WEAKLY NONCOLLAPSED AND COLLAPSED RCD SPACES WITH UPPER CURVATURE BOUNDS

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Abstract. We show that if a $CD(K,n)$ space $(X,d,f\mathcal{H}_n)$ with $n \geq 2$ has curvature bounded above by $\kappa$ in the sense of Alexandrov then $f = const$.

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1. Introduction

In [DPG18] Gigli and De Philippis introduced the following notion of a noncollapsed $RCD(K,n)$ space. An $RCD(K,n)$ space $(X,d,m)$ is noncollapsed if $n$ is a natural number and $m = \mathcal{H}_n$. A similar notion was considered by Kitabeppu in [Kit17].

Noncollapsed $RCD(K,n)$ give a natural intrinsic generalization of noncollapsing limits of manifolds with lower Ricci curvature bounds which are noncollapsed in the above sense by work of Cheeger–Colding [CC97].

In [DPG18] Gigli and De Philippis also considered the following a-priori weaker notion. An $RCD(K,n)$ space $(X,d,m)$ is weakly noncollapsed if $n$ is a natural number and $m \ll \mathcal{H}_n$. Gigli and De Philippis gave several equivalent characterizations of weakly noncollapsed $RCD(K,n)$ spaces and studied their properties. By work of Gigli–Pasqualetto [GP10], Mondino–Kell [KM18], De Philippis–Marchese–Rindler [DPMR17] and Brué–Semola [BS18] it follows that an $RCD(K,n)$ space is weakly noncollapsed iff $\mathcal{R}_n \neq \emptyset$ where $\mathcal{R}_n$ is the rectifiable set of $n$-regular points in $X$.
It is well-known that if \((X, d, m) = (M^n, g, e^{-f} \, \text{vol}_g)\) where \((M^n, g)\) is a smooth \(n\)-dimensional Riemannian manifold and \(f\) is a smooth function on \(M\) then \((X, d, m)\) is RCD\((K, n)\) if \(f = \text{const}\).

More precisely, the classical Bakry-Emery condition \(\text{BE}(K, N, t)\) is satisfied if \(\frac{1}{2} L|\nabla u|^2_g \geq \langle \nabla Lu, \nabla u \rangle_g + \frac{1}{N} (Lu)^2 + K|\nabla u|^2_g\), \(\forall u \in C^\infty(M)\),

where \(L = \Delta - \nabla f\). In [Bak94, Proposition 6.2] Bakry shows that \(\text{BE}(K, N)\) holds if and only if

\[
\nabla f \otimes \nabla f \leq (N - n) \left( \text{ric}_g + \nabla^2 f - Kg \right).
\]

In particular, if \(N = n\), then \(f\) is locally constant.

On the other hand, it was shown that a metric measure space \((X, d, m)\) satisfies RCD\((K, n)\) if and only if the corresponding Cheeger energy satisfies a weak version of \(\text{BE}(K, N)\) that is equivalent to the classical version for \((M, g, e^{-f} \, \text{vol}_g)\) from above.

In [DPG18] Gigli and De Philippis conjectured that a weakly noncollapsed RCD\((K, n)\) space is already noncollapsed up to rescaling of the measure by a constant. Our main result is that this conjecture holds if a weakly noncollapsed space has curvature bounded above in the sense of Alexandrov.

**Theorem 1.1.** Let \(n \geq 2\) and let \((X, d, f\mathcal{H}_n)\) (where \(f = L^1_{\text{loc}}\) with respect to \(\mathcal{H}_n\) and \(\text{supp}(f\mathcal{H}_n) = X\)) be a complete metric measure space which is \(CBA(\kappa)\) (has curvature bounded above by \(\kappa\) in the sense of Alexandrov) and satisfies \(CD(K, n)\). Then \(f = \text{const}\).

Since smooth Riemannian manifolds locally have curvature bounded above this immediately implies

**Corollary 1.2.** Let \((M^n, g)\) be a smooth Riemannian manifold and suppose \((M^n, g, f\mathcal{H}_n)\) is \(CD(K, n)\) where \(K\) is finite and \(f \geq 0\) is \(L^1_{\text{loc}}\) with respect to \(\mathcal{H}_n\) and \(\text{supp}(f\mathcal{H}_n) = M\). Then \(f = \text{const}\).

As was mentioned above, this corollary was well-known in case of smooth \(f\) but was not known in case of general locally integrable \(f\).

In [KK18] it was shown that if a \((X, d, m)\) is RCD\((K, n)\) and has curvature bounded above then \(X\) is RCD\((K, n)\) and if in addition \(m = \mathcal{H}_n\) then \(X\) is Alexandrov with two sided curvature bounds. Combined with Theorem 1.1 this implies that the same remains true if the assumption on the measure is weakened to \(m \ll \mathcal{H}_n\).

**Corollary 1.3.** Let \(n \geq 2\) and let \((X, d, f\mathcal{H}_n)\) where \(f = L^1_{\text{loc}}\) with respect to \(\mathcal{H}_n\) and \(\text{supp}(f\mathcal{H}_n) = X\) be a complete metric measure space which is \(CBA(\kappa)\) (has curvature bounded above by \(\kappa\) in the sense of Alexandrov) and satisfies \(CD(K, n)\). Then \(X\) is RCD\((K, n)\), \(f = \text{const}\), \(\kappa(n - 1) \geq K\), and \((X, d)\) is an Alexandrov space of curvature bounded below by \(K - \kappa(n - 2)\).

**Remark 1.4.** Note that since a space \((X, d, f\mathcal{H}_n)\) satisfying the assumptions of Theorem 1.1 is automatically RCD\((K, n)\), as was remarked in [DPG18] it follows from the results of [KM18] that \(n\) must be an integer.

Bakry’s proof for smooth manifolds does not easily generalize to a non-smooth context. But let us describe a strategy that does generalize to RCD + CAT spaces.

Assume that \((X, d)\) is induced by a smooth manifold \((M^n, g)\) and the density function \(f\) is smooth and positive such that \((X, d, fm)\) satisfies RCD\((K, n)\). Then, by integration by parts on \((M, g)\) the induced Laplace operator \(L\) is given by

\[
Lu = \Delta u - \langle \nabla \log f, \nabla u \rangle, \quad u \in C^\infty(M),
\]

where \(\Delta u\) is the classical Laplace-Beltrami operator of \((M, g)\) for smooth functions. By a recent result of Han one has for any RCD\((K, n)\) space that the operator \(L\) is equal to the trace of

\[1\text{Here and in all applications by } f = \text{const} \text{ we mean } f = \text{const a.e. with respect to } \mathcal{H}_n.\]
Gigli’s Hessian \cite{Gig18} on the set of $n$-regular points $\mathcal{R}_n$. Hence, after one identifies the trace of Gigli’s Hessian with the Laplace-Beltrami operator $\Delta$ of $M$ (what is true on $(M^n, g)$), one obtains immediately that $\nabla \log f = 0$. If $M$ is connected, this yields the claim.

The advantage of this approach is that it does not involve the Ricci curvature tensor and in non-smooth context one might follow the same strategy. However, we have to overcome several difficulties that arise from the non-smoothness of the density function $f$ and of the space $(X, d, m)$.

