WEAKLY NONCOLLAPSED AND COLLAPSED RCD SPACES WITH UPPER CURVATURE BOUNDS

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Abstract. We show that if a $CD(K,n)$ space $(X,d,f\mathcal{H}_n)$ with $n \geq 2$ has curvature bounded above by κ in the sense of Alexandrov then $f = const$.

Contents

1. Introduction

In [DPG18] Gigli and De Philippis introduced the following notion of a *noncollapsed* $RCD(K,n)$ space. An $RCD(K,n)$ space (X,d,m) is noncollapsed if n is a natural number and $m = \mathcal{H}_n$. A similar notion was considered by Kitabeppu in [Kit17].

Noncollapsed $RCD(K,n)$ give a natural intrinsic generalization of noncollapsing limits of manifolds with lower Ricci curvature bounds which are noncollapsed in the above sense by work of Cheeger–Colding [CC97].

In [DPG18] Gigli and De Philippis also considered the following a-priori weaker notion. An $RCD(K,n)$ space (X,d,m) is *weakly noncollapsed* if n is a natural number and $m \ll \mathcal{H}_n$. Gigli and De Philippis gave several equivalent characterizations of weakly noncollapsed $RCD(K,n)$ spaces and studied their properties. By work of Gigli–Pasqualetto [GP16], Mondino–Kell [KM18], De Philippis-Marchese-Rindler [DPMR17] and Brué–Semola [BS18] it follows that an $RCD(K,n)$ space is weakly noncollapsed iff $\mathcal{R}_n \neq \emptyset$ where $\mathcal{R}_n$ is the rectifiable set of n-regular points in X.

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It is well-known that if $(X,d,m)=(M^n,g,e^{-f}d\,\mathrm{vol}_g)$ where (M^n,g) is a smooth n-dimensional Riemannian manifold and f is a smooth function on M then (X,d,m) is $RCD(K,n)$ iff $f=\mathrm{const}$. More precisely, the classical Bakry-Emery condition $BE(K,N)$, $K\in\mathbb{R}$ and $N\geq n$, for a (compact) smooth metric measure space $(M^n,g,e^{-f}d\,\mathrm{vol}_g)$, $f\in C^\infty(M)$, is

$$\frac{1}{2}L|\nabla u|_g^2\geq\langle\nabla Lu,\nabla u\rangle_g+\frac{1}{N}(Lu)^2+K|\nabla u|_g^2,\quad\forall u\in C^\infty(M),$$

where $L=\Delta-\nabla f$. In [Bak94, Proposition 6.2] Bakry shows that $BE(K,N)$ holds if and only if

$$\nabla f\otimes\nabla f\leq(N-n)\left(\mathrm{ric}_g+\nabla^2 f-Kg\right).$$

In particular, if $N=n$, then f is locally constant.

On the other hand, in [EKS15, AGS15] it was proven that a metric measure space (X,d,m) satisfies $RCD(K,N)$ if and only if the corresponding Cheeger energy satifies a weak version of $BE(K,N)$ that is equivalent to the classical version for $(M,g,e^{-f}\,\mathrm{vol}_g)$ from above.

In [DPG18] Gigli and De Philippis conjectured that a weakly noncollapsed $RCD(K,n)$ space is already noncollapsed up to rescaling of the measure by a constant. Our main result is that this conjecture holds if a weakly noncollapsed space has curvature bounded above in the sense of Alexandrov.

Theorem 1.1. *Let $n\geq 2$ and let $(X,d,f\mathcal{H}_n)$ (where f is L^1_{loc} with respect to $\mathcal{H}_n$ and $\mathrm{supp}(f\mathcal{H}^n)=X$) be a complete metric measure space which is $CBA(\kappa)$ (has curvature bounded above by κ in the sense of Alexandrov) and satisfies $CD(K,n)$. Then $f=\mathrm{const}$* [1] *.*

Since smooth Riemannian manifolds locally have curvature bounded above this immediately implies

Corollary 1.2. *Let (M^n,g) be a smooth Riemannian manifold and suppose $(M^n,g,f\mathcal{H}_n)$ is $CD(K,n)$ where K is finite and $f\geq 0$ is L^1_{loc} with respect to $\mathcal{H}_n$ and $\mathrm{supp}(f\mathcal{H}^n)=M$. Then $f=\mathrm{const}$.*

As was mentioned above, this corollary was well-known in case of smooth f but was not known in case of general locally integrable f.

In [KK18] it was shown that if a (X,d,m) is $CD(K,n)$ and has curvature bounded above then X is $RCD(K,n)$ and if in addition $m=\mathcal{H}_n$ then X is Alexandrov with two sided curvature bounds. Combined with Theorem 1.1 this implies that the same remains true if the assumption on the measure is weakened to $m\ll\mathcal{H}_n$.

Corollary 1.3. *Let $n\geq 2$ and let $(X,d,f\mathcal{H}_n)$ where f is L^1_{loc} with respect to $\mathcal{H}_n$ and $\mathrm{supp}(f\mathcal{H}^n)=X$ be a complete metric measure space which is $CBA(\kappa)$ (has curvature bounded above by κ in the sense of Alexandrov) and satisfies $CD(K,n)$. Then X is $RCD(K,n)$, $f=\mathrm{const}$, $\kappa(n-1)\geq K$, and (X,d) is an Alexandrov space of curvature bounded below by $K-\kappa(n-2)$.*

Remark 1.4. Note that since a space $(X,d,f\mathcal{H}_n)$ satisfying the assumptions of Theorem 1.1 is automatically $RCD(K,n)$, as was remarked in [DPG18] it follows from the results of [KM18] that n must be an integer.

Bakry's proof for smooth manifolds does not easily generalize to a non-smooth context. But let us describe a strategy that does generalize to $RCD+CAT$ spaces.

Assume that (X,d) is induced by a smooth manifold (M^n,g) and the density function f is smooth and positive such that (X,d,fm) satisfies $RCD(K,n)$. Then, by integration by parts on (M,g) the induced Laplace operator L is given by

$$Lu=\Delta u-\langle\nabla\log f,\nabla u\rangle,\quad u\in C^\infty(M),\tag{1}$$

where Δu is the classical Laplace-Beltrami operator of (M,g) for smooth functions. By a recent result of Han one has for any $RCD(K,n)$ space that the operator L is equal to the trace of

[1] Here and in all applications by $f=\mathrm{const}$ we mean $f=\mathrm{const}$ a.e. with respect to $\mathcal{H}_n$.

Gigli's Hessian [Gig18] on the set of n-regular points $\mathcal{R}_n$. Hence, after one identifies the trace of Gigli's Hessian with the Laplace-Beltrami operator Δ of M (what is true on (M^n, g)), one obtains immediately that $\nabla \log f = 0$. If M is connected, this yields the claim.

The advantage of this approach is that it does not involve the Ricci curvature tensor and in non-smooth context one might follow the same strategy. However, we have to overcome several difficulties that arise from the non-smoothness of the density function f and of the space (X, d, m).

In particular, we apply the recently developed DC-calculus by Lytchak-Nagano for spaces with upper curvature bounds to show that on the regular part of X the Laplace operator with respect to $\mathcal{H}_n$ is equal to the trace of the Hessian. We also show that the combination of CD and CAT condition implies that f is locally semiconcave [KK18] and hence locally Lipschitz on the regular part of X. This allows us to generalize the above argument for smooth Riemannian manifolds to the general case.

