lambda^* u)\phi dx = -\sum_{k=1}^{n-1} c_k \int_\Omega (\Delta u - \lambda^* u) v_k dx$$

$$\text{2nd Green Id} = -\sum_{k=1}^{n-1} c_k (\Delta v_k - \lambda^* v_k) u dx = \sum_{k=1}^{n-1} c_k \int_\Omega (\lambda_k - \lambda^*) v_k u dx = 0.$$

Hence

$$-\int_\Omega \Delta u\phi dx = \lambda^* \int_\Omega u\phi dx$$

and since $\phi \in C^1(\Omega)$ with compact support in Ω was arbitrary, we get that $-\Delta u = \lambda^* u$ on Ω and $\lambda_k \leq \lambda^*$.

Moreover, since $\{v_1, \dots, v_{n-1}, u\}$ are linear independent, $\lambda^* = \lambda_n$ is the nth EV. □

How can we compute the eigenvalues?

13.1. **Min-Max principle for the $n$the eigenvalue.** Let $v_1, \dots, v_n$ be orthogonal eigenfunctions for the first n EV $\lambda_1, \dots, \lambda_n$. Assume $\|v_i\|_2 = 1$ and set

$$V = \text{ span } \{v_1, \dots, v_n\} \subset \mathcal{E}_0$$

that is linear space of dimension n. Assume (v_i), $i = 1, \dots, n$, are orthogonal.

If $w = \sum_{k=1}^n c_k v_k \in V$, then

$$E(w) = \sum_{l,k=1}^n c_k c_l \int_\Omega \langle \nabla v_k, \nabla v_l \rangle dx = \sum_{k=1}^n c_k^2 \int_\Omega |\nabla v_k|^2 dx = \sum_{k=1}^n c_k^2 \lambda_k.$$

Hence, on V the energy E is represented by the matrix $A = \text{Diag}(\lambda_1, \dots, \lambda_k)$, Precisely

$$E(w) = (c_1, \dots, c_n) A (c_1, \dots, c_n)^T \text{ for } w = \sum_{k=1}^n c_k v_k.$$

Linear Algebra tells us

$$\lambda_n = \max_{c \in \mathbb{R}^n} \frac{cAc^T}{|c|_2^2} = \max_{w \in V} \frac{E(w)}{\|w\|_2^2}.$$

Now, consider arbitrary $w_1, \dots, w_n \in \mathcal{E}_0$ that are linear independent and $W = \text{ span}\{w_1, \dots, w_n\}$.

We want to find $w = \sum_{k=1}^n x_k w_k \in W$ such that $(w, v_i) = 0$ $\forall i = 1, \dots, n-1$. For this consider the set of equations

$$(w, v_i) = \sum_{k=1}^n x_k (w_k, v_i), \ i = 1, \dots, n-1$$

or equivalently $A \cdot x^T = 0$ for the $n \times (n-1)$ matrix $A = ((w_k, v_i))_{k=1,\dots,n; i=1,\dots,n-1}$. Clearly, we always find a solution $x \in \mathbb{R}^n \backslash \{0\}$.

Hence, there exists $w \in W \backslash \{0\}$ such that

$$R(w) = \frac{E(w)}{\|w\|_2^2} \geq \lambda_n.$$

Hence

$$\max_{w \in W \backslash \{0\}} R(w) \geq \lambda_k$$

where $W = \text{span}(w_1, \dots, w_n)$. This proves the following theorem.

Theorem 13.3 (MinMax principle for EVs)**.** *The nth EV of $-\Delta$ on Ω with hom. Dir. BC is given by*

$$\lambda_n = \min_{w_1, \dots, w_n \in \mathcal{E}_0, \ lin.indep.} \left(\max_{w \in W} \frac{E(w)}{\|w\|_2^2} \right).$$

Ritz-Rayleigh approximation.

The MinMax principle is useful to approximate EVs. For instance, if we have that $v_1, \dots, v_n$ are linear independent in $\mathcal{E}_0$, then we define

$$a_{i,j} = \int_\Omega \langle \nabla v_i, \nabla v_j \rangle dx \implies A = (a_{i,j})_{i,j=1,\dots,n}$$
$$b_{i,j} = \int_\Omega v_i v_k dx \implies B = (b_{i,j})_{i,j=1,\dots,n}$$

For $w^l = \sum_{k=1}^l c_k^l v_k$ we write $R(w^l) = \lambda_l^* = \frac{c^l A (c^l)^T}{c^l B (c^l)^T} \geq \lambda_l$ where $c^l = (c_1^l, \dots, c_k^l, 0, \dots, 0) \in \mathbb{R}^n$.

$$\implies c^l A (c^l)^T - c^l B (c^l)^T = (c^l)\,(A - \lambda_l^* B)\,(c^l)^T = 0 \text{ with } c^l \neq 0 \implies \det(A - \lambda_l^* B) = 0$$

Hence, by solving $0 = P(\lambda) = \det(A - \lambda B)$ we find $\lambda_1^*, \dots, \lambda_n^*$ that approximate the first n EVs .