In particular, we apply the recently developed $DC$-calculus by Lytchak-Nagano for spaces with upper curvature bounds to show that on the regular part of $X$ the Laplace operator with respect to $\mathcal{H}_n$ is equal to the trace of the Hessian. We also show that the combination of CD and CAT condition implies that $f$ is locally semiconcave \cite{KK18} and hence locally Lipschitz on the regular part of $X$. This allows us to generalize the above argument for smooth Riemannian manifolds to the general case.

In section 2 we provide necessary preliminaries. We present the setting of $RCD$ spaces and the calculus for them. We state important results by Mondino-Cavalletti (Theorem 2.11), Han (Theorem 2.11) and Gigli (Theorem 2.7, Proposition 2.9). We also give a brief introduction to the calculus of $BV$ and $DC$ function for spaces with upper curvature bounds.

In section 3 we develop a structure theory for general $RCD + CAT$ spaces where we adapt the $DC$-calculus of Lytchak-Nagano \cite{LN18}. This might be of independent interest.

Finally, in section 4 we prove our main theorem following the above idea.

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2. Preliminaries

2.1. Curvature-dimension condition. A metric measure space is a triple $(X, d, m)$ where $(X, d)$ is a complete and separable metric space and $m$ is a locally finite measure.

$\mathcal{P}^2(X)$ denotes the set of Borel probability measures $\mu$ on $(X, d)$ such that $\int_X d(x_0, x)^2 d\mu(x) < \infty$ for some $x_0 \in X$ equipped with $L^2$-Wasserstein distance $W_2$. The sub-space of $m$-absolutely continuous probability measures in $\mathcal{P}^2(X)$ is denoted by $\mathcal{P}^2(X, m)$.

The $N$-Rényi entropy is

$S_N(\cdot|m) : \mathcal{P}^2_0(X) \to (-\infty, 0], \quad S_N(\mu|m) = - \int \rho^{1 - \frac{1}{N}} d\mu$ if $\mu = \rho m$, and 0 otherwise.

$S_N$ is lower semi-continuous, and $S_N(\mu) \geq -m(\text{supp } \mu)^{\frac{1}{N}}$ by Jensen’s inequality.

For $\kappa \in \mathbb{R}$ we define

$$
\cos_\kappa(x) = \begin{cases} 
\cosh(\sqrt{|\kappa|} x) & \text{if } \kappa < 0 \\
1 & \text{if } \kappa = 0 \\
\cos(\sqrt{|\kappa|} x) & \text{if } \kappa > 0
\end{cases}
$$

$$
\sin_\kappa(x) = \begin{cases} 
\sinh(\sqrt{|\kappa|} x) & \text{if } \kappa < 0 \\
x & \text{if } \kappa = 0 \\
\sin(\sqrt{\kappa} x) & \text{if } \kappa > 0
\end{cases}
$$

Let $\pi_\kappa$ be the diameter of a simply connected space form $S^2_\kappa$ of constant curvature $\kappa$, i.e.

$$
\pi_\kappa = \begin{cases} 
\infty & \text{if } \kappa \leq 0 \\
\frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0
\end{cases}
$$

For $K \in \mathbb{R}$, $N \in (0, \infty)$ and $\theta \geq 0$ we define the distortion coefficient as

$$
t \in [0, 1] \mapsto \sigma^{(l)}_{K,N}(\theta) = \begin{cases} 
\frac{\sin_{K/N}(\theta)}{\sin_{K/N}(\theta)} & \text{if } \theta \in \left[0, \pi_{K/N}\right), \\
\infty & \text{otherwise}.
\end{cases}
$$
Note that \( \sigma^{(t)}_{K,N}(0) = t \). For \( K \in \mathbb{R}, N \in [1, \infty) \) and \( \theta \geq 0 \) the modified distortion coefficient is

\[
t \in [0,1] \mapsto \tau^{(t)}_{K,N}(\theta) = \begin{cases} 
0 \quad & \text{if } K > 0 \text{ and } N = 1, \\
\frac{1}{\tau} \left[ \sigma^{(t)}_{K,N-1}(\theta) \right]^{1 - \frac{1}{\theta}} \quad & \text{otherwise}.
\end{cases}
\]

**Definition 2.1** ([Stu06] [LV09] [BS10]). We say \((X,d,m)\) satisfies the curvature-dimension condition \( CD(K,N) \) for \( K \in \mathbb{R} \) and \( N \in [1,\infty) \) if for every \( \mu_0, \mu_1 \in \mathcal{P}_2(X,m) \) there exists an \( L^2\)-Wasserstein geodesic \((\mu_t)_{t \in [0,1]}\) and an optimal coupling \( \pi \) between \( \mu_0 \) and \( \mu_1 \) such that

\[
S_N(\mu_t|m) \leq - \int \left[ \tau^{(1-t)}_{K,N}(d(x,y)) \rho_0(x)^{-\frac{1}{\theta}} + \tau^{(t)}_{K,N}(d(x,y)) \rho_1(y)^{-\frac{1}{\theta}} \right] d\pi(x,y)
\]

where \( \mu_i = \rho_i dm, i = 0,1 \).

**Remark 2.2.** If \((X,d,m)\) is complete and satisfies the condition \( CD(K,N) \) for \( N < \infty \), then \((\text{supp } m,d)\) is a geodesic space and \((\text{supp } m,d,m)\) is \( CD(K,N) \).

In the following we always assume that \( \text{supp } m = X \).

**Remark 2.3.** For the variants \( CD^*(K,N) \) and \( CD^p(K,N) \) of the curvature-dimension condition we refer to [BS10] [EKST15].

2.2. Calculus on metric measure spaces. For further details about this section we refer to [AGS13] [AGS14] [AGS14b] [Gil15].

Let \((X,d,m)\) be a metric measure space, and let \( \operatorname{Lip}(X) \) be the space of Lipschitz functions. For \( f \in \operatorname{Lip}(X) \) the local slope is

\[
\operatorname{Lip}(f)(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x,y)}, \quad x \in X.
\]

If \( f \in L^2(m) \), a function \( g \in L^2(m) \) is called relaxed gradient if there exists sequence of Lipschitz functions \( f_n \) which \( L^2\)-converges to \( f \), and there exists \( h \) such that \( \operatorname{Lip} f_n \) weakly converges to \( h \) in \( L^2(m) \) and \( h \leq g \) m-a.e.. \( g \in L^2(m) \) is called the minimal relaxed gradient of \( f \) and denoted by \( |\nabla f| \). If \( f \) is a relaxed gradient and minimal w.r.t. the \( L^2\)-norm amongst all relaxed gradients. The space of \( L^2\)-Sobolev functions is then

\[
W^{1,2}(X) := D(\operatorname{Ch}^X) := \left\{ f \in L^2(m) : \int |\nabla f|^2 dm < \infty \right\}.
\]

\( W^{1,2}(X) \) equipped with the norm \( \|f\|^2_{W^{1,2}}(X) = \|f\|^2_{L^2} + \|\nabla f\|^2_{L^2} \) is a Banach space. If \( W^{1,2}(X) \) is a Hilbert space, we say \((X,d,m)\) is infinitesimally Hilbertian. In this case we can define

\[
(f,g) \in W^{1,2}(X)^2 \mapsto \langle \nabla f, \nabla g \rangle := \frac{1}{4} |\nabla(f + g)|^2 - \frac{1}{4} |\nabla(f-g)|^2 \in L^1(m).
\]

Assuming \( X \) is locally compact, if \( U \) is an open subset of \( X \), we say \( f \in W^{1,2}(X) \) is in the domain \( D(\Delta,U) \) of the measure valued Laplace \( \Delta \) on \( U \) if there exists a signed Radon functional \( \Delta f \) on the set of Lipschitz function \( g \) with bounded support in \( U \) such that

\[
\int \langle \nabla g, \nabla f \rangle dm = - \int g \Delta f.
\]

If \( U = X \) and \( \Delta f = [\Delta f]_{ac} dm \) with \( [\Delta f]_{ac} \in L^2(m) \), we write \( [\Delta f]_{ac} =: \Delta f \) and \( D(\Delta,X) = D_{L^2(m)}(\Delta) \). \( [\Delta f]_{ac} \) denotes the \( m \)-absolutely continuous part in the Lebesgue decomposition of a Borel measure \( \mu \). If \( V \) is any subspace of \( L^2(m) \) and \( f \in D_{L^2(m)}(\Delta) \) with \( \Delta f \in V \), we write \( f \in D_{V}(\Delta) \).