In section 2 we provide necessary preliminaries. We present the setting of RCD spaces and the calculus for them. We state important results by Mondino-Cavalletti (Theorem 2.4), Han (Theorem 2.11) and Gigli (Theorem 2.7, Proposition 2.9). We also give a brief introduction to the calculus of BV and DC function for spaces with upper curvature bounds.

In section 3 we develop a structure theory for general $RCD + CAT$ spaces where we adapt the DC-calculus of Lytchak-Nagano [LN18]. This might be of independent interest.

Finally, in section 4 we prove our main theorem following the above idea.

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2. Preliminaries

2.1. **Curvature-dimension condition.** A *metric measure space* is a triple (X, d, m) where (X, d) is a complete and separable metric space and m is a locally finite measure.

$\mathcal{P}^2(X)$ denotes the set of Borel probability measures μ on (X, d) such that $\int_X d(x_0, x)^2 d\mu(x) < \infty$ for some $x_0 \in X$ equipped with L^2-Wasserstein distance W_2. The sub-space of m-absolutely continuous probability measures in $\mathcal{P}^2(X)$ is denoted by $\mathcal{P}^2(X, m)$.

The N-*Renyi entropy* is

$$S_N(\cdot|m) : \mathcal{P}_b^2(X) \to (-\infty, 0], \quad S_N(\mu|m) = -\int \rho^{1-\frac{1}{N}} dm \ \text{ if } \mu = \rho m, \text{ and 0 otherwise.}$$

S_N is lower semi-continuous, and $S_N(\mu) \geq -m(\operatorname{supp}\mu)^{\frac{1}{N}}$ by Jensen's inequality.

For $\kappa \in \mathbb{R}$ we define

$$\cos_\kappa(x) = \begin{cases} \cosh(\sqrt{|\kappa|}x) & \text{if } \kappa < 0 \\ 1 & \text{if } \kappa = 0 \\ \cos(\sqrt{\kappa}x) & \text{if } \kappa > 0 \end{cases} \quad \sin_\kappa(x) = \begin{cases} \frac{\sinh(\sqrt{|\kappa|}x)}{\sqrt{|\kappa|}} & \text{if } \kappa < 0 \\ x & \text{if } \kappa = 0 \\ \frac{\sin(\sqrt{\kappa}x)}{\sqrt{\kappa}} & \text{if } \kappa > 0 \end{cases}$$

Let π_κ be the diameter of a simply connected space form $\mathbb{S}^2_\kappa$ of constant curvature κ, i.e.

$$\pi_\kappa = \begin{cases} \infty & \text{if } \kappa \leq 0 \\ \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0 \end{cases}$$

For $K \in \mathbb{R}$, $N \in (0, \infty)$ and $\theta \geq 0$ we define the *distortion coefficient* as

$$t \in [0, 1] \mapsto \sigma^{(t)}_{K,N}(\theta) = \begin{cases} \frac{\sin_{K/N}(t\theta)}{\sin_{K/N}(\theta)} & \text{if } \theta \in [0, \pi_{K/N}), \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\sigma^{(t)}_{K,N}(0) = t$. For $K \in \mathbb{R}$, $N \in [1,\infty)$ and $\theta \geq 0$ the *modified distortion coefficient* is

$$t \in [0,1] \mapsto \tau^{(t)}_{K,N}(\theta) = \begin{cases} \theta \cdot \infty & \text{if } K > 0 \text{ and } N = 1, \\ t^{\frac{1}{N}} \left[\sigma^{(t)}_{K,N-1}(\theta)\right]^{1-\frac{1}{N}} & \text{otherwise.} \end{cases}$$

Definition 2.1 ([Stu06, LV09, BS10])**.** We say (X,d,m) satisfies the *curvature-dimension condition* $CD(K,N)$ for $K \in \mathbb{R}$ and $N \in [1,\infty)$ if for every $\mu_0, \mu_1 \in \mathcal{P}^2_b(X,m)$ there exists an L^2-Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ and an optimal coupling π between μ_0 and μ_1 such that

$$S_N(\mu_t|m) \leq -\int \left[\tau^{(1-t)}_{K,N}(d(x,y))\rho_0(x)^{-\frac{1}{N}} + \tau^{(t)}_{K,N}(d(x,y))\rho_1(y)^{-\frac{1}{N}}\right] d\pi(x,y)$$

where $\mu_i = \rho_i dm$, $i = 0,1$.

Remark 2.2. If (X,d,m) is complete and satisfies the condition $CD(K,N)$ for $N < \infty$, then $(\operatorname{supp} m, d)$ is a geodesic space and $(\operatorname{supp} m, d, m)$ is $CD(K,N)$.

In the following we always assume that $\operatorname{supp} m = X$.

Remark 2.3. For the variants $CD^*(K,N)$ and $CD^e(K,N)$ of the curvature-dimension condition we refer to [BS10, EKS15].

2.2. **Calculus on metric measure spaces.** For further details about this section we refer to [AGS13, AGS14a, AGS14b, Gig15].

Let (X,d,m) be a metric measure space, and let $\mathrm{Lip}(X)$ be the space of Lipschitz functions. For $f \in \mathrm{Lip}(X)$ the local slope is

$$\mathrm{Lip}(f)(x) = \limsup_{y\to x} \frac{|f(x)-f(y)|}{d(x,y)}, \quad x \in X.$$

If $f \in L^2(m)$, a function $g \in L^2(m)$ is called *relaxed gradient* if there exists sequence of Lipschitz functions f_n which L^2-converges to f, and there exists h such that $\mathrm{Lip} f_n$ weakly converges to h in $L^2(m)$ and $h \leq g$ m-a.e. . $g \in L^2(m)$ is called the *minimal relaxed gradient* of f and denoted by $|\nabla f|$ if it is a relaxed gradient and minimal w.r.t. the L^2-norm amongst all relaxed gradients. The space of L^2-*Sobolev functions* is then

$$W^{1,2}(X) := D(\mathrm{Ch}^X) := \left\{ f \in L^2(m) : \int |\nabla f|^2 dm < \infty \right\}.$$

$W^{1,2}(X)$ equipped with the norm $\|f\|^2_{W^{1,2}(X)} = \|f\|^2_{L^2} + \||\nabla f|\|^2_{L^2}$ is a Banach space. If $W^{1,2}(X)$ is a Hilbert space, we say (X,d,m) is *infinitesimally Hilbertian.* In this case we can define

$$(f,g) \in W^{1,2}(X)^2 \mapsto \langle \nabla f, \nabla g\rangle := \frac{1}{4}|\nabla(f+g)|^2 - \frac{1}{4}|\nabla(f-g)|^2 \in L^1(m).$$

Assuming X is locally compact, if U is an open subset of X, we say $f \in W^{1,2}(X)$ is in the domain $D(\mathbf{\Delta}, U)$ of the *measure valued Laplace* $\mathbf{\Delta}$ on U if there exists a signed Radon functional $\mathbf{\Delta} f$ on the set of Lipschitz function g with bounded support in U such that

$$\int \langle \nabla g, \nabla f\rangle dm = -\int g d\mathbf{\Delta} f. \tag{2}$$

If $U = X$ and $\mathbf{\Delta} f = [\mathbf{\Delta} f]_{ac} m$ with $[\mathbf{\Delta} f]_{ac} \in L^2(m)$, we write $[\mathbf{\Delta} f]_{ac} =: \Delta f$ and $D(\mathbf{\Delta}, X) = D_{L^2(m)}(\Delta)$. μ_{ac} denotes the m-absolutely continuous part in the Lebesgue decomposition of a Borel measure μ. If $\mathbb{V}$ is any subspace of $L^2(m)$ and $f \in D_{L^2(m)}(\Delta)$ with $\Delta f \in \mathbb{V}$, we write $f \in D_{\mathbb{V}}(\Delta)$.