13.2. **Homogeneous Neumann Boundary conditions.** Consider

$$-\Delta u = \tilde{\lambda} u \text{ on } \Omega \text{ with } \frac{\partial u}{\partial N} = 0 \text{ on } \partial\Omega$$

where N is the unit normal vector along Ω.

In contrast to the Dirichlet problem $u = c \neq \mathbb{R}$ solves the equation for $\tilde{\lambda} = 0$. Hence, the smallest eigenvalue is $\tilde{\lambda}_1 = 0$.

The all previous considerations hold as well for the Neumann boundary problem where the set of trial functions is $\mathcal{E} = C^1(\overline{\Omega})$ instead of $\mathcal{E}_0$. For instance the minimization problem for the second eigenvalue is

$$\min \frac{E(w)}{\|w\|_2^2} \text{ where } w \in C^1(\overline{\Omega}) \text{ with } \int_\Omega w dx = 0.$$

Remark 13.4. Since there is no boundary condition for the trial functions one calls the Neumann boundary condition also free boundary condition

The minimization principle for higher eigenvalues and the MinMax principle are left as an exercise.

13.3. **Completeness.**

Theorem 13.5. *Let $(v_n)_{n\in\mathbb{N}}$ be a set of eigenfunctions (orthogonal) for EVs λ_n of $-\Delta$ on Ω with hom. Dir. BC. If f is L^2-integrable (square integrable) ($\int_\Omega f^2 dx < \infty$), then*

$$f = \sum_{n=1}^\infty c_n v_n \text{ in mean square sense, where } c_n = \frac{(f, v_n)}{(v_n, v_n)}$$

Proof. We assume $\lambda_n \to \infty$ for $n \to \infty$. We prove the theorem only for $f \in C^2(\overline{Omega})$ (as we did for Fourier series).

Let $r_N = f - \sum_{n=1}^\infty c_n v_n$. It follows

$$(r_N, v_j) = (f, v_j) - \sum_{n=1}^N c_n (v_n, v_j) = (f, v_j) - c_j (v_j, v_j) = 0$$

Hence

$$R(r_n) = \frac{E(r_N)}{\|r_N\|_2^2} \geq \lambda_{N+1}.$$

Then

$$E(\nabla r_n) = \int_\Omega |\nabla (f - \sum_{n=1}^N c_n v_n)|^2 dx = \int_\Omega \left(|\nabla f|^2 + 2 \sum_{n=1}^N c_n \langle \nabla f, \nabla v_n \rangle + \sum_{n,m=1}^n c_n c_m \langle \nabla v_n, \nabla v_m \rangle \right) dx$$

Since $v_n, f \in C^2(\Omega)$ it follows with Green's first identity that

$$\int_\Omega \langle \nabla f, \nabla v_n \rangle dx = -\int_\Omega f \Delta v_n dx = \lambda_n \int_\Omega f v_n dx$$

$$\int_\Omega \langle \nabla v_m, \nabla_n \rangle dx = -\int_\Omega v_m \Delta v_n = \lambda_n \int_\Omega v_m v_n dx = \delta_{m,n} \lambda_n \int_\Omega v_n^2 dx$$

Hence

$$E(r_N) = \int_\Omega \left(|\nabla f|^2 - 2\sum_{n=1}^N c_n \lambda_n (f, v_n) + \sum_{n=1}^N c_n^2 \lambda_n (v_n, v_n) \right) dx = \int_\Omega \left(|\nabla f|^2 - \sum_{n=1}^N c_n^2 \lambda_n (v_n, v_n) \right) dx \le E(\nabla f)$$

and therefore

$$\lambda_{N+1} \le \frac{E(\nabla r_n)}{\|r_N\|_2^2} \le \frac{E(\nabla f)}{\|\nabla r_n\|_2^2} \implies \|r_n\|_2^2 \le \frac{E(\nabla f)}{\lambda_{N+1}} \to 0.$$

□

Lecture 42

Let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of $-\Delta$ on Ω with homegeneous Dirichlet BC where $\Omega \subset \mathbb{R}^n$ is open and bounded with $\partial\Omega$ smooth. Let $(v_n)_{n\in\mathbb{N}}$ be the corresponding Eigenfunctions. In the follwing we always will assume that these eigenfunctions exist and are in $\mathcal{E}_0 \cap C^2(\overline{\Omega})$.