**Theorem 2.4** (Cavalletti-Mondino, [CM18]). Let \((X,d,m)\) be an essentially non-branching \( CD(K,N) \) space for some \( K \in \mathbb{R} \) and \( N > 1 \). For \( p \in X \) consider \( d_p = d(p,\cdot) \) and the associated disintegration \( \mu = \int_Q h_{\alpha} \mathcal{H}^1|_{X_\alpha} \rho(d\alpha) \).
Remark 2.5. The sets $X_\alpha$ in the previous disintegration are geodesic segments $[a(X_\alpha), p]$ with initial point $a(X_\alpha)$ and endpoint $p$. In particular, the set of points $q \in X$ such that there exists a geodesic connecting $p$ and $q$ that is extendible beyond $q$, is a set of full measure.

Definition 2.6 (AGS14b, Gig15). A metric measure space $(X, d, m)$ satisfies the Riemannian curvature-dimension condition $RCD(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty]$ if it satisfies a curvature-dimension conditions $CD(K, N)$ and is infinitesimally Hilbertian.

In Gigli introduced a notion of Hess $f$ in the context of $RCD$ spaces. Hess $f$ is tensorial and defined for $f \in W^{2,2}(X)$ that is the second order Sobolev space. An important property of $W^{2,2}(X)$ that we will need in the following is

Theorem 2.7 (Corollary 3.3.9 in Gig18, Sav14). $D_{L^2(m)}(\Delta) \subset W^{2,2}(X)$.

Remark 2.8. The closure of $D_{L^2(m)}(\Delta)$ in $W^{2,2}(X)$ is denoted $H^{2,2}(X)$ [Gig18, Proposition 3.3.18].

The next proposition [Gig18, Proposition 3.3.22 i)] allows to compute the Hess $f$ explicitly.

Proposition 2.9. Let $f, g_1, g_2 \in H^{2,2}(X)$. Then $\langle \nabla f, \nabla g_i \rangle \in W^{1,2}(X)$, and

\begin{align}
(3) \quad 2 \text{Hess } f(\nabla g_1, \nabla g_2) = \langle \nabla g_1, \nabla (\nabla f, \nabla g_2) \rangle + \langle \nabla g_2, \nabla (\nabla f, \nabla g_1) \rangle + \langle \nabla f, \nabla \langle \nabla g_1, \nabla g_2 \rangle \rangle
\end{align}

holds m.a.e. where the two sides in this expression are well-defined in $L^0(m)$.

Theorem 2.10 (BS18). Let $(X, d, m)$ be a metric measure space satisfying $RCD(K, N)$ with $N < \infty$. Then, there exist $n \in \mathbb{N}$ and such that set of $n$-regular points $R_n$ has full measure.

Theorem 2.11 (Han18). Let $(X, d, m)$ be as in the previous theorem. If $N = n \in \mathbb{N}$, then for any $f \in D_\infty$ we have that $\Delta f = \text{tr Hess } f$ m.a.e. More precisely, if $B \subset R_n$ is a set of finite measure and $(e_i)_{i=1,...,n}$ is a unit orthogonal basis on $B$, then

$$\Delta f|_B = \sum_{i=1}^n \text{Hess } f(e_i, e_i)1_B =: [\text{tr Hess } f]|_B.$$ 

Corollary 2.12. Let $(X, d, m)$ be a metric measure space as before. If $f \in D_{L^2(m)}(\Delta)$, we have that $\Delta f = \text{tr Hess } f$ m.a.e. in the sense of the previous theorem.

Theorem 2.13 (GP16, KM18, DPMR17). Let $(X, d, m)$ be an $RCD(K, N)$ for some $K \in \mathbb{R}$ and $N \in (0, \infty)$. Denote

$$R^*_n := \left\{ x \in R_k : \exists \lim_{r \to 0^+} m(B_r(x) \cap K \omega_k) \in (0, \infty) \right\}.$$ 

Then, $m(R_n \backslash R^*_n) = 0$, there exists $n$ such that $R^*_n$ has full measure, and $m|_{R^*_n}$ and $H^n|_{R^*_n}$ are mutually absolutely continuous and

$$\lim_{r \to 0^+} \frac{m(B_r(x))}{\omega_n r^n} = \frac{dm|_{R^*_n}}{dH^n|_{R^*_n}}(x) = \theta(x)$$

for m.a.e. $x \in R^*_n$. Hence $\theta$ is locally in $L^1(H^n)$.
2.3. Spaces with upper curvature bounds. We will assume familiarity with the notion of $CAT(\kappa)$ spaces. We refer to [BB10], [BH99] or [KK18] for the basics of the theory.

Definition 2.14. Given a point $p$ in a $CAT(\kappa)$ space $X$ we say that two unit speed geodesics starting at $p$ define the same direction if the angle between them is zero. This is an equivalence relation by the triangle inequality for angles and the angle induces a metric on the set $\Sigma_p^X$ of equivalence classes. The metric completion $\Sigma_1(X)$ is called the space of geodesic directions at $p$. The Euclidean cone $C(\Sigma_p^X)$ is called the geodesic tangent cone at $p$ and will be denoted by $T_p^X$.

The following theorem is due to Nikolaev [BH99] Theorem 3.19:

Theorem 2.15. $T_p^X$ is $CAT(0)$ and $\Sigma_p^X$ is $CAT(1)$.

Note that this theorem in particular implies that $T_p^X$ is a geodesic metric space which is not obvious from the definition. More precisely, it means that each path component of $\Sigma_p^X$ is $CAT(1)$ (and hence geodesic) and the distance between points in different components is $\pi$. Note however, that $\Sigma_p^X$ itself need not be path connected.

2.4. BV-functions and DC-calculus. Recall that a function $g : V \subset \mathbb{R}^n \to \mathbb{R}$ of bounded variation $(BV)$ admits a derivative in the distributional sense [EG15] Theorem 5.1 that is a signed vector valued Radon measure $[Dg] = (\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}) = [Dg]_{ac} + [Dg]_s$. Moreover, if $g$ is $BV$, then it is $L^1$-differentiable [EG15] Theorem 6.1] a.e. with $L^1$-derivative $[Dg]_{ac}$, and approximately differentiable a.e. [EG15] Theorem 6.4] with approximate derivative $D^{ap}g = (\frac{\partial^{ap}g}{\partial x_1}, \ldots, \frac{\partial^{ap}g}{\partial x_n})$ that coincides almost everywhere with $[Dg]_{ac}$. The set of $BV$-functions $BV(V)$ on $V$ is closed under addition and multiplication [Per95 Section 4]. We’ll call $BV$ functions $BV_0$ if they are continuous.