Theorem 2.4 (Cavalletti-Mondino, [CM18])**.** *Let (X,d,m) be an essentially non-branching $CD(K,N)$ space for some $K \in \mathbb{R}$ and $N > 1$. For $p \in X$ consider $d_p = d(p,\cdot)$ and the associated disintegration $m = \int_Q h_\alpha \mathcal{H}^1|_{X_\alpha} q(d\alpha)$.*

Then $d_p \in D(\mathbf{\Delta}, X\setminus\{p\})$ and $\mathbf{\Delta} d_p|_{X\setminus\{p\}}$ has the following representation formula:

$$\mathbf{\Delta} d_p|_{X\setminus\{p\}} = -(\log h_\alpha)' m - \int_Q h_\alpha \delta_{a(X_\alpha)} q(d\alpha).$$

Moreover

$$\mathbf{\Delta} d_p|_{X\setminus\{p\}} \leq (N-1)\frac{\sin'_{K/(N-1)}(d_p(x))}{\sin_{K/(N-1)}(d_p(x))} m \quad \& \quad \left[\mathbf{\Delta} d_p|_{X\setminus\{p\}}\right]^{reg} \geq -(N-1)\frac{\sin'_{K/(N-1)}(d_p(x))}{\sin_{K/(N-1)}(d_p(x))} m.$$

Remark 2.5. The sets X_α in the previous disintegration are geodesic segments $[a(X_\alpha), p]$ with initial point $a(X_\alpha)$ and endpoint p. In particular, the set of points $q \in X$ such that there exists a geodesic connecting p and q that is extendible beyond q, is a set of full measure.

Definition 2.6 ([AGS14b, Gig15])**.** A metric measure space (X, d, m) satisfies the Riemannian curvature-dimension condition $RCD(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty]$ if it satisfies a curvature-dimension conditions $CD(K, N)$ and is infinitesimally Hilbertian.

In [Gig18] Gigli introduced a notion of Hess f in the context of RCD spaces. Hess f is tensorial and defined for $f \in W^{2,2}(X)$ that is the second order Sobolev space. An important property of $W^{2,2}(X)$ that we will need in the following is

Theorem 2.7 (Corollary 3.3.9 in [Gig18], [Sav14])**.** $D_{L^2(m)}(\Delta) \subset W^{2,2}(X)$.

Remark 2.8. The closure of $D_{L^2(m)}(\Delta)$ in $W^{2,2}(X)$ is denoted $H^{2,2}(X)$ [Gig18, Proposition 3.3.18].

The next proposition [Gig18, Proposition 3.3.22 i)] allows to compute the Hess f explicitely.

Proposition 2.9. *Let $f, g_1, g_2 \in H^{2,2}(X)$. Then $\langle \nabla f, \nabla g_i\rangle \in W^{1,2}(X)$, and*

$$2\operatorname{Hess} f(\nabla g_1, \nabla g_2) = \langle \nabla g_1, \nabla\langle \nabla f, \nabla g_2\rangle\rangle + \langle \nabla g_2, \nabla\langle \nabla f, \nabla g_1\rangle\rangle + \langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\rangle \tag{3}$$

holds m-a.e. where the two sides in this expression are well-defined in $L^0(m)$.

Theorem 2.10 ([BS18])**.** *Let (X, d, m) be a metric measure space satisfying $RCD(K, N)$ with $N < \infty$. Then, there exist $n \in \mathbb{N}$ and such that set of n-regular points $\mathcal{R}_n$ has full measure.*

Theorem 2.11 ([Han18])**.** *Let (X, d, m) be as in the previous theorem. If $N = n \in \mathbb{N}$, then for any $f \in \mathbb{D}_\infty$ we have that $\Delta f = \operatorname{tr}\operatorname{Hess} f$ m-a.e. . More precisely, if $B \subset \mathcal{R}_n$ is a set of finite measure and $(e_i)_{i=1,\dots,n}$ is a unit orthogonal basis on B, then*

$$\Delta f|_B = \sum_{i=1}^n \operatorname{Hess} f(e_i, e_i) 1_B =: [\operatorname{tr}\operatorname{Hess} f]|_B.$$

Corollary 2.12. *Let (X, d, m) be a metric measure space as before. If $f \in D_{L^2(m)}(\Delta)$, we have that $\Delta f = \operatorname{tr}\operatorname{Hess} f$ m-a.e. in the sense of the previous theorem.*

Theorem 2.13 ([GP16], [KM18], [DPMR17])**.** *Let (X, d, m) be an $RCD(K, N)$ for some $K \in \mathbb{R}$ and $N \in (0, \infty)$. Denote*

$$\mathcal{R}_n^* := \left\{ x \in \mathcal{R}_k : \exists \lim_{r\to 0^+} \frac{m(B_r(x)}{r^k \omega_k} \in (0, \infty) \right\}.$$

Then, $m(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$, there exists n such that $\mathcal{R}_n^*$ has full measure, and $m|_{\mathcal{R}_n^*}$ and $\mathcal{H}^n|_{\mathcal{R}_n^*}$ are mutually absolutely continuous and

$$\lim_{r\to 0^+} \frac{m(B_r(x))}{\omega_n r^n} = \frac{dm|_{\mathcal{R}_n^*}}{d\mathcal{H}^n|_{\mathcal{R}_n^*}}(x) = \theta(x)$$

for m-a.e. $x \in \mathcal{R}_n^*$. Hence θ is locally in $L^1(\mathcal{H}^n)$.

2.3. **Spaces with upper curvature bounds.** We will assume familiarity with the notion of $CAT(\kappa)$ spaces. We refer to [BBI01, BH99] or [KK18] for the basics of the theory.

Definition 2.14. Given a point p in a $CAT(\kappa)$ space X we say that two unit speed geodesics starting at p define the same direction if the angle between them is zero. This is an equivalence relation by the triangle inequality for angles and the angle induces a metric on the set $S_p^g(X)$ of equivalence classes. The metric completion $\Sigma_p^g X$ of $S_p^g X$ is called the *space of geodesic directions* at p. The Euclidean cone $C(\Sigma_p^g X)$ is called the *geodesic tangent cone* at p and will be denoted by $T_p^g X$.

The following theorem is due to Nikolaev [BH99, Theorem 3.19]:

Theorem 2.15. *$T_p^g X$ is $CAT(0)$ and $\Sigma_p^g X$ is $CAT(1)$.*

Note that this theorem in particular implies that $T_p^g X$ is a geodesic metric space which is not obvious from the definition. More precisely, it means that each path component of $\Sigma_p^g X$ is $CAT(1)$ (and hence geodesic) and the distance between points in different components is π. Note however, that $\Sigma_p^g X$ itself need not be path connected.