Minimum principle for the nth eigenvalue. We showed that if $u \in \mathcal{E}_0 \cap C^2(\overline{\Omega})$ satisfies

$$R(u) = \min R(w) \quad w \in \mathcal{E}_0\backslash\{0\}, \quad 0 = (w, v_i) \ \forall i = 1, \dots, n-1$$

then $u = v_n$ and $\lambda_n = R(u)$.

Now, let $w_1, \dots, w_{n-1}$ be arbitrary square integrable (L^2) functions on Ω. In the following we can assume that w_i is piecewiese continuous: $\exists \Omega_j, j = 1, \dots, l$ such that $\bigcup_{j=1}^{l} \overline{\Omega}_j = \overline{\Omega}$ such that $w_i|_{\Omega_j} \in C^0(\overline{\Omega}_j)$. Define

$$\lambda_n^* = \inf R(w), \ w \in \mathcal{E}_0\backslash\{0\}, \ (w, w_i) = 0 \ \forall i = 1, \dots, n-1.$$

Lemma 13.6. $\lambda_n^* \leq \lambda_n$

Proof. $\exists c_k$, $k = 1, \dots, n$ s.t. $w = \sum_{k=1}^n c_k v_k$ satisfies $(w, w_i) = \forall i = 1, \dots, n-1$. Such c_k can be found by solving a system of linear equations. Hence

$$\lambda_n^* \leq R(w) = \frac{E(w)}{\|w\|_2^2} = \frac{\sum_{k,l=1}^n c_k c_l \int_\Omega \langle \nabla v_k, \nabla v_l \rangle dx}{\sum_{k,l=1}^n c_k c_l \int_\Omega v_k v_l dx} = \frac{\sum_{k,l=1} c_k c_l \int_\Omega \Delta v_k v_l dx}{\sum_{k=1}^n c_k^2} = \frac{\sum_{k=1}^n c_k^2 \lambda_k}{\sum_{k=1}^n c_k} \leq \lambda_n$$

$\square$

Since the eigenfunctions $v_1, \dots, v_{n-1}$ are square integrable, we proved the following theorem.

Theorem 13.7 (Max-Min Principle)**.**

$$\lambda_n = \max_{w_1,\dots,w_{n-1}} \left(\inf_{w \in \mathcal{E}_0\backslash\{0\}, \ (w,w_i)=0 \ \ \forall i=1,\dots,n-1} R(w) \right)$$

Remark 13.8. An analog theorem holds for the eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \dots$ of the corresponding problem with homogeneous Neumann BC, but we have to replace $\mathcal{E}_0$ with $C^1(\overline{\Omega})\backslash\{0\}$.

Corollary 13.9. $\tilde{\lambda}_n \leq \lambda_n \ \forall n \in \mathbb{N}$.

Proof. Note that $\{w \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\} = \mathcal{E}_0 \subset \mathcal{E} = C^1(\overline{\Omega})$. Hence $\tilde{\lambda}_n^* \leq \lambda_n^* \Rightarrow \tilde{\lambda}_n \leq \lambda_n$. $\square$

Assymptotics of EVs. We want to show that $\lambda_n \to \infty$. This will follow from

Theorem 13.10 (Weyl)**.** *Let Ω be open and bounded with $\partial\Omega$ smooth.*

(1) *Assume $\Omega \subset \mathbb{R}^2$: Then* $\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{Area(\Omega)}$.

(2) *If $\Omega \subset R^3$, then* $\lim_{n\to\infty} \frac{\lambda_n^{\frac{3}{2}}}{n} = \frac{6\pi^2}{vol(\Omega)}$.

Before we begin with the proof let us explore the following examples.

Example 13.11. For $\Omega \subset \mathbb{R}$ (that is $\Omega = (0, L)$) we have the following. The Dirichlet eigenfunctions are $v_n(x) = \sin(n\frac{\pi}{L}x)$ with EVs $\lambda_n = \frac{n^2\pi^2}{L^2}$. Hence $\lim_{n\to\infty} \frac{\sqrt{\lambda_n}}{n} = \frac{\pi}{L}$.

Now we consider $\Omega = (0, a) \times (0, b)$. The Dirichlet eigenfunctions are $v_{n,m}(x, y) = v_n(x)v_n(y)$ with EVs $\lambda_{n,m} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b}$. We can relabel the EVs as an increasing sequence $\hat{\lambda}_n$, $n \in \mathbb{N}$.