Remark 2.16. In [Per95] and [AB18] $BV$ functions are called $BV_0$ if they are continuous away from an $H_{n-1}$-negligible set. However, for the purposes of the present paper it will be more convenient to work with the more restrictive definition above.

Then for $f, g \in BV_0(V)$ we have

\begin{equation}
\frac{\partial f g}{\partial x_i} = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}
\end{equation}

as signed Radon measures [Per95 Section 4, Lemma]. By taking the $L^n$-absolutely continuous part of this equality it follows that (4) also holds a.e. in the sense of approximate derivatives. In fact, it holds at all points of approximate differentiability of $f$ and $g$. This easily follows by a minor variation of the standard proof that $d(fg) = fdg + gdf$ for differentiable functions.

A function $f : V \subset \mathbb{R}^n \to \mathbb{R}$ is called a $DC$–function if in a small neighborhood of each point $x \in V$ one can write $f$ as a difference of two semi-convex functions. The set of $DC$–functions on $V$ is denoted by $DC(V)$ and contains the class $C^{1,1}(V)$. $DC(V)$ is closed under addition and multiplication. The first partial derivatives $\frac{\partial f}{\partial x_i}$ of a $DC$–function $f : V \to \mathbb{R}$ are $BV$, and hence the second partial derivatives $\frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j}$ exist as signed Radon measure that satisfy

\begin{equation}
\frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}
\end{equation}

[EG15 Theorem 6.8], and hence

\begin{equation}
\frac{\partial^{ap}}{\partial x_i} \frac{\partial f}{\partial x_j} = \frac{\partial^{ap}}{\partial x_j} \frac{\partial f}{\partial x_i} \quad \text{a.e. on } V.
\end{equation}

A map $F : V \to \mathbb{R}^l$, $l \in \mathbb{N}$, is called a $DC$–map if each coordinate function $F_i$ is $DC$. The composition of two $DC$–maps is again $DC$. A function $f$ on $V$ is called $DC_0$ if it’s $DC$ and $C^1$.

Let $(X, d)$ be a geodesic metric space. A function $f : X \to \mathbb{R}$ is called a $DC$–function if it can be locally represented as the difference of two Lipschitz semi-convex functions. A map $F : Z \to Y$
between metric spaces $Z$ and $Y$ that is locally Lipschitz is called a $DC$-map if for each $DC$-function $f$ that is defined on an open set $U \subset Y$ the composition $f \circ F$ is $DC$ on $F^{-1}(U)$. In particular, a map $F: Z \to \mathbb{R}^l$ is $DC$ if and only if its coordinates are $DC$. If $F$ is a bi-Lipschitz homeomorphism and its inverse is $DC$, we say $F$ is a $DC$-isomorphism.

2.5. $DC$-coordinates in $CAT$-spaces. The following was developed in [LN18] based on previous work by Perelman [Per95].

Assume $(X, d)$ is a $CAT$-space, let $p \in X$ such that there exists an open neighborhood $\hat{U}$ of $p$ that is homeomorphic to $\mathbb{R}^n$. It is well known (see e.g. [KK18] Lemma 3.1) that this implies that geodesics in $\hat{U}$ are locally extendible.

Suppose $T^a_p X \cong \mathbb{R}^n$.

Then, there exist $DC$ coordinates near $p$ with respect to which the distance on $\hat{U}$ is induced by a $BV$ Riemannian metric $g$.

More precisely, let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be points near $p$ such that $d(p, a_i) = d(p, b_i) = r$, $p$ is the midpoint of $[a_i, b_i]$ and $\angle a_i p a_j = \pi/2$ for all $i \neq j$ and all comparison angles $\angle a_i p a_j, \angle a_i p b_j, \angle b_i p b_j$ are sufficiently close to $\pi/2$ for all $i \neq j$.

Let $x: \hat{U} \to \mathbb{R}^n$ be given by $x = (\langle x, \ldots, x \rangle) = (d(\cdot, a_1), \ldots, d(\cdot, a_n))$.

Then by [LN18] Corollary 11.12 for any sufficiently small $0 < \varepsilon < \pi/4$ the restriction $x|_{B_{0}(p)}$ is Lipschitz onto an open subset of $\mathbb{R}^n$. Let $U = B_{\varepsilon}(p)$ and $V = x(U)$. By [LN18] Proposition 14.4 $x: U \to V$ is a $DC$-equivalence in the sense that $h: U \to \mathbb{R}$ is $DC$ iff $h \circ x^{-1}$ is $DC$ on $V$.

Further, the distance on $U$ is induced by a $BV$ Riemannian metric $g$ which in coordinates is given by a $2$-tensor $g^{ij}(p) = \cos \alpha_{ij}$ where $\alpha_{ij}$ is the angle at $p$ between geodesics connecting $p$ and $a_i$ and $a_j$ respectively. By the first variation formula $g^{ij}$ is the derivative of $d(a_i, \gamma(t))$ at $0$ where $\gamma$ is the geodesic with $\gamma(0) = p$ and $\gamma(1) = a_j$. Since $\gamma(a_i, \cdot)$, $i = 1, \ldots, n$, are Lipschitz, $g^{ij}$ is in $L^\infty$. We denote $\langle v, w \rangle_g(p) = g^{ij}(p)v_iw_j$ the inner product of $v, w \in \mathbb{R}^n$ at $p$. $g^{ij}$ induces a distance function $d_g$ on $V$ such that $x$ is a metric space isomorphism for $\varepsilon > 0$ sufficiently small.

If $u$ is a Lipschitz function on $U$, $u \circ x^{-1}$ is a Lipschitz function on $V$, and therefore differentiable $\mathcal{L}^n$-a.e. in $V$ by Rademacher’s theorem. Hence, we can define the gradient of $u$ at points of differentiability of $u$ in the usual way as the metric dual of its differential. Then the usual Riemannian formulas hold and $\nabla u = g^{ij}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}$ and $|\nabla u|^2_g = g^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$ a.e.

3. Structure theory of $RCD+CAT$ spaces

In this section we study metric measure spaces $(X, d, m)$ satisfying

\[(6) \quad (X, d, m) \text{ is } CAT(\kappa) \text{ and satisfies the conditions } RCD(K, N) \text{ for } 1 \leq N < \infty, K, \kappa < \infty.\]

The following result was proved in [KK18]

**Theorem 3.1** ([KK18]). Let $(X, d, m)$ satisfy $CD(K, N)$ for $1 \leq N < \infty, K, \kappa \in \mathbb{R}$. Then $X$ is infinitesimally Hilbertian. In particular, $(X, d, m)$ satisfies $RCD(K, N)$.

**Remark 3.2.** It was shown in [KK18] that the above theorem also holds if the $CD(K, N)$ assumption in (6) is replaced by $CD^*(K, N)$ or $CD^e(K, N)$ conditions (see [KK18] for the definitions). Moreover, in a recent paper [MGPS18] Di Marino, Gigli, Pasqualetto and Soultanis show that a $CAT(\kappa)$ space with any Radon measure is infinitesimally Hilbertian. For these reasons (6) is equivalent to assuming that $X$ is $CAT(\kappa)$ and satisfies one of the assumptions $CD(K, N), CD^*(K, N)$ or $CD^e(K, N)$ with $1 \leq N < \infty, K, \kappa < \infty$.