2.4. **BV-functions and DC-calculus.** Recall that a function $g : V \subset \mathbb{R}^n \to \mathbb{R}$ of bounded variation (BV) admits a derivative in the distributional sense [EG15, Theorem 5.1] that is a signed vector valued Radon measure $[Dg] = (\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}) = [Dg]_{ac} + [Dg]_s$. Moreover, if g is BV, then it is L^1–differentiable [EG15, Theorem 6.1] a.e. with L^1-derivative $[Dg]_{ac}$, and approximately differentiable a.e. [EG15, Theorem 6.4] with approximate derivative $D^{ap}g = (\frac{\partial^{ap} g}{\partial x_1}, \dots, \frac{\partial^{ap} g}{\partial x_n})$ that coincides almost everywhere with $[Dg]_{ac}$. The set of BV-functions $BV(V)$ on V is closed under addition and multiplication [Per95, Section 4]. We'll call BV functions BV_0 if they are continuous.

Remark 2.16. In [Per95] and [AB18] BV functions are called BV_0 if they are continuous away from an $\mathcal{H}_{n-1}$-negligible set. However, for the purposes of the present paper it will be more convenient to work with the more restrictive definition above.

Then for $f, g \in BV_0(V)$ we have

$$\frac{\partial fg}{\partial x_i} = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_j} \tag{4}$$

as signed Radon measures [Per95, Section 4, Lemma]. By taking the $\mathcal{L}^n$-absolutely continuous part of this equality it follows that (4) also holds a.e. in the sense of approximate derivatives. In fact, it holds at *all* points of approximate differentiability of f and g. This easily follows by a minor variation of the standard proof that $d(fg) = fdg + gdf$ for differentiable functions.

A function $f : V \subset \mathbb{R}^n \to \mathbb{R}$ is called a DC–function if in a small neighborhood of each point $x \in V$ one can write f as a difference of two semi-convex functions. The set of DC–functions on V is denoted by $DC(V)$ and contains the class $C^{1,1}(V)$. $DC(V)$ is closed under addition and multiplication. The first partial derivatives $\frac{\partial f}{\partial x_i}$ of a DC–function $f : V \to \mathbb{R}$ are BV, and hence the second partial derivatives $\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}$ exist as signed Radon measure that satisfy

$$\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}$$

[EG15, Theorem 6.8], and hence

$$\frac{\partial^{ap}}{\partial x_i}\frac{\partial f}{\partial x_j} = \frac{\partial^{ap}}{\partial x_j}\frac{\partial f}{\partial x_i} \quad \text{a.e. on } V. \tag{5}$$

A map $F : V \to \mathbb{R}^l$, $l \in \mathbb{N}$, is called a DC–map if each coordinate function F_i is DC. The composition of two DC–maps is again DC. A function f on V is called DC_0 if it's DC and C^1.

Let (X, d) be a geodesic metric space. A function $f : X \to \mathbb{R}$ is called a DC-function if it can be locally represented as the difference of two Lipschitz semi-convex functions. A map $F : Z \to Y$

between metric spaces Z and Y that is locally Lipschitz is called a DC-map if for each DC-function f that is defined on an open set $U \subset Y$ the composition $f \circ F$ is DC on $F^{-1}(U)$. In particular, a map $F : Z \to \mathbb{R}^l$ is DC if and only if its coordinates are DC. If F is a bi-Lipschitz homeomorphism and its inverse is DC, we say F is a DC-isomorphism.

2.5. **DC-coordinates in CAT-spaces.** The following was developed in [LN18] based on previous work by Perelman [Per95].

Assume (X, d) is a CAT-space, let $p \in X$ such that there exists an open neighborhood $\hat{U}$ of p that is homeomorphic to $\mathbb{R}^n$. It is well known (see e.g. [KK18, Lemma 3.1]) that this implies that geodesics in $\hat{U}$ are locally extendible.

Suppose $T_p^g X \cong \mathbb{R}^n$.

Then, there exist DC coordinates near p with respect to which the distance on $\hat{U}$ is induced by a BV Riemannian metric g.

More precisely, let $a_1, \dots, a_n, b_1, \dots, b_n$ be points near p such that $d(p, a_i) = d(p, b_i) = r$, p is the midpoint of $[a_i, b_i]$ and $\angle a_i p a_j = \pi/2$ for all $i \neq j$ and all comparison angles $\widetilde{\angle} a_i p a_j, \widetilde{\angle} a_i p b_j, \widetilde{\angle} b_i p b_j$ are sufficiently close to $\pi/2$ for all $i \neq j$.

Let $x\colon \hat{U} \to \mathbb{R}^n$ be given by $x = (x_1, \dots, x_n) = (d(\cdot, a_1), \dots, d(\cdot, a_n))$.

Then by [LN18, Corollary 11.12] for any sufficiently small $0 < \varepsilon < \pi_k/4$ the restriction $x|_{B_{2\varepsilon}(p)}$ is Bilipschitz onto an open subset of $\mathbb{R}^n$. Let $U = B_\varepsilon(p)$ and $V = x(U)$. By [LN18, Proposition 14.4] $x\colon U \to V$ is a DC-equivalence in the sense that $h\colon U \to \mathbb{R}$ is DC iff $h \circ x^{-1}$ is DC on V.

Further, the distance on U is induced by a BV Riemannian metric g which in x coordinates is given by a 2-tensor $g^{ij}(p) = \cos \alpha_{ij}$ where α_{ij} is the angle at p between geodesics connecting p and a_i and a_j respectively. By the first variation formula g^{ij} is the derivative of $d(a_i, \gamma(t))$ at 0 where γ is the geodesic with $\gamma(0) = p$ and $\gamma(1) = a_j$. Since $d(a_i, \cdot)$, $i = 1, \dots n$, are Lipschitz, g^{ij} is in L^∞. We denote $\langle v, w \rangle_g(p) = g^{ij}(p) v_i w_j$ the inner product of $v, w \in \mathbb{R}^n$ at p. g^{ij} induces a distance function d_g on V such that x is a metric space isomorphism for $\epsilon > 0$ sufficiently small.

If u is a Lipschitz function on U, $u \circ x^{-1}$ is a Lipschitz function on V, and therefore differentiable $\mathcal{L}^n$-a.e. in V by Rademacher's theorem. Hence, we can define the gradient of u at points of differentiability of u in the usual way as the metric dual of its differential. Then the usual Riemannian formulas hold and $\nabla u = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j}$ and $|\nabla u|_g^2 = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$ a.e. .

3. Structure theory of RCD+CAT spaces

In this section we study metric measure spaces (X, d, m) satisfying

(6) (X, d, m) is $CAT(\kappa)$ and satisfies the conditions $RCD(K, N)$ for $1 \leq N < \infty$, $K, \kappa < \infty$.

The following result was proved in [KK18]

Theorem 3.1 ([KK18])**.** *Let (X, d, m) satisfy $CD(K, N)$ for $1 \leq N < \infty$, $K, \kappa \in \mathbb{R}$. Then X is infinitesimally Hilbertian. In particular, (X, d, m) satisfies $RCD(K, N)$.*

Remark 3.2. It was shown in [KK18] that the above theorem also holds if the $CD(K, N)$ assumption in (6) is replaced by $CD^*(K, N)$ or $CD^e(K, N)$ conditions (see [KK18] for the definitions). Moreover, in a recent paper [MGPS18] Di Marino, Gigli, Pasqualetto and Soultanis show that a $CAT(\kappa)$ space with *any* Radon measure is infinitesimally Hilbertian. For these reasons (6) is equivalent to assuming that X is $CAT(\kappa)$ and satisfies one of the assumptions $CD(K, N), CD^*(K, N)$ or $CD^e(K, N)$ with $1 \leq N < \infty$, $K, \kappa < \infty$.