Define the *counting function* N as

$$N(\lambda) = \text{ The number of EVs } \lambda_n \leq \lambda.$$

Hence, $N(\lambda_n) = n$.

Consider $\lambda_{n,m} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} = \frac{n^2}{(\frac{a}{\pi})^2} + \frac{m^2}{(\frac{b}{\pi})^2} \leq \lambda$. How many such $\lambda_{n,m}$ are there?

Consider the ellipse $E = \{(x,y) \in \mathbb{R}^2 : \frac{n^2}{(\frac{a}{\pi})^2} + \frac{m^2}{(\frac{b}{\pi})^2} \le \lambda\}$. Then, we observe that

$$N(\lambda) \le \frac{\text{Area}(E)}{4} = \frac{1}{4}\pi \frac{a\sqrt{\lambda}}{\pi}\frac{b\sqrt{\lambda}}{\pi} = \frac{ab\lambda}{4\pi}.$$

Similar

$$N(\lambda) \ge \frac{\text{Area}(E)}{4} - C \cdot \text{Length}(\partial E) = \frac{ab\lambda}{4\pi} - C\sqrt{\lambda}.$$

Hence

$$\frac{ab\lambda}{4\pi} - c\sqrt{\lambda} \le N(\lambda) \le \frac{ab\lambda}{4\pi}$$

and therefore

$$\frac{ab}{4\pi} - \frac{C}{\sqrt{\lambda}} \le \frac{N(\lambda)}{\lambda} \le \frac{ab}{4\pi} \;\Rightarrow\; \frac{\lambda}{N(\lambda)} \to \frac{4\pi}{ab} \text{ if } \lambda \to \infty.$$

Since $N(\hat{\lambda}_k) = k$, the claim follows.

Lecture 43
Goal:

Theorem 13.12 (Weyl)**.** *Let $\Omega \subset \mathbb{R}^2$ be open and bounded, $\partial\Omega$ smooth. Let $(\lambda_n)_{n\in\mathbb{N}}$ be the increasing sequence of Dirichlet eigenvalues. Then*

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{Area(\Omega)}$$

In particular, $\lambda_n \to \infty$.

Lemma 13.13. *Let $\Omega_1 \subset \Omega_2$. Then*
(1) $\lambda_n^{\Omega_2} \le \lambda_n^{\Omega_1}$
(2) $\tilde{\lambda}_n^{\Omega_1} \le \tilde{\lambda}_n^{\Omega_2}$
$\forall n \in \mathbb{N}$.

Proof. (1) Let $w_1, \dots, w_{n-1}$ be square integrable ($\int_{\Omega_2} w_i^2 dx < \infty$) and piecewise continuous. Then

$$\lambda_n^{*,\Omega_2} = \min R(w), \quad w \in \mathcal{E}_0(\Omega_2),\ 0 = (w, w_i)\ \forall i = 1, \dots, n-1.$$

Let $w \in \mathcal{E}_0(\Omega_1)$.

We use the following fact: there exsits a sequence of $w^\alpha \in C_c^1(\Omega_1)$, $\alpha \in \mathbb{N}$, such that $R(w_\alpha) \to R(w)$. From this we can see that we can replace the set of trial functions $\mathcal{E}_0(\Omega_1)$ in the definition of λ_n^{*,Ω_1} with $C_c^1(\Omega)$.

Now, if $w \in C_c^1(\Omega_1)$, then $w \in \mathcal{E}_0(\Omega_2)$. Moreover $(w, w_i) = 0\ \forall i = 1, \dots, n-1$. In particular, if $\tilde{w}_i = w_i|_{\Omega_1}$, then w_i is square integrable and $(w, \tilde{w}_i) = 0\ \forall i = 1, \dots, n-1$.

Hence, the set of trial functions for λ_n^{*,Ω_1} is contained in the set of trial functions for λ_n^{*,Ω_2}. Therefore

$$\lambda_n^{*,\Omega_2} \le \lambda_n^{*,\Omega_1}.$$

Taking the maximum w.r..t square integrable functions $w_1, \dots, w_{n-1}$ on Ω_2 yields $\lambda_n^{\Omega_2} \le \lambda_n^{\Omega_1}$.

