CD MEETS CAT

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ABSTRACT. We show that if a noncollapsed $CD(K,n)$ space X with $n \geq 2$ has curvature bounded above by κ in the sense of Alexandrov then $K \leq (n-1)\kappa$ and X is an Alexandrov space of curvature bounded below by $K - \kappa(n-2)$. We also show that if a $CD(K,n)$ space Y with finite n has curvature bounded above then it is infinitesimally Hilbertian.

CONTENTS

1. INTRODUCTION

It trivially follows from the definitions of sectional and Ricci curvature that if (M^n, g) is a Riemannian manifold with $n \geq 2$ satisfying $\mathrm{Ric}_M \geq K, \sec_M \leq \kappa$ then $\kappa(n-1) \geq K$ and M also satisfies $\sec_M \geq K - \kappa(n-2)$. The main purpose of this paper is to show that the same holds true for metric measure spaces with intrinsically defined sectional and Ricci curvature bounds.

Theorem 1.1. *Let $n \geq 2$ be a natural number and let $(X, d, \mathcal{H}_n)$ be a complete metric measure space which is $CBA(\kappa)$ (has curvature bounded above by κ in the sense of Alexandrov) and satisfies $CD(K,n)$. Then $\kappa(n-1) \geq K$, and (X,d) is an Alexandrov space of curvature bounded below by $K - \kappa(n-2)$. In particular X is infinitesimally Hilbertian.*

Examples given by manifolds of constant sectional curvature show that the lower curvature bound provided by this theorem is optimal.

Theorem 1.1 shows that X has two sided curvature bounds in Alexandrov sense. By work of Alexandrov, Berestovsky and Nikolaev (see [BN93]) this immediately gives the following corollary:

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Corollary 1.2. *Let $(X, d, \mathcal{H}_n)$ be as in Theorem 1.1 then X is a topological n-dimensional manifold with boundary and $\operatorname{Int} X$ has a canonical open C^3 atlas of harmonic coordinates and a Riemannian metric g which induces d and such that g is in $C^{1,\alpha} \cap W^{2,p}$ in local harmonic charts for every $1 \le p < \infty, 0 < \alpha < 1$.*

Let us comment on the assumptions in the main theorem.

$CD(K, N)$ spaces for $N \in [1, \infty)$ were introduced by Lott and Villani for $K = 0$ in [LV09], and independently by Sturm for general $K \in \mathbb{R}$ in [Stu06b]. Other curvature-dimension conditions are the *reduced curvature-dimension condition* $CD^*(K, N)$ [BS10] and the *entropic curvature dimension condition* $CD^e(K, N)$ [EKS15] that are simpler from an analytical viewpoint. For a Riemannian manifold each condition characterizes lower Ricci curvature bounds. However it is not known if the conditions CD^* or CD^e are in general equivalent to the original one by Sturm, or to each other. Moreover, in general they do not produce sharp estimates in geometric inequalities. Conditions $CD(K, N)$, $CD^*(K, N)$ and $CD^e(K, N)$ are known to be equivalent under the extra assumption that the space is *essentially non-branching* [EKS15, CM16]. In Proposition 6.9 we prove that a $CBA(\kappa)$ space which satisfies any of the conditions $CD(K, N)$, $CD^*(K, N)$ or $CD^e(K, N)$ with $N < \infty$ is *non-branching* and therefore for $CBA(\kappa)$ spaces all these curvature-dimension conditions are equivalent.

In the original version of this paper the main theorem had an extra assumption that X is infinitesimally Hilbertian which might be considered a natural assumption in this setting, and under which all of the previous curvature-dimension conditions are equivalent as well. However, as we show in Theorem 6.2 a space satisfying any of the curvature dimension conditions $CD(K, N)$, $CD^*(K, N)$ or $CD^e(K, N)$, and $CBA(\kappa)$ for $1 \le N < \infty$, $K, \kappa < \infty$ is automatically infinitesimally Hilbertian and hence $RCD(K, N)$. (Note that this includes the case $N = 1$).

Next, we exclude $n = 1$ in the statement of the main theorem because if (X, d, m) is $RCD(K, 1)$ then by [KL16] it is a point or a smooth Riemannian 1-dimensional manifold (possibly with boundary) and thus is an Alexandrov space with curvature bounded below and above without an extra assumption of an a priori upper curvature bound.