In [KK18] we also established the following property of spaces satisfying (6):

Proposition 3.3 ([KK18])**.** *Let X satisfy* (6)*. Then X is non-branching.*

Next we prove

Proposition 3.4. *Let X satisfy* (6)*. Then for almost all $p \in X$ it holds that $T_p^g X \cong \mathbb{R}^k$ for some $k \leq N$.*

Remark 3.5. Note that from the fact that X is an RCD space it follows that T_pX is an Euclidean space for almost all $p \in X$ [GMR15]. However, at this point in the proof we don't know if $T_pX \cong T_p^gX$ at all such points (we expect this to be true for all p).

Proof. First, recall that by the CAT condition, geodesics of length less than π_κ in X are unique. Moreover, since X is nonbranching and CD, for any $p \in X$ the set E_p of points q, such that the geodesic which connects p and q is not extendible, has measure zero (Remark 2.5).

Let $A = \{p_i\}_{i=1}^\infty$ be a countable dense set of points in X, and let $C = \bigcup_{i\in\mathbb{N}} E_{p_i}$. For any $q \in X\backslash C$ and any i with $d(p_i, q) < \pi_\kappa$ the geodesic $[p_iq]$ can be extended slightly past q. Since A is dense this implies that for any $q \in X\backslash C$ there is a dense subset in T_q^gX consisting of directions v which have "opposites" (i.e. making angle π with v).

For every $p \in X$ and every tangent cone T_pX the geodesic tangent cone T_p^gX is naturally a closed convex subset of T_pX. Since X is RCD this means that for almost all p the geodesic tangent cone T_p^gX is a convex subset of a Euclidean space. Thus, for almost all $p \in X$ it holds that T_p^gX is a convex subset in $\mathbb{R}^m$ for some $m \le N$, is a metric cone over Σ_p^gX and contains a dense subset of points with opposites also in T_p^gX. In particular, Σ_p^gX is a convex subset of $\mathbb{S}^m$. Since a closed convex subset of $\mathbb{S}^m$ is either $\mathbb{S}^k$ with $k \le m$ or has boundary this means that for any such p T_p^gX is isometric to a Euclidean space of dimension $k \le m$. □

Proposition 3.6. *Let X satisfy* (6).

i) Let $p \in X$ satisfy $T_p^gX \cong \mathbb{R}^m$ for some $m \le N$.
Then an open neighbourhood W of p is homeomorphic to $\mathbb{R}^m$.

ii) If an open neighborhood W of p is homeomorphic to $\mathbb{R}^m$ then for any $q \in W$ it holds that $T_q^gX \cong T_qX \cong \mathbb{R}^m$.
Moreover, for any compact set $C \subset W$ there is $\varepsilon = \varepsilon(C) > 0$ such that every geodesic starting in C can be extended to length at least ε.

Proof. Let us first prove part i). Suppose $T_p^gX \cong \mathbb{R}^m$. By [Kra11, Theorem A] there is a small $R > 0$ such that $B_R(p)\backslash\{p\}$ is homotopy equivalent to $\mathbb{S}^{m-1}$. Since $\mathbb{S}^{m-1}$ is not contractible, by [LS07, TRheorem 1.5] there is $0 < \varepsilon < \pi_\kappa/2$ such that every geodesic starting at p extends to a geodesic of length ε. The natural "logarithm" map $\Phi\colon \bar{B}_\varepsilon(p) \to \bar{B}_\varepsilon(0) \subset T_p^gX$ is Lipschitz since X is $CAT(\kappa)$. By the above mentioned result of Lytchak and Schroeder [LS07, Theorem 1.5] Φ is *onto*.

We also claim that Φ is 1-1. If Φ is not 1-1 then there exist two distinct unit speed geodesics γ_1, γ_2 of the same length $\varepsilon' \le \varepsilon$ such that $p = \gamma_1(0) = \gamma_2(0)$, $\gamma_1'(0) = \gamma_2'(0)$ but $\gamma_1(\varepsilon') \ne \gamma_2(\varepsilon')$.

Let $v = \gamma_1'(0) = \gamma_2'(0)$. Since $T_p^gX \cong \mathbb{R}^m$ the space of directions T_p^gX contains the "opposite" vector $-v$. Then there is a geodesic γ_3 of length εstarting at p in the direction $-v$. Since X is $CAT(\kappa)$ and $2\varepsilon < \pi_k$, the concatenation of γ_3 with γ_1 is a geodesic and the same is true for γ_2. This contradicts the fact that X is nonbranching.

Thus, Φ is a continuous bijection and since both $\bar{B}_\varepsilon(p)$ and $\bar{B}_\varepsilon(0)$ are compact and Hausdorff it's a homeomorphism. This proves part i).

Let us now prove part ii). Suppose an open neighborhood W of p is homeomorphic to $\mathbb{R}^m$. By [KK18, Lemma 3.1] or by the same argument as above using [Kra11] and [LS07], for any $q \in W$ all geodesics starting at q can be extended to length at least $\varepsilon(q) > 0$. Therefore $T_q^gX \cong T_qX$. By the splitting theorem $T_qX \cong \mathbb{R}^l$ where $l = l(q) \le N$ might a priori depend on q. However, using part i) we conclude that an open neighbourhood of q is homeomorphic to $\mathbb{R}^{l(q)}$. Since W is homeomorphic to $\mathbb{R}^m$ this can only happen if $l(q) = m$.

The last part of ii) immediately follows from above and compactness of C.

□

3.1. ***DC*-coordinates in *RCD*+*CAT*-spaces.** Let X_{reg}^g be the set of points p in X with $T_pX \cong T_p^gX \cong \mathbb{R}^n$. Then by Proposition 3.6 there is an open neighbourhood $\hat{U}$ of p homeomorphic to $\mathbb{R}^n$

such that every $q \in \hat{U}$ also lies in X^g_{reg}. In particular, X^g_{reg} is open. Further, geodesics in $\hat{U}$ are locally extendible by Proposition 3.6.

Thus the theory of Lytchak–Nagano from [LN18] applies, and let $x : U \to V$ with $U = B_{2\epsilon}(p) \subset \hat{U}$ be DC-coordinates as in Subsection 2.5. The pushforward of the Hausdorff measure $\mathcal{H}^n$ on U under x coordinates is given by $\sqrt{|g|}\mathcal{L}$ where $|g|$ is the determinant of g_{ij} Consequently, the map $x\colon (U, d, \mathcal{H}_n) \to (V, d_g, \sqrt{|g|}\mathcal{L}_n)$ is a metric-measure isomorphism.

With a slight abuse of notations we will identify these metric-measure spaces as well as functions on them, i.e we will identify any function u on U with $u \circ x^{-1}$ on V.

Lemma 3.7. *Angles between geodesics in U are continuous. That is if $q_i \to q \in U, [q_i s_i] \to [qs], [q_i t_i] \to [qt]$ are converging sequences with $q \neq s, q \neq t$ then $\angle s_i q_i t_i \to \angle sqt$.*

Proof. Without loss of generality we can assume that $q_i \in U$ for all i. Let $\alpha_i = \angle s_i q_i t_i, \alpha = \angle sqt$. Let $\{\alpha_{i_k}\}$ be a converging subsequence and let $\bar{\alpha} = \lim_{k\to\infty} \alpha_{i_k}$. Then by upper semicontinuity of angles in $CAT(\kappa)$ spaces it holds that $\alpha \geq \bar{\alpha}$. We claim that $\alpha = \bar{\alpha}$.