(2) For the Neumann case we first observe

$$\tilde{\lambda}_n^{*,\Omega_1} = \inf R(w), \quad w \in \mathcal{E}(\Omega_1),\ 0 = (w, w_i)\ \forall i = 1, \dots, n-1.$$

where $w_1, \dots, w_{n-1}$ are square integrable functions on Ω_1. We define an extension $\tilde{w}_i$ of each w_i to Ω_2 by setting $\tilde{w}_i \equiv 0$ on $\Omega_2 \backslash \Omega_1$. Then $\tilde{w}_1, \dots, \tilde{w}_{n-1}$ are sqare integrable functions on Ω_2. Hence, for

$$\tilde{\lambda}_n^{*,\Omega_2} = \inf R(w), \quad w \in \mathcal{E}(\Omega_2),\ 0 = (w, \tilde{w}_i) = (w, w_i)\ \forall i = 1, \dots, n-1.$$

we have $\lambda_n^{*,\Omega_1} \le \lambda_n^{*,\Omega_2}$ since any $w \in \mathcal{E}(\Omega_2)$ yields $w|_{\Omega_1} \in \mathcal{E}(\Omega_1)$ (all the trial functions on Ω_2 yield a trial function on Ω_1, and hence the infinimum that defines $\lambda_n^{\Omega_1}$ is smaller than the minimum that defines $\lambda_n^{\Omega_2}$.

Taking the maximum w.r.t. all square integrable function $w_1, \dots, w_{n-1}$ on Ω_1 yields $\tilde{\lambda}_n^{\Omega_1} \le \lambda_n^{\Omega_2}$. □

Remark 13.14. Consider $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. If $v_n^{\Omega_1}$ and $v_m^{\Omega_2}$ are linear independent Dirichlet eigenfunctions of Ω_1 and Ω_2 respectively (with EVs $\lambda_n^{\Omega_1}$ and $\lambda_m^{\Omega_2}$), then $\{v_n^{\Omega_1}, v_m^{\Omega_2}\}_{n,m\in\mathbb{N}^2}$ are Dirichlet eigenfunctions of $\Omega = \Omega_1 \cup \Omega_2$ where, for instance, one sets $v_n^{\Omega_1} \equiv 0$ on Ω_2.

Proof of Weyl's theorem. **1st step.** Let $\underline{\Omega} = \bigcup Q_j \subset \Omega$ with $Q_j \equiv (0, a_j) \times (0, b_j)$ disjoint cubes in Ω. Then the previous lemma yields that $\lambda_n^{\underline{\Omega}} \ge \lambda_n^{\Omega}$. This implies for the counting functions $N(\lambda), N^{\underline{\Omega}}(\lambda)$ (the number of eigenvalue smaller than λ) that

$$N^{\underline{\Omega}}(\lambda) \le N(\lambda)$$

What is the counting function $N^{\underline{\Omega}}$ of $\underline{\Omega}$. By the remark we have

$$N^{\underline{\Omega}}(\lambda) = \sum_{j=1}^{l} N^j(\lambda)$$

where N^j is the counting function of Q_j.

Moreover, we already computed that $\frac{N^j(\lambda)}{\lambda} \to \frac{a_j b_j}{4\pi}$. Hence

$$\liminf_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} \geq \sum_{j=1}^{l} \frac{N^j(\lambda)}{\lambda} = \sum_{j=1}^{l} \frac{\text{Area}(Q_j)}{4\pi} = \frac{\text{Area}(\underline{\Omega})}{4\pi}.$$

2nd step. Now, consider $Q_j \equiv (0, a_j) \times (0, b_j)$ such that Q_j are disjoint, but $\Omega \subset \bigcup_{j=1}^{l} \overline{Q}_j$.

If $\tilde{v}_n^j$ are the Neumann eigenfunctions of Q_j, then $\bigcup_{j=1}^{l}\{v_n^j : n \in \mathbb{N}\}$ are the Neumann eigenfunctions of $\bigcup_{j=1}^{l} Q_j = Q$ (with corresponding EVs λ_n^j). We reorder these EVs and rename them as μ_n, $n \in \mathbb{N}$, such that μ_n is an increasing sequence. Let w_n be the corresponding eigenfunctions.

On the other hand, we can consider $\hat{\Omega} = \left(\bigcup_{j=1}^{l} \overline{Q}\right)^{\circ} = \left(\overline{Q}\right)^{\circ}$ ($(\cdot)^{\circ}$ gives the open interior) and $Q \subset \hat{\Omega}$.