In [KK18] we also established the following property of spaces satisfying (6):

**Proposition 3.3** ([KK18]). Let $X$ satisfy (6). Then $X$ is non-branching.

Next we prove

**Proposition 3.4.** Let $X$ satisfy (6). Then for almost all $p \in X$ it holds that $T_p^a X \cong \mathbb{R}^k$ for some $k \leq N$. 

Remark 3.5. Note that from the fact that $X$ is an $RCD$ space it follows that $T_pX$ is an Euclidean space for almost all $p \in X$ [GMR13]. However, at this point in the proof we don’t know if $T_pX \cong T^\#_pX$ at all such points (we expect this to be true for all $p$).

Proof. First that by the $\text{CAT}$ condition, geodesics of length less than $\pi_\kappa$ in $X$ are unique. Moreover, since $X$ is nonbranching and $CD$, for any $p \in X$ the set $E_p$ of points $q$, such that the geodesic which connects $p$ and $q$ is not extendible, has measure zero (Remark [2.5]).

Let $A = \{p_i\}_{i=1}^\infty$ be a countable dense set of points in $X$, and let $C = \bigcup_{i \in \mathbb{N}} E_{p_i}$. For any $q \in X \setminus C$ and any $i$ with $d(p_i, q) < \pi_\kappa$ the geodesic $[p_i, q]$ can be extended slightly past $q$. Since $A$ is dense this implies that for any $q \in X \setminus C$ there is a dense subset in $T^\#_qX$ consisting of directions $\nu$ which have "opposites" (i.e. making angle $\pi$ with $\nu$).

For every $p \in X$ and every tangent cone $T_pX$ the geodesic tangent cone $T^\#_pX$ is naturally a closed convex subset of $T_pX$. Since $X$ is $RCD$ this means that for almost all $p$ the geodesic tangent cone $T^\#_pX$ is a convex subset of a Euclidean space. Thus, for almost all $p \in X$ it holds that $T^\#_pX$ is a convex subset in $\mathbb{R}^m$ for some $m \leq N$, is a metric cone over $\Sigma^m_\kappa$ and contains a dense subset of points with "opposites" also in $T^\#_pX$. In particular, $\Sigma^m_\kappa$ is a convex subset of $\mathbb{S}^m$. Since a closed convex subset of $\mathbb{S}^m$ is either $\mathbb{S}^k$ with $k \leq m$ or has boundary this means that for any such $p$ $T^\#_pX$ is isometric to a Euclidean space of dimension $k \leq m$.

\begin{proposition}
Let $X$ satisfy $\mathbf{[6]}$. 

i) Let $p \in X$ satisfy $T^\#_pX \cong \mathbb{R}^m$ for some $m \leq N$.

Then an open neighborhood $W$ of $p$ is homeomorphic to $\mathbb{R}^m$.

ii) If an open neighborhood $W$ of $p$ is homeomorphic to $\mathbb{R}^m$ then for any $q \in W$ it holds that $T^\#_qX \cong T^\#_pX \cong \mathbb{R}^m$.

Moreover, for any compact set $C \subset W$ there is $\varepsilon = \varepsilon(C) > 0$ such that every geodesic starting in $C$ can be extended to length at least $\varepsilon$.

\end{proposition}

Proof. Let us first prove part i). Suppose $T^\#_pX \cong \mathbb{R}^m$. By [Kra11], Theorem A there is a small $R > 0$ such that $B_R(p) \setminus \{p\}$ is homotopy equivalent to $\mathbb{S}^{m-1}$. Since $\mathbb{S}^{m-1}$ is not contractible, by [LS07], Theorem 1.5 there is $0 < \varepsilon < \pi_\kappa/2$ such that every geodesic starting $p$ extends to a geodesic of length $\varepsilon$. The natural "logarithm" map $\Phi: \tilde{B}_\varepsilon(p) \to \tilde{B}_\varepsilon(0) \subset T^\#_pX$ is Lipschitz since $X$ is $\text{CAT}(\kappa)$. By the above mentioned result of Lytchak and Schroeder [LS07], Theorem 1.5 $\Phi$ is onto.

We also claim that $\Phi$ is 1-1. If $\Phi$ is not 1-1 then there exist two distinct unit speed geodesics $\gamma_1, \gamma_2$ of the same length $\varepsilon' < \varepsilon$ such that $p = \gamma_1(0) = \gamma_2(0)$, $\gamma_1'(0) = \gamma_2'(0)$ but $\gamma_1(\varepsilon') \neq \gamma_2(\varepsilon')$.

Let $v = \gamma_1'(0) = \gamma_2'(0)$. Since $T^\#_pX \cong \mathbb{R}^m$ the space of directions $T^\#_pX$ contains the "opposite" vector $-v$. Then there is a geodesic $\gamma_3$ of length starting at $p$ in the direction $-v$. Since $X$ is $\text{CAT}(\kappa)$ and $2\varepsilon < \pi_\kappa$, the concatenation of $\gamma_3$ with $\gamma_1$ is a geodesic and the same is true for $\gamma_2$. This contradicts the fact that $X$ is nonbranching.

Thus, $\Phi$ is a continuous bijection and since both $\tilde{B}_\varepsilon(p)$ and $\tilde{B}_\varepsilon(0)$ are compact and Hausdorff it’s a homeomorphism. This proves part i).

Let us now prove part ii). Suppose an open neighborhood $W$ of $p$ is homeomorphic to $\mathbb{R}^m$. By [KK18], Lemma 3.1 or by the same argument as above using [Kra11] and [LS07], for any $q \in W$ all geodesics starting at $q$ can be extended to length at least $\varepsilon(q) > 0$. Therefore $T^\#_qX \cong T^\#_pX$. By the splitting theorem $T_qX \cong \mathbb{R}^l$ where $l = l(q) \leq N$ might a priori depend on $q$. However, using part i) we conclude that an open neighbourhood of $q$ is homeomorphic to $\mathbb{R}^{l(q)}$. Since $W$ is homeomorphic to $\mathbb{R}^m$ this can only happen if $l(q) = m$.

The last part of ii) immediately follows from above and compactness of $C$.

3.1. $\text{DC}$-coordinates in $\text{RCD} + \text{CAT}$-spaces. Let $X^\#_{p,q}$ be the set of points $p$ in $X$ with $T_pX \cong T^\#_pX \cong \mathbb{R}^n$. Then by Proposition 3.6 there is an open neighbourhood $\tilde{U}$ of $p$ homeomorphic to $\mathbb{R}^n$.
such that every $q \in \tilde{U}$ also lies in $X^\theta_{\text{reg}}$. In particular, $X^\theta_{\text{reg}}$ is open. Further, geodesics in $\tilde{U}$ are locally extendible by Proposition 3.6.

Thus the theory of Lytchak–Nagano from [LNT18] applies, and let $x : U \to V$ with $U = B_\alpha(p) \subset \tilde{U}$ be $DC$-coordinates as in Subsection 2.5. The pushforward of the Hausdorff measure $\nu^n$ on $U$ under $x$ coordinates is given by $\sqrt{|g|} \nu$ where $|g|$ is the determinant of $g_{ij}$. Consequently, the map $x : (U, d, H_n) \to (V, dg, \sqrt{|g|} \nu^n)$ is a metric-measure isomorphism.