By Proposition 3.6 we can extend $[s_i q_i]$ past q_i as geodesics a definite amount δ to geodesics $[s_i z_i]$. Let $\beta_i = \angle z_i q_i t_i$. By possibly passing to a subsequence of $\{i_k\}$ we can assume that $[s_{i_k} z_{i_k}] \to [sz]$. Let $\beta = \angle zqt$. Then since all spaces of directions $T^g_{q_i} X$ and $T^g_q X$ are Euclidean by Proposition 3.6, we have that $\alpha_i + \beta_i = \alpha + \beta = \pi$ for all i. Again using semicontinuity of angles we get that $\beta \geq \bar{\beta}$.

We therefore have

$$\pi = \alpha + \beta \geq \bar{\alpha} + \bar{\beta} = \pi$$

Hence all the inequalities above are equalities and $\alpha = \bar{\alpha}$. Since this holds for an arbitrary converging subsequence $\{\alpha_{i_k}\}$ it follows that $\lim_{\imath\to\infty} \alpha_i = \alpha$. □

Let $\widetilde{\mathcal{A}}$ be the algebra of functions of the form $\varphi(f_1, \dots, f_m)$ where $f_i = d(\cdot, q_i)$ for some $q_1, \dots, q_m$ with $|q_i p| > \varepsilon$ and φ is smooth. Together with the first variation formula for distance functions Lemma 3.7 implies that for any $u, h \in \widetilde{\mathcal{A}}$ it holds that $\langle \nabla u, \nabla h\rangle_g$ is continuous on V. In particular, $g^{ij} = \langle \nabla x_i, \nabla x_j\rangle_g$ is continuous and hence g is BV_0 and not just BV.

Furthermore, since $\frac{\partial}{\partial x_i} = \sum_j g_{ij} \nabla x_j$ where g_{ij} is the pointwise inverse of g^{ij}, Lemma 3.7 also implies that any $u \in \widetilde{\mathcal{A}}$ is C^1 on V. Hence, any such u is DC_0 on V.

Recall that for a Lipschitz function u on V we have two a-priori different notions of the norm of the gradient defined m-a.e.: the "Riemannian" norm of the gradient $|\nabla u|^2_g = g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x_j}$ and the minimal weak upper gradient $|\nabla u|$ when u is viewed as a Sobolev functions in $W^{1,2}(m)$. We observe that these two notions are equivalent.

Lemma 3.8. *Let $u, h : U \to \mathbb{R}$ be Lipschitz functions. Then $|\nabla u| = |\nabla u|_g$, $|\nabla h| = |\nabla h|_g$ m-a.e. and $\langle \nabla u, \nabla h\rangle = \langle \nabla u, \nabla h\rangle_g$ m-a.e..*

In particular, $g^{ij} = \langle \nabla x_i, \nabla x_j\rangle_g = \langle \nabla x_i, \nabla x_j\rangle$ m-a.e..

Proof. First note that since both $\langle \nabla u, \nabla h\rangle$ and $\langle \nabla u, \nabla h\rangle_g$ satisfy the parallelogram rule, it's enough to prove that $|\nabla u| = |\nabla u|_g$ a.e..

Recall that g^{ij} is continuous on U. Fix a point p where u is differentiable. Then

$$\begin{aligned}\operatorname{Lip} u(p) = \limsup_{q\to p} \frac{|u(p) - u(q)|}{d(p, q)} &= \limsup_{q\to p} \frac{|u(p) - u(q)|}{|p - q|_{g(p)}} \\ &= \sup_{|v|_{g(p)}=1} D_v u = \sup_{|v|_{g(p)}=1} \langle v, \nabla u\rangle_{g(p)} = |\nabla u|_{g(p)}.\end{aligned}$$

In the second equality we used that d is induced by g^{ij}, and that g^{ij} is continuous. Since (U, d, m) admits a local 1-1 Poincaré inequality and is doubling, the claim follows from [Che99] where it is proved that for such spaces $\operatorname{Lip} u = |\nabla u|$ a.e.. □

In view of the above Lemma from now on we will not distinguish between $|\nabla u|$ and $|\nabla u|_g$ and between $\langle \nabla u, \nabla h\rangle$ and $\langle \nabla u, \nabla h\rangle_g$.

Proposition 3.9. *If $u \in W^{1,2}(m) \cap BV(U)$, then $|\nabla u|^2 = g^{ij} \frac{\partial^{ap} u}{\partial x_i} \frac{\partial^{ap} u}{\partial x_j}$ m-a.e. .*

Proof. We choose a set $S \subset U$ of full measure such that u and $|\nabla u|$ are defined pointwise on S and u is approximately differentiable at every $x \in S$. Since u is $BV(U)$, for $\eta > 0$ there exist $\hat{u}_\eta \in C^1(U)$ such that for the set

$$B_\eta = \{x \in S : u(x) \neq \hat{u}_\eta(x), D^{ap}u(x) \neq D\hat{u}(x)\} \cap S$$

one has $m(B_\eta) \leq \eta$ [EG15, Theorem 6.13]. Note, since f is continuous, there exists a constant $\lambda > 0$ such that $\lambda^{-1} m \leq \mathcal{H}^n \leq \lambda m$ on U. Moreover, since g^{ij} is continuous, one can check that $\hat{u}_\eta$ is Lipschitz w.r.t. d_g, and hence $\hat{f} \in W^{1,2}(m)$.

By [AGS14a, Proposition 4.8] we know that $|\nabla u||_{A_\eta} = |\nabla \hat{u}||_{A_\eta}$ m-a.e. for $A_\eta = S \backslash B_\eta$. On the other hand, uniqueness of approximative derivatives also yields that $g^{ij} \frac{\partial^{ap} u}{\partial x_i} \frac{\partial^{ap} u}{\partial x_j}|_{A_\eta} = g^{ij} \frac{\partial^{ap} \hat{u}_\eta}{\partial x_i} \frac{\partial^{ap} \hat{u}_\eta}{\partial x_j}|_{A_\eta}$ m-a.e. . Hence, since $\hat{u}$ is Lipschitz w.r.t. d,

$$|\nabla u| 1_{A_\eta} = g^{ij} \frac{\partial^{ap} u}{\partial x_i} \frac{\partial^{ap} u}{\partial x_j} 1_{A_\eta} \quad m\text{-a.e. .}$$

by Lemma 3.8.

Now, we pick a sequence η_k for $k \in \mathbb{N}$ such that $\sum_{k=1}^\infty \eta_k < \infty$. Then, by the Borel-Cantelli Lemma the set

$$B = \{x \in S : \exists \text{ infinitely many } k \in \mathbb{N} \text{ s.t. } x \in B_{\eta_k}\}$$

is of m-measure 0. Consequently, for $x \in A = S \backslash B$ we can pick a $k \in \mathbb{N}$ such that $x \in A_{\eta_k} \subset S$. It follows

$$|\nabla u|^2(x) = g^{ij} \frac{\partial^{ap} u}{\partial x_i} \frac{\partial^{ap} u}{\partial x_j}(x) \ \forall x \in S$$

and hence m-a.e. . □

4. Density functions

4.1. **Noncollapsed case.** Let $(X, d, f\mathcal{H}_n)$ be $RCD(K, n)$ and $CAT(\kappa)$ where $0 \leq f \in L^1_{loc}(\mathcal{H}^n)$.