Hence, by (2) in the previous Lemma we have

$$\mu_n \leq \tilde{\lambda}_n^{\hat{\Omega}}.$$

As in the first step, one can see that $\tilde{N}^Q(\lambda) = \sum_{j=1}^{l} \tilde{N}^j(\lambda)$ where $\tilde{N}^Q$ and $\tilde{N}^j$ are the counting functions for the Neumann eigenvalues of Q and Q_j respectively. Hence

$$\tilde{N}^{\hat{\Omega}}(\lambda) \leq \tilde{N}^Q(\lambda) = \sum_{j=1}^{l} \tilde{N}^j(\lambda)$$

Therefore

$$\limsup_{\lambda\to\infty} \frac{\tilde{N}^{\hat{\Omega}}(\lambda)}{\lambda} \leq \sum_{j=1}^{l} \frac{\tilde{N}^j(\lambda)}{\lambda} \to \frac{\text{Area}(Q)}{4\pi}$$

Combining the previous two step together with the fact that the Neumann EVs are smaller than the Dirichlet EVs gives

$$\frac{\text{Area}(\hat{\Omega})}{4\pi} \geq \limsup \frac{\tilde{N}^{\hat{\Omega}}(\lambda)}{\lambda} \geq \liminf \frac{N^{\tilde{\Omega}}(\lambda)}{\lambda} \geq \frac{\text{Area}(\underline{\Omega})}{4\pi}.$$

Since we can approximate Ω from the inside and from the outside by unions of squares Q_j such that the area converges to the area of Ω, this proves the theorem. □

14. Fourier Transform

The Fourier series is useful concepts whenever we have periodic boundary conditions, for instance after a change to spherical coordinates.

Let $f : \mathbb{R} \to \mathbb{R}$ be a $2l$ periodic, continuous function. Its Fourier series (the complex version) was defines as

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi\frac{x}{l}} = \mathcal{F}(f)$$

and $\mathcal{F}(f)$ converges uniformily to f where the coefficients are $c_n = \frac{1}{2l}\int_{-l}^{l} f(y) e^{-in\pi\frac{y}{l}} dy$.

Question: Can we drop the $2l$ periodicity somehow? Or can we let $l \to \infty$?

Let's write

$$f(x) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\int_{-l}^{l} f(y) e^{-i\pi y \frac{n}{l}} dy \right) e^{i\pi x \frac{n}{l}} \frac{1}{l} = \dagger$$

If $l \to \infty$, then formally we expect

$$f(x) = \dagger \text{ "} \to \text{" } \frac{1}{2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-i\pi y z} dy \right) e^{i\pi x z} dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-iyz} dy \right) e^{ixz} dz$$

where the last inequality is change of variables w.r.t. z.

Although this limit is not justified yet, we can make the following definition:

Definition 14.1. Let $f : \mathbb{R} \to \mathbb{C}$ be integrable. Its Fourier transform is defined as

$$F(\xi) := \hat{f}(\xi) := \int_{-\infty}^{\infty} f(y) e^{-i\xi y} dy.$$

Remark 14.2. Note that $y \in \mathbb{R} \mapsto f(y)e^{-i\xi y} \in \mathbb{C}$ is complex valued, and hence the integral is understood as a complex valued integral.

Since

$$\left| \int_{-\infty}^{\infty} f(y) e^{-i\xi y} dy \right| \leq \int_{-\infty}^{\infty} \left| f(y) e^{-i\xi y} \right| dy \leq \int_{-\infty}^{\infty} |f(y)| \left| e^{-i\xi y} \right| dy = \int_{-\infty}^{\infty} |f(y)| dy < \infty$$

the Fourier transform for an integrable f (that ist $\int_{-\infty}^{\infty} |f(y)| dy < \infty$) is well-defined.

We would like to justify a formula of the form

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} \frac{dx}{2\pi}.$$

However $\hat{f}$ might not be integrable no more.

To circumvent this problem one introduces the Schwartz space $\mathcal{S}(\mathbb{R})$.

Definition 14.3. A funciton $f \in C^{\infty}(\mathbb{R}, \mathbb{C})$ is in the Schwartz space $\mathcal{S}(\mathbb{R})$ if f is rapidly decreasing in the sense that

$$\sup_{x \in \mathbb{R}} |x|^k \left| \frac{d^l}{dx^l} f(x) \right| < \infty \ \forall k, l \in \mathbb{N}.$$

One can check that $\mathcal{S}$ is a vector space, and $\frac{d^l}{dx^l} f \in \mathcal{S}(\mathbb{R})$ for all $l \in \mathbb{N}$ and $xf(x) \in \mathcal{S}(\mathbb{R})$ for all $f \in \mathcal{S}(\mathbb{R})$. In particular $\mathcal{S}(\mathbb{R})$ is closed under multiplication with polynomials.

A simple example for a function that is in $\mathcal{S}(\mathbb{R})$ is e^{-x^2}.