With a slight abuse of notations we will identify these metric-measure spaces as well as functions on them, i.e. we will identify any function $u$ on $U$ with $u \circ x^{-1}$ on $V$.

**Lemma 3.7.** Angles between geodesics in $U$ are continuous. That is if $q_i \to q \in U, [q_is_i] \to [qt]$ are converging sequences with $q \neq s, q \neq t$ then $\angle s_i q_i t_i \to \angle sqt$.

**Proof.** Without loss of generality we can assume that $q_i \in U$ for all $i$. Let $\alpha_i = \angle s_i q_i t_i, \alpha = \angle sqt$.

Let $\{\alpha_i\}$ be a converging subsequence and let $\bar{\alpha} = \lim_{k \to \infty} \alpha_i$. Then by upper semicontinuity of angles in $CAT(\kappa)$ spaces it holds that $\alpha \geq \bar{\alpha}$. We claim that $\alpha = \bar{\alpha}$.

By Proposition 3.6 we can extend $[s_iq_i]$ past $q_i$ as geodesics a definite amount $\delta$ to geodesics $[s_i z_i]$. Let $\beta_i = \angle z_i q_i t_i$. By possibly passing to a subsequence of $\{i_k\}$ we can assume that $[s_k z_k] \to [sz]$. Let $\beta = \angle szt$. Then since all spaces of directions $T^n_q X$ and $T^n_U X$ are Euclidean by Proposition 3.6 we have that $\alpha_i + \beta_i = \alpha + \beta = \pi$ for all $i$. Again using semicontinuity of angles we get that $\beta \geq \beta_i$.

We therefore have $\pi = \alpha + \beta \geq \bar{\alpha} + \bar{\beta} = \pi$.

Hence all the inequalities above are equalities and $\alpha = \bar{\alpha}$. Since this holds for an arbitrary converging subsequence $\{\alpha_i\}$ it follows that $\lim_{i \to \infty} \alpha_i = \alpha$.

Let $\tilde{A}$ be the algebra of functions of the form $\varphi(f_1, \ldots, f_m)$ where $f_i = d(\cdot, q_i)$ for some $q_1, \ldots, q_m$ with $|q_i| > \varepsilon$ and $\varphi$ is smooth. Together with the first variation formula for distance functions Lemma 3.7 implies that for any $u, h \in \mathcal{A}$ it holds that $\langle \nabla u, \nabla h \rangle_g$ is continuous on $U$. In particular, $g^{ij} = \langle \nabla x_i, \nabla x_j \rangle_g$ is continuous and hence $g$ is $B\nu_0$ and not just BV.

Furthermore, since $\frac{\partial}{\partial x_i} = \sum_j g^{ij} \nabla x_j$ where $g^{ij}$ is the pointwise inverse of $g^{ij}$, Lemma 3.7 also implies that any $u \in \mathcal{A}$ is $C^1$ on $V$. Hence, any such $u$ is $DC_0$ on $V$.

Recall that for a Lipschitz function $u$ on $V$ we have two a-priori different notions of the norm of the gradient defined $m$-a.e.: the "Riemannian" norm of the gradient $|\nabla u|_g^2 = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$ and the minimal weak upper gradient $|\nabla u|$ when $u$ is viewed as a Sobolev functions in $W^{1,2}(m)$. We observe that these two notions are equivalent.

**Lemma 3.8.** Let $u, h : U \to \mathbb{R}$ be Lipschitz functions. Then $|\nabla u| = |\nabla u|_g, |\nabla h| = |\nabla h|_g$ m-a.e. and $\langle \nabla u, \nabla h \rangle = \langle \nabla u, \nabla h \rangle_g$ m-a.e.

In particular, $g^{ij} = \langle \nabla x_i, \nabla x_j \rangle_g = \langle \nabla x_i, \nabla x_j \rangle_g$ m-a.e.

**Proof.** First note that since both $\langle \nabla u, \nabla h \rangle_g$ and $\langle \nabla u, \nabla h \rangle_g$ satisfy the parallelogram rule, it’s enough to prove that $|\nabla u| = |\nabla u|_g$ a.e.

Recall that $g^{ij}$ is continuous on $U$. Fix a point $p$ where $u$ is differentiable. Then

$$\text{Lip} u(p) = \limsup_{q \to p} \frac{|u(p) - u(q)|}{d(p, q)} = \limsup_{q \to p} \frac{|u(p) - u(q)|}{|p - q|_{g(p)}} = \sup_{|v| = 1} D_{v} u = \sup_{|v| = 1} \langle v, \nabla u \rangle_{g(p)} = |\nabla u|_{g(p)}.$$

In the second equality we used that $d$ is induced by $g^{ij}$, and that $g^{ij}$ is continuous. Since $(U, d, m)$ admits a local 1-1 Poincaré inequality and is doubling, the claim follows from [Che99] where it is proved that for such spaces $\text{Lip} u = |\nabla u|$ a.e.

In view of the above Lemma from now on we will not distinguish between $|\nabla u|$ and $|\nabla u|_g$ and between $\langle \nabla u, \nabla h \rangle$ and $\langle \nabla u, \nabla h \rangle_g$. 


4.1. Proposition 3.9. If \( u \in W^{1,2}(m) \cap BV(U) \), then \(|\nabla u|^2 = g^{ij} \frac{\partial^p u}{\partial x_i} \frac{\partial^p u}{\partial x_j} \) m.a.e. .

Proof. We choose a set \( S \subset U \) of full measure such that \( u \) and \(|\nabla u|\) are defined pointwise on \( S \) and \( u \) is approximately differentiable at every \( x \in S \). Since \( u \) is \( BV(U) \), for \( \eta > 0 \) there exist \( \hat{u}_\eta \in C^1(U) \) such that for the set

\[
B_\eta = \{ x \in S : u(x) \neq \hat{u}_\eta(x), D^p u(x) \neq D \hat{u}(x) \} \cap S
\]
one has \( m(B_\eta) \leq \eta \). By Lemma 3.8, we choose a set \( \eta \)

\[
\text{Remark 4.1.}
\]

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\[
\text{Remark 4.1.}
\]

by Lemma 3.8.

Now, we pick a sequence \( \eta_k \) for \( k \in \mathbb{N} \) such that \( \sum_{k=1}^{\infty} \eta_k < \infty \). Then, by the Borel-Cantelli Lemma the set

\[
B = \{ x \in S : \exists \text{ infinitely many } k \in \mathbb{N} \text{ s.t. } x \in B_{\eta_k} \}
\]
is of \( m \)-measure 0. Consequently, for \( x \in A = S \setminus B \) we can pick a \( k \in \mathbb{N} \) such that \( x \in A_{\eta_k} \subset S \). It follows

\[
|\nabla u|^2(x) = g^{ij} \frac{\partial^p u}{\partial x_i} \frac{\partial^p u}{\partial x_j}(x) \forall x \in S
\]

and hence m.a.e. .

\[
\square
\]

4. Density functions

4.1. Noncollapsed case. Let \( (X, d, f\mathcal{H}_n) \) be \( RCD(K, n) \) and \( CAT(\kappa) \) where \( 0 \leq f \in L^1(\mathcal{H}^n) \).

Remark 4.1. If \( (X, d, m) \) is a weakly non-collapsed \( RCD \)-space in the sense of \([DPG18]\) or a space satisfying the generalized Bishop inequality in the sense of \([Kit17]\) and if \( (X, d) \) is \( CAT(\kappa) \), the assumptions are satisfied by \([DPG18]\) Theorem 1.10].