Remark 4.1. If (X, d, m) is a weakly non-collapsed RCD-space in the sense of [DPG18] or a space satisfying the generalized Bishop inequality in the sense of [Kit17] and if (X, d) is $CAT(\kappa)$, the assumptions are satisfied by [DPG18, Theorem 1.10].

Following Gigli and De Philippis [DPG18] for any $x \in X$ we consider the monotone quantity $\frac{m(B_r(x))}{v_{k,n}(r)}$ which is non increasing in r by the Bishop-Gromov volume comparison. Let $\theta_{n,r}(x) = \frac{m(B_r(x))}{\omega_n r^n}$. Consider the density function $\theta_n(x) = \lim_{r \to 0} \theta_{n,r}(x) = \lim_{r \to 0} \frac{m(B_r(x))}{\omega_n r^n}$.

Since n is fixed throughout the proof we will drop the subscripts n and from now on use the notations $\theta(x)$ and $\theta_r(x)$ for $\theta_n(x)$ and $\theta_{n,r}(x)$ respectively.

By Propositions 3.4, 3.6 and [DPG18, Theorem 1.10] we have that for almost all $p \in X$ it holds that $T_p X \cong T_p^g X \cong \mathbb{R}^n$ and $\theta(x) = f(x)$.

Therefore we can and will assume from now on that $f = \theta$ everywhere.

Remark 4.2. Monotonicity of $r \mapsto \frac{m(B_r(x))}{v_{k,n}(r)}$ immediately implies that $f(x) = \theta(x) > 0$ for all x.

Let $x \in X^g_{reg}$. Then $T_p^g X \cong \mathbb{R}^m$ for some $m \leq n$. We claim that $m = n$. By Proposition 3.6 X^g_{reg} is an m-manifold near p and by section 2.5 DC coordinates near p give a *biLipschitz* homeomorphism of an open neighborhood of p onto an open set in $\mathbb{R}^m$. Since $m = f\mathcal{H}_n$ this can only happen if $m = n$.

Lemma 4.3. [KK18, Lemma 5.4] *$\theta = f$ is semiconcave on X.*

Corollary 4.4. *$\theta = f$ is locally Lipschitz near any $p \in X^g_{reg}$.*

Proof. First observe that semiconcavity of θ, the fact that $\theta \geq 0$ and local extendability of geodesics on X^g_{reg} imply that θ must be locally bounded on X^g_{reg}. Now the corollary becomes an easy consequence of Lemma 4.3, the fact that geodesics are locally extendible a definite amount near p by Proposition 3.6 and the fact that a semiconcave function on $(0,1)$ is locally Lipschitz. □

4.2. **General case.** Let (X,d,m) be $RCD(K,N)$ and $CAT(\kappa)$.

Proposition 4.5. *$-\log\theta : X \to (-\infty,\infty]$ is well-defined and semi-convex.*

Proof. There exists a set of full measure S where θ is defined.

We extend θ via

$$[0,\infty] \ni \limsup_{r\to 0} \frac{m(B_r(x)}{\omega_n r^n} = \hat{\theta}(x)$$

to a function that is defined everywhere on X. In particular, $-\log\hat{\theta}$ takes values in $[-\infty,\infty]$.

Let $\gamma : [0,1] \to X$ be any geodesic in X, and recall that the condition $RCD(K,N)$ implies the condition $CD(K,\infty)$. More precisely, the Shanon-Boltzmann entropy is weakly K-convex on $\mathcal{P}^2(X)$. In particular, defining $\mu_i = m(B_r(\gamma(0)))^{-1} m|_{B_r(\gamma(0))}$, $i = 0,1$, there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathcal{P}^2(X)$ such that

$$\operatorname{Ent}\mu_t \leq (1-t)\operatorname{Ent}\mu_0 + t\operatorname{Ent}\mu_t - \frac{K}{2}t(1-t)W_2(\mu_0,\mu_1)^2.$$

The left hand side is bounded from below by $-\log m(A_t)$ where A_t is the set of all t-midpoints of geodesic that start in $B_r(\gamma(0))$ and end in $B_r(\gamma(1))$. Then, by Lemma 5.5 in [KK18] $-\log m(A_t)$ is bounded from below $-\log m(B_{r(1+cl^2)}(\gamma(t))$ for some constant $c > 0$ and $l = d(\gamma(0),\gamma(1))$. The constant c only depends on the upper sectional curvature bound.

Hence

$$\begin{aligned} -\log\frac{m(B_{r(1+cl^2)}(\gamma(t))}{\omega_n r^n(1+cl^2)^n} &+ \frac{K}{2}t(1-t)W_2(\mu_0,\mu_1)^2 \\ &\leq -(1-t)\log\frac{\mu(B_r(\gamma(0)))}{\omega_n r^n} - t\log\frac{\mu(B_r(\gamma(1)))}{\omega_n r^n} + n\log(1+cl^2). \end{aligned}$$

Now, we note that $m(B_r(\gamma(i)))^{-1} m|_{B_r(\gamma(i))} \to \delta_{\gamma(i)}$, $i = 0,1$, weakly, and therefore

$$\liminf_{r\to 0} W_2(\mu_0,\mu_1) \geq W_2(\delta_{\gamma(0)},\delta_{\gamma(1)}) = d(\gamma(0),\gamma(1)).$$

Moreover, $\log(1+cl^2) \leq cl^2$. It follows for $t = \frac{1}{2}$

$$-\log\hat{\theta}(\gamma(t)) \leq -\frac{1}{2}\log\hat{\theta}(\gamma(0)) - \frac{1}{2}\log\hat{\theta}(\gamma(1)) + (\frac{\frac{nc}{2}-K}{2})\frac{1}{4}d(\gamma(0),\gamma(1))^2.$$

Finally, let $x \in X$ be arbitrary. Then, there exist a geodesic $\gamma : [0,1] \to X$ with $\gamma(0) = x$ such that there exists $t \in [0,1]$ with $\hat{\theta}(\gamma(t)) = \theta(\gamma(t)) < \infty$. Otherwise, as consequence of the disintegration in Theorem 2.4 we would obtain a contradiction from Theorem 2.13. Hence $-\log\hat{\theta}(\gamma(t)) > -\infty$, and therefore $-\log\hat{\theta}(x) > -\infty$ and $\hat{\theta}(x) < \infty$. □

Corollary 4.6. *$-\log\theta$ is locally Lipschitz near any $p \in X^g_{reg}$.*

In particular, $-\log\theta$ is finite and defined everywhere on X^g_{reg}.

Proof. This follows again by extendability of geodesics on X^g_{reg} as in Corollary 4.4. □

5. Laplace operator in DC coordinates

1. Since small balls in spaces with curvature bounded above are geodesically convex, we can assume that $\operatorname{diam} X < \pi_\kappa$. Let $p \in X$, $x : U \to \mathbb{R}^n$ and $\widetilde{\mathcal{A}}$ be as in the previous subsection.