Proposition 14.4. *Let $f \in \mathcal{S}(\mathbb{R})$. The following properties hold.*

(1) *$f \mapsto \hat{f}$ is a linear operator,*
(2) *$\frac{d}{dx} f \mapsto i\xi \hat{f}(\xi)$,*
(3) *$xf(x) \mapsto i \frac{d}{d\xi} \hat{f}$*
(4) *$f(x-a) \mapsto e^{-ia\xi} \hat{f}(\xi)$,*
(5) *$e^{iax} f(x) \mapsto \hat{f}(\xi - a)$,*
(6) *$f(ax) \mapsto \frac{1}{|a|} \hat{f}(\xi/a)$.*

Proof. Let us prove (2). With integration by parts we compute

$$\int_{-N}^{N} f'(x) e^{-i\xi x} dx = f(x) e^{-i\xi x} \Big|_{-N}^{N} + \int_{-N}^{N} f(x) i\xi e^{-i\xi x} dx$$

Since f rapidly decays, it follows when $N \to \infty$

$$\left(\frac{d}{dx} f(\xi) \right)^{\wedge} = i\xi \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx = i\xi \hat{f}(\xi).$$

For the proof of (3) consider

$$\frac{i}{h}\left(\hat{f}(\xi+h)-\hat{f}(\xi)\right)-(xf(x))^{\wedge}=\int_{-\infty}^{\infty}f(x)e^{-i\xi x}\left[i\frac{e^{-ixh}-1}{h}-x\right]dx$$

The integral on the left can be split into a part $\int_{|x|\geq N}$ and a part $\int_{-N}^{N}$. Since $f(x)$ and $xf(x)$ rapidly decrease and since
$\left|i\frac{e^{-ixh}-1}{h}\right|\leq 1$, we can estimate the first integral by $C\epsilon$ for a constant that does not depende on h, provided N is sufficiently large.

For the second integral we note that $\left|i\frac{e^{-ixh}-1}{h}-x\right|\leq\epsilon$ for h sufficiently small. Hence, we obtain that

$$\left|\frac{1}{h}\left(\hat{f}(\xi+h)-\hat{f}(\xi)\right)-(ixf(x))^{\wedge}\right|\leq C\epsilon$$

for h sufficiently small. Since $\epsilon>0$ is arbitrary, (3) follows. $\square$

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Corollary 14.5. *If $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$.*

Proof. By the previous propositon we have

$$g(k) = (i^{t+s} k^t \frac{d^s}{dk^s} \hat{f}(k) = \left(\frac{d^t}{dx^t} \left(x^s f(x) \right) \right)^{\wedge} (k)$$

Then

$$|g(k)| \leq \int_{-\infty}^{\infty} \left| \frac{d^t}{dx^t} \left(x^s f(x) \right) \right| dx < \infty$$

since $f \in \mathcal{S}(\mathbb{R})$. □

Example 14.6. (1) The Fourier transform of $f(x) = e^{-x^2/2}$ is $\hat{f}(k) = \sqrt{2\pi} e^{-\frac{k^2}{2}}$.
Indeed we compute

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ixk} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ik)^2} dx e^{-\frac{1}{2}k^2} = \sqrt{2\pi} e^{-\frac{k^2}{2}}$$

where we used $-\frac{1}{2}\left(x^2 + ikx\right) + \frac{1}{2}k^2 - \frac{1}{2}k^2 = -\frac{1}{2}\left(x + ik\right)^2 - \frac{1}{2}k^2$.

(2) The Fourier transform of $f(x) = e^{-a|x|}$ is $\hat{f}(k) = \frac{2a}{a^2+k^2}$ $a > 0$. However note that f is not in $\mathcal{S}(\mathbb{R})$ since it is not differentiable in $x = 0$.

Corollary 14.7. *If $f(x) = e^{-x^2/2}$, then $\hat{f}(\xi) = \sqrt{2\pi} e^{-\xi^2/2}$.*

Theorem 14.8. *Let $f \in \mathcal{S}(\mathbb{R})$.*

(1) *Plancherel forumal:* $\int_{-\infty}^{\infty} (f(x))^2 dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 d\xi / 2\pi$

(2) *Inversion formula:* $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} dx =: \left(\hat{f}(k) \right)^{\vee} (x)$

(3) *Convolution rule:* $(f \star g)^{\wedge} = \hat{f} \cdot \hat{g}$.

14.1. **Tempered Distribution.** Recall that $\mathcal{F} : C_c^{\infty}(\mathbb{R}) \to \mathbb{R}$ is called distribution if

(1) $\mathcal{F}$ is linear,

(2) $\mathcal{F}(f_n) \to \mathcal{F}(f)$ if $\frac{d^t}{dx^t} f_n \to \frac{d^t}{dx^t} f$ uniformily $\forall t \in \mathbb{N}$.