Following Gigli and De Philippis \([DPG18]\) for any \( x \in X \) we consider the monotone quantity \( \frac{m(B_{r}(x))}{m(B_{r}(y))} \), which is non increasing in \( r \) by the Bishop-Gromov volume comparison. Let \( \theta_{n,r}(x) = \frac{m(B_{r}(x))}{m(B_{r}(y))} \). Consider the density function \( \theta_n(x) = \lim_{r \to 0} \theta_{n,r}(x) = \lim_{r \to 0} \frac{m(B_{r}(x))}{m(B_{r}(y))} \).

Since \( n \) is fixed throughout the proof we will drop the subscripts \( n \) and from now on use the notations \( \theta(x) \) and \( \theta_r(x) \) for \( \theta_n(x) \) and \( \theta_{n,r}(x) \) respectively.

By Propositions 3.4, 3.6, and \([DPG18]\) Theorem 1.10] we have that for almost all \( p \in X \) it holds that \( T_pX \cong T_p^0X \cong \mathbb{R}^n \) and \( \theta(x) = f(x) \).

Therefore we can and will assume from now on that \( f = \theta \) everywhere.

Remark 4.2. Monotonicity of \( r \to \frac{m(B_{r}(x))}{m(B_{r}(y))} \) immediately implies that \( f(x) = \theta(x) > 0 \) for all \( x \).

Let \( x \in X^{\text{reg}} \). Then \( T_pX \cong \mathbb{R}^m \) for some \( m \leq n \). We claim that \( m = n \). By Proposition 3.6, \( X^{\text{reg}} \) is an \( m \)-manifold near \( p \) and by section 2.5 DC coordinates near \( p \) give a biLipschitz homeomorphism of an open neighborhood of \( p \) onto an open set in \( \mathbb{R}^m \). Since \( m = f\mathcal{H}_n \) this can only happen if \( m = n \).

Lemma 4.3. \([KK18]\) Lemma 5.4] \( \theta = f \) is semiconcave on \( X \).

Corollary 4.4. \( \theta = f \) is locally Lipschitz near any \( p \in X^{\text{reg}} \).
Proof. First observe that semiconcavity of \( \theta \), the fact that \( \theta \geq 0 \) and local extendability of geodesics on \( X_{\text{reg}}^g \) imply that \( \theta \) must be locally bounded on \( X_{\text{reg}}^g \). Now the corollary becomes an easy consequence of Lemma 3.6, the fact that geodesics are locally extendible a definite amount near \( p \) by Proposition 3.6 and the fact that a semiconcave function on \((0,1)\) is locally Lipschitz. \( \square \)

4.2. General case. Let \((X,d,m)\) be \( RCD(K,N) \) and \( CAT(\kappa) \).

Proposition 4.5. \(- \log \theta : X \to (-\infty, \infty)\) is well-defined and semi-convex.

Proof. There exists a set of full measure \( S \) where \( \theta \) is defined.

We extend \( \theta \) via

\[
[0, \infty] \ni \limsup_{r \to 0} \frac{m(B_r(x))}{\omega_n r^n} = \hat{\theta}(x)
\]
to a function that is defined everywhere on \( X \). In particular, \(- \log \hat{\theta} \) takes values in \([ -\infty, \infty]\).

Let \( \gamma : [0,1] \to X \) be any geodesic in \( X \), and recall that the condition \( RCD(K,N) \) implies the condition \( CD(K,\infty) \). More precisely, the Schanuel-Boltzmann entropy is weakly \( K \)-convex on \( \mathcal{P}^2(X) \). In particular, defining \( \mu_i = (m(B_r(\gamma(0))))^{-1}m|_{B_r(\gamma(0))}, \; i = 0, 1 \), there exists a \( W_2 \)-geodesic \((\mu_i)_{t \in [0,1]} \) in \( \mathcal{P}^2(X) \) such that

\[
\text{Ent} \mu_i \leq (1-t) \text{Ent} \mu_0 + t \text{Ent} \mu_t - \frac{K}{2} t(1-t)W_2(\mu_0, \mu_1)^2.
\]

The left hand side is bounded from below by \(- \log m(A_i) \) where \( A_i \) is the set of all \( t \)-midpoints of geodesic that start in \( B_r(\gamma(0)) \) and end in \( B_r(\gamma(1)) \). Then, by Lemma 5.5 in \[KK18\] \(- \log m(A_i) \) is bounded from below by \(- \log m(B_{r(1+ct^2)}(\gamma(t))) \) for some constant \( c > 0 \) and \( l = d(\gamma(0),\gamma(1)) \). The constant \( c \) only depends on the upper sectional curvature bound.

Hence

\[
- \log m(B_{r(1+ct^2)}(\gamma(t))) \leq \frac{K}{2} t(1-t)W_2(\mu_0, \mu_1)^2
\]

\[
\leq -(1-t) \log m(B_{r(\gamma(0)))} \omega_n r^n - t \log m(B_{r(\gamma(1)))} \omega_n r^n + n \log(1+ct^2).
\]

Now, we note that \( m(B_{r(\gamma(0)))} \omega_n r^n \to \delta_{\gamma(0)}, \; i = 0, 1 \), weakly, and therefore

\[
\liminf_{r \to 0} W_2(\mu_0, \mu_1) \geq W_2(\delta_{\gamma(0)}, \delta_{\gamma(1)}) = d(\gamma(0), \gamma(1)).
\]

Moreover, \( \log(1+ct^2) \leq ct^2 \). It follows for \( t = \frac{1}{2} \)

\[
- \log \hat{\theta}(\gamma(t)) \leq - \frac{1}{2} \log \hat{\theta}(\gamma(0)) - \frac{1}{2} \log \hat{\theta}(\gamma(1)) + \left( \frac{nc}{2} - \frac{K}{4} \right) d(\gamma(0), \gamma(1))^2.
\]

Finally, let \( x \in X \) be arbitrary. Then, there exist a geodesic \( \gamma : [0,1] \to X \) with \( \gamma(0) = x \) such that there exists \( t \in [0,1] \) with \( \hat{\theta}(\gamma(t)) = \theta(\gamma(t)) < \infty \). Otherwise, as consequence of the disintegration in Theorem 2.3 we would obtain a contradiction from Theorem 2.13. Hence \(- \log \hat{\theta}(\gamma(t)) \to -\infty \), and therefore \(- \log \hat{\theta}(x) \to -\infty \) and \( \theta(x) < \infty \). \( \square \)

Corollary 4.6. \(- \log \theta \) is locally Lipschitz near any \( p \in X_{\text{reg}}^g \).

In particular, \(- \log \theta \) is finite and defined everywhere on \( X_{\text{reg}}^g \).