By the same argument as in [Per95, Section 4] (cf. [Pet11], [AB18]) it follows that any $u \in \widetilde{\mathcal{A}}$ lies in $D(\mathbf{\Delta}, U, \mathcal{H}_n)$ and the $\mathcal{H}^n$-absolutely continuous part of $\mathbf{\Delta}_0 u$ can be computed using standard Riemannian geometry formulas that is

$$\mathbf{\Delta}_0^a(u) = \frac{1}{\sqrt{|g|}} \frac{\partial^{ap}}{\partial x_j} (g^{jk} \sqrt{|g|} \frac{\partial u}{\partial x_k}) \tag{7}$$

where $|g|$ denotes the pointwise determinant of g^{ij}. Here $\mathbf{\Delta}_0$ denotes the measure valued Laplacian on $(U, d, \mathcal{H}_n)$. Note that g, $\sqrt{|g|}$ and $\frac{\partial u}{\partial x_i}$ are BV_0-functions, and the derivatives on the right are understood as approximate derivatives.

Indeed, w.l.o.g. let $u \in DC_0(U)$, and let v be Lipschitz with compact support in U. As before we identify u and v with their representatives in x coordinates. First, we note that, since g, $\sqrt{|g|}$ and $\frac{\partial u}{\partial x_i}$ are BV_0, their product is also in BV_0, as well as the product with v. Then, the Leibniz rule (4) for the approximate partial derivatives yieds that

$$\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial^{ap} v}{\partial x_j} = -\frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \right) v + \frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) \ \mathcal{L}^n\text{-a.e. .}$$

Again using (4) we also have that

$$\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} = -\frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \right) v + \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) \text{as measures} \tag{8}$$

and the absolutely continuous with respect to $\mathcal{L}^n$ part of this equation is given by the previous identity.

The fundamental theorem of calculus for BV functions (see [EG15, Theorem 5.6]) yields that

$$\int_V \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) = 0. \tag{9}$$

Moreover, by Lemma 3.8 $\langle \nabla v, \nabla u \rangle$ is given in x coordinates by $g^{ij} \frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_i}$ $\mathcal{L}^n$-a.e. .

Combining the above formulas gives that

$$\begin{aligned} -\int_V \langle \nabla u, \nabla v \rangle \sqrt{|g|} d\mathcal{L}^n &= \int_V \left[\frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \right) v - \frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) \right] d\mathcal{L}^n \\ &= \int_V \frac{1}{\sqrt{|g|}} \frac{\partial^{ap}}{\partial x_j} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \right) v \sqrt{|g|} d\mathcal{L}^n + \int_V v d\mu \end{aligned}$$

where μ is some signed measure such that $\mu \perp \mathcal{L}^n$. This implies (7).

4. It follows that

$$\begin{aligned} \operatorname{Tr} \operatorname{Hess}(\chi u)|_{B_\delta(p)} &= \operatorname{Hess}(\chi u)(\nabla x_i, \nabla x_j) g_{ij}|_{B_\delta(p)} \\ &= \operatorname{Hess}(\chi u)(\nabla \chi x_i, \nabla \chi x_j) g_{ij}|_{B_\delta(p)} = Hu(x_i, x_j) g_{ij} = \Delta_0 u|_{B_\delta(p)}. \end{aligned} \tag{10}$$

for every $u \in \widetilde{\mathcal{A}}$ where Hess is the Hessian in the sense of Gigli, and $H(u)$ is denotes the RHS of (**??**). The first equality in (10) is the definition of Tr, the second equality is the L^∞-homogeneity of the tensor $\operatorname{Hess}(\chi u)$, and the third equality is the identity (**??**).

Since f is locally Lipschitz and positive on $B_\delta(p)$, we can perform the following integration by parts in DC_0 coordinates. Let $u \in \widetilde{\mathcal{A}}$ and let g be Lipschitz with compact support in $B_\delta(p)$.

$\chi u \in D_{L^2(m)}(\Delta)$ implies $u|_{B_\delta(p)} \in D(\mathbf{\Delta}, B_\delta(p))$. Then

$$\begin{aligned}\int_{B_\delta(p)} g\Delta u dm &= -\int_{B_\delta(p)} \langle \nabla u, \nabla g\rangle dm = -\int_{B_\delta(p)} \langle \nabla u, \nabla g\rangle f d\mathcal{H}^n \\ &= -\int_{B_\delta(p)} \langle \nabla u, \nabla (gf)\rangle d\mathcal{H}^n + \int_{B_\delta(p)} \langle \nabla u, \nabla \log f\rangle g f d\mathcal{H}^n \\ &= \int_{B_\delta(p)} (\Delta_0 u + \langle \nabla u, \nabla \log f\rangle) g dm\end{aligned}$$

yields

$$\Delta u = \Delta_0 u + \langle \nabla u, \nabla \log f\rangle$$

on $B_\delta(p)$ for any $u \in \widetilde{\mathcal{A}}$. Note again that only χu is in $D_{L^2(m)}(\Delta)$.

On the other hand, by Corollary 2.12 it holds that $\Delta(\chi u) = \operatorname{Tr}\operatorname{Hess}(\chi u)$ m-a.e. . Thus

$$0 = \operatorname{Tr}\operatorname{Hess}(\chi u)|_{B_\delta(p)} - \operatorname{Tr}\operatorname{Hess}(\chi u)|_{B_\delta(p)} = \Delta u|_{B_\delta(p)} - \Delta_0(\chi u)|_{B_\delta(p)} = \langle \nabla u, \nabla \log f\rangle|_{B_\delta(p)}$$

a.e. for any $u \in \widetilde{\mathcal{A}}$.

6. Weakly noncollpased implies noncollapsed

5. Therefore $f\nabla \log f|_{B_\delta(p)} = \nabla f|_{B_\delta(p)} = 0$. Indeed, since f is semiconcave, $f \circ x^{-1}$ is DC by [LN18]. Hence $\nabla f = g^{ij}\frac{\partial f}{\partial x_i}$ is continuous on a set of full measure Z in $B_\delta(p)$ since this is true for convex functions on $\mathbb{R}^n$. Let $q \in Z$ be a point of continuity of $\nabla f|_Z$ and $v = \nabla f(q)$. Assume $v \neq 0$. Then due to extendability of geodesics there exists $z \notin U$ such that $\nabla d_z(q) = \frac{v}{|v|}$. Since ∇d_z is continuous near q and ∇f is continuous on Z it follows $\langle \nabla f, \nabla d_z\rangle \neq 0$ on a set of positive measure. Hence $\nabla f|_{B_\delta(p)} = 0$ and $f|_{B_\delta(p)} = const$.

6. We claim that this implies that f is constant on X^g_{reg}. (This is not immediate since we don't know yet that X^g_{reg} is connected.) Indeed, since X is essentially nonbranching, radial disintegration of m centered at p (Theorem 2.4) implies that for almost all $q \in X$ the set $[pq] \cap X^g_{reg}$ has full measure in $[p, q]$. It is also open in $[p, q]$ since X^g_{reg} is open.

Suppose $q \in X^g_{reg}$ is as above.

Since θ is semiconcave on X and locally constant on X^g_{reg} it is locally Lipschitz (and hence Lipschitz) on the geodesic segment $[p, q]$. A Lipschitz function on $[0, 1]$ which is locally constant on an open set of full $\mathcal{L}^1$ measure is constant. Therefore θ is constant on $[p, q]$ and hence θ is constant on X^g_{reg} which has full measure. Therefore $f = \theta = const$ a.e. globally. □

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