Definition 14.9. If we replace $C_c^{\infty}(\mathbb{R})$ with $\mathcal{S}(\mathbb{R})$ in the definition of a distribution $\mathcal{F}$, then we call $\mathcal{F}$ a tempered distribution.

Definition 14.10 (Fourier transform of tempered distribution)**.** The Fourier transform of a tempered distribution $\mathcal{F}$ is again a tempered distribution $\hat{\mathcal{F}}$ define as

$$\hat{\mathcal{F}}(\phi) = \mathcal{F}(\hat{\phi}) \ \forall \phi \in \mathcal{S}(\mathbb{R})$$

Example 14.11. (1) $\mathcal{F}(\phi) = \int_{\mathbb{R}} f \phi dx$ with f integrable. Then

$$\hat{\mathcal{F}}(\phi) = \mathcal{F}(\hat{\phi}) = \int \int \phi(x) e^{-ikx} dx f(f) dk = \int \int f(k) e^{-ikx} dk \phi(x) dx = \int \hat{f}(x) \phi(x) dx.$$

Hence $\hat{\mathcal{F}} \sim \hat{f}$.

(2) $\mathcal{F}(\phi) = \delta_x(\phi) = \phi(x)$. Then

$$\hat{\delta_x}(\phi) = \delta_x(\hat{\phi}) = \hat{\phi}(x) = \int \phi(y) e^{-iyx} dy$$

Hence $\hat{\delta_x} \sim e^{-ixy}$. In particular $\hat{\delta_0} \sim 1$.

(3) $\mathcal{F}(\phi) = \int \phi(x) dx$. Note that $f(x) = 1$ is not integrable.
Define $f_n = e^{-\frac{|x|}{n}}$. Then $f_n \sim \mathcal{F}_n \to \mathcal{F} \sim 1$. Hence

$$\hat{\mathcal{F}_n} \sim \hat{f}_n(k) = \frac{\frac{2}{n}}{\frac{1}{n^2} + k} \to \delta_0 \sim \hat{\mathcal{F}}.$$

14.2. Application of Fourier transformation to PDEs.

(1) Fundamental solution of the diffusion equation on $\mathbb{R}$:

$$S_t = S_{x,x} \quad \text{on } \mathbb{R} \times [0, \infty),$$
$$\text{“}S(x,0) = \delta_0 \quad \text{on } \mathbb{R}''$$

After Fourier transformation the PDE becomes

$$\hat{S}_t(k,t) = (ik)^2 \hat{S}(k,t) = -k^2 S(k,t) \quad \text{on } \mathbb{R} \times (0,\infty)$$
$$\hat{S}(k,0) = 1 \quad \text{on } \mathbb{R}$$

The soluiton of this ODE is

$$\hat{S}(k,t) = e^{-k^2 t}$$

Then the previous proposition easily gives $S(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$.

(2) Fundamental solution of the Laplace equation on the half plan $\{(x,y) \in \mathbb{R}^2 : y > 0\} =: \mathbb{H}$. Consider

$$\Delta u = u_{x,x} + u_{y,y} = 0 \quad \text{on } \mathbb{H}$$
$$\text{“}u(x,0) = \delta_0''$$

After a Fourier transformation w.r.t. x the PDE becomes

$$-k^2 \hat{u}(k,y) + \hat{u}_{y,y}(k,y) = 0 \quad \text{on } \mathbb{H}$$
$$\hat{u}(k,0) = 1 \quad \text{on } \mathbb{R}$$

Solutions are given by

$$\hat{u}(k,y) = e^{\pm yk}.$$

We dismiss the solution that will not tend to 0 for $|k| \to \infty$. So $\hat{u}(k,y) = e^{-|k|y}$. For this we compute

$$\begin{aligned}(\hat{u})^\vee(x,y) &= \frac{1}{2\pi}\int e^{ikx} e^{-y|k|} dk \\ &= \frac{1}{2\pi}\int_0^\infty e^{k(ix-y)} dk + \frac{1}{2\pi}\int_{-\infty}^0 e^{k(ix+y)} dk \\ &= \frac{1}{2\pi}\left(\frac{1}{ix-y} e^{k(ix-y}\right)\Big|_0^\infty + \frac{1}{2\pi}\frac{1}{ix-y} e^{(ix+y)k}\Big|_{-\infty}^0 = \frac{y}{\pi(x^2+y^2)}\end{aligned}$$

this is exactly the Poisson kernel of $\mathbb{H}$ that we have already computed before.