Proof. This follows again by extendability of geodesics on \( X_{\text{reg}}^g \) as in Corollary 4.4. \( \square \)

5. Laplace operator in \( DC \) coordinates

1. Since small balls in spaces with curvature bounded above are geodesically convex, we can assume that \( \text{diam } X \leq \pi_\kappa \). Let \( p \in X, \; x : U \to \mathbb{R}^n \) and \( \bar{\mathcal{A}} \) be as in the previous subsection.
By the same argument as in [Per95, Section 4] (cf. [Pet11], [AB18]) it follows that any \( u \in \tilde{A} \) lies in \( D(\Delta, U, \mathcal{H}_n) \) and the \( \mathcal{H}^n \)-absolutely continuous part of \( \Delta_0 u \) can be computed using standard Riemannian geometry formulas that is

\[
\Delta_0^a(u) = \frac{1}{\sqrt{|g|}} \partial^{ap} \left( g^{ij} \sqrt{|g|} \partial^a u \right)
\]

where \(|g|\) denotes the pointwise determinant of \( g^{ij} \). Here \( \Delta_0 \) denotes the measure valued Laplacian on \((U, d, \mathcal{H}_n)\). Note that \( g, \sqrt{|g|} \) and \( \frac{\partial g}{\partial x_i} \) are BV\(_0\)-functions, and the derivatives on the right are understood as approximate derivatives.

Indeed, w.l.o.g. let \( u \in DC_0(U) \), and let \( v \) be Lipschitz with compact support in \( U \). As before we identify \( u \) and \( v \) with their representatives in \( x \) coordinates. First, we note that, since \( g, \sqrt{|g|} \) and \( \frac{\partial g}{\partial x_i} \) are BV\(_0\), their product is also in BV\(_0\), as well as the product with \( v \). Then, the Leibniz rule (4) for the approximate partial derivatives yields that

\[
\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} = - \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) + \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) \quad \mathcal{L}^n\text{-a.e. .}
\]

Again using (4) we also have that

\[
\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} = - \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) + \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) \text{as measures}
\]

and the absolutely continuous with respect to \( \mathcal{L}^n \) part of this equation is given by the previous identity.

The fundamental theorem of calculus for BV functions (see [EG15, Theorem 5.6]) yields that

\[
\int_V \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) = 0.
\]

Moreover, by Lemma 3.8 \( \langle \nabla v, \nabla u \rangle \) is given in \( x \) coordinates by \( g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \mathcal{L}^n\text{-a.e. .} \)

Combining the above formulas gives that

\[
- \int_V \langle \nabla u, \nabla v \rangle \sqrt{|g|} d\mathcal{L}^n = \int_V \left[ \frac{\partial^{ap}}{\partial x_j} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) - \frac{\partial^{ap}}{\partial x_j} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x_i} v \right) \right] d\mathcal{L}^n
\]

where \( \mu \) is some signed measure such that \( \mu \perp \mathcal{L}^n \). This implies (7).

4. It follows that

\[
\text{Tr Hess}(\chi u)|_{B_\delta(p)} = \text{Hess}(\chi u)(\nabla x_i, \nabla x_j)g_{ij}|_{B_\delta(p)} = \text{Hess}(\chi u)(\nabla x_i, \nabla x_j)g_{ij}|_{B_\delta(p)} = H_{uv}(x_i, x_j)g_{ij} = \Delta_0 u|_{B_\delta(p)}.
\]

for every \( u \in \tilde{A} \) where Hess is the Hessian in the sense of Gigli, and \( H(u) \) denotes the RHS of (7). The first equality in (10) is the definition of \( \text{Tr} \), the second equality is the \( L^\infty \)-homogeneity of the tensor Hess(\( \chi u \)), and the third equality is the identity (4).

Since \( f \) is locally Lipschitz and positive on \( B_\delta(p) \), we can perform the following integration by parts in \( DC_0 \) coordinates. Let \( u \in \tilde{A} \) and let \( g \) be Lipschitz with compact support in \( B_\delta(p) \).
\( \chi u \in D_{L^2(m)}(\Delta) \) implies \( u|_{B_\delta(p)} \in D(\Delta, B_\delta(p)) \). Then

\[
\int_{B_\delta(p)} g \Delta u \, dm = -\int_{B_\delta(p)} \langle \nabla u, \nabla g \rangle \, dm = -\int_{B_\delta(p)} (\nabla u, \nabla g) f \, d\mathcal{H}^n
\]

\[
= -\int_{B_\delta(p)} \langle \nabla u, \nabla (gf) \rangle \, d\mathcal{H}^n + \int_{B_\delta(p)} \langle \nabla u, \nabla \log f \rangle g \, f \, d\mathcal{H}^n
\]

\[
= \int_{B_\delta(p)} (\Delta_0 u + \langle \nabla u, \nabla \log f \rangle) g \, dm
\]

yields

\[ \Delta u = \Delta_0 u + \langle \nabla u, \nabla \log f \rangle \]

on \( B_\delta(p) \) for any \( u \in \bar{A} \). Note again that only \( \chi u \) is in \( D_{L^2(m)}(\Delta) \).

On the other hand, by Corollary 2.12 it holds that \( \Delta(\chi u) = \text{Tr} \text{Hess}(\chi u) \) a.e. Thus

\[ 0 = \text{Tr} \text{Hess}(\chi u)|_{B_\delta(p)} - \text{Tr} \text{Hess}(\chi u)|_{B_\delta(p)} = \Delta u|_{B_\delta(p)} - \Delta_0(\chi u)|_{B_\delta(p)} = \langle \nabla u, \nabla \log f \rangle|_{B_\delta(p)} \]

a.e. for any \( u \in \bar{A} \).

6. WEAKLY NONCOLLAPSED IMPLIES NONCOLLAPSED

5. Therefore \( f \nabla \log f|_{B_\delta(p)} = \nabla f|_{B_\delta(p)} = 0 \). Indeed, since \( f \) is semiconcave, \( f \circ x^{-1} \) is DC by LNTS. Hence \( \nabla f = g^{ij} \frac{\partial f}{\partial x_i} \) is continuous on a set of full measure \( Z \) in \( B_\delta(p) \) since this is true for convex functions on \( \mathbb{R}^n \). Let \( q \in Z \) be a point of continuity of \( \nabla f|_Z \) and \( v = \nabla f(q) \). Assume \( v \neq 0 \). Then due to extendability of geodesics there exists \( z \notin U \) such that \( \nabla d_z(q) = \frac{1}{|v|} v \). Since \( \nabla d_z \) is continuous near \( q \) and \( \nabla f \) is continuous on \( Z \) it follows \( \langle \nabla f, \nabla d_z \rangle \neq 0 \) on a set of positive measure. Hence \( \nabla f|_{B_\delta(p)} = 0 \) and \( f|_{B_\delta(p)} = \text{const} \).

6. We claim that this implies that \( f \) is constant on \( X^\theta_{reg} \). (This is not immediate since we don’t know yet that \( X^\theta_{reg} \) is connected.) Indeed, since \( X \) is essentially nonbranching, radial disintegration of \( m \) centered at \( p \) (Theorem 2.4) implies that for almost all \( q \in X \) the set \([p, q] \cap X^\theta_{reg} \) has full measure in \([p, q] \). It is also open in \([p, q] \) since \( X^\theta_{reg} \) is open.

Suppose \( q \in X^\theta_{reg} \) is as above.

Since \( \theta \) is semiconcave on \( X \) and locally constant on \( X^\theta_{reg} \) it is locally Lipschitz (and hence Lipschitz) on the geodesic segment \([p, q] \). A Lipschitz function on \([0, 1] \) which is locally constant on an open set of full \( L^1 \) measure is constant. Therefore \( \theta \) is constant on \([p, q] \) and hence \( \theta \) is constant on \( X^\theta_{reg} \) which has full measure. Therefore \( f = \theta = \text{const} \) a.e. globally. \( \square \)

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