Further, some assumptions on the measure m in relation to n in the main theorem are obviously necessary as the following simple example indicates

Example 1.3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x) = 4|x|^2$. Let $X = \overline{B}_{\frac{1}{100}}(0)$. A simple computation shows that $\mathrm{Ric}_f^3 \ge 2$ on X where Ric_f^3 is the 3-Bakry-Emery Ricci tensor of $(X, g_{Eucl}, e^{-f}\mathcal{H}_2)$. Since all balls $B_r(0)$ are convex this easily implies that $(X, d_{Eucl}, e^{-f}\mathcal{H}_2)$ is $RCD(2, 3)$. On the other hand, (X, d_{Eucl}) is obviously $CBA(0)$. Thus, X is $RCD(K, n)$ and $CBA(\kappa)$ with $n = 3, K = 2, \kappa = 0$ but $K > \kappa(n - 1)$.

Note that while the space X constructed in the above example violates the conclusion of Theorem 1.1, it nevertheless *is* an Alexandrov space of curvature bounded below (it obviously has curv ≥ 0), just with a different lower curvature bound that the one claimed in Theorem 1.1. In section 6 we construct an example of a compact $CBA(0)$, $RCD(-100, 3)$ space which is not Alexandrov of curv $\ge \hat{\kappa}$ for any $\hat{\kappa}$ (Example 6.8).

In [DPG17] De Philippis and Gigli (cf. also [Kit17]) considered the class of $RCD(K, n)$ spaces where the background measure is $\mathcal{H}_n$. Following De Philippis and Gigli we will call such spaces *noncollapsed*.

It follows from work of Cheeger–Colding [CC97] that a measured Gromov-Hausdorff limit of a sequence of complete n-dimensional Riemannian manifolds (M_i, p_i) with convex boundary satisfying $\mathrm{Ric}_{M_i} \ge K, \mathrm{vol}(B(p_i, 1)) \ge v > 0$ for some $K \in \mathbb{R}, v > 0$ is a noncollapsed $RCD(K, n)$ space in the above sense. This also follows from [DPG17] where it is shown more generally, that for any $v > 0$ the class of noncollapsed $RCD(K, n)$ spaces (X, d, m, p) satisfying $m(B_1(p)) \ge 1$ is *compact* in the pointed measured Gromov-Hausdorff topology. This includes the nontrivial statement that for a sequence $(X_i, d, \mathcal{H}_n, p_i)$ in the above class converging to (X, d, m, p) the limit measure m is automatically $\mathcal{H}_n$.

Thus, noncollapsed $RCD(K,n)$ spaces are a natural synthetic generalization of noncollapsing Ricci limits.

The above discussion shows that requiring that the background measure be $\mathcal{H}_n$ in the context of Theorem 1.1 is a natural assumption.

Let us outline the structure of the proof of the main theorem. It consists of two independent parts. Part one is to show that a space satisfying $CD(K,n)$ and $CBA(\kappa)$ with finite n is infinitesimally Hilbertian and hence $RCD(K,n)$. Part two is to show that a noncollapsed $RCD(K,n)$ space which is also $CBA(\kappa)$ is Alexandrov with $curv \geq K-\kappa(n-2)$.

Since small balls in X are convex, using local-to-global results for RCD and Alexandrov spaces it's enough to prove both parts for small balls in X which are $CAT(\kappa)$ i.e. satisfy the upper curvature triangle comparison *globally*. Thus, for most of the paper we only consider spaces X which have small diameter and are $CAT(\kappa)$ rather than $CBA(\kappa)$.

The proof that X satisfying $CD(K,n)$ and $CAT(\kappa)$ with finite n must be infinitesimally Hilbertian consists of several steps. The main step is proving the splitting theorem (Proposition 6.5) which says that if a space X which is $CD(0,n)$ and $CAT(0)$ with $n<\infty$ then it must metrically split as $Y\times\mathbb{R}$.

Recall that the usual scheme for proving the splitting theorem under various versions of non-negative Ricci curvature involves a variation of the following argument [CG72, Gig13].

Let $\gamma\colon \mathbb{R}\to X$ be a line in X. Consider the rays $\gamma_+(t)=\gamma(t)$ and $\gamma_-(t)=\gamma(-t)$ for $t\geq 0$. Let $b_\pm$ be the corresponding Busemann functions. From the triangle inequality it holds that $b=b_++b_-\geq 0$ on X. Also, $b|_\gamma\equiv 0$. Then the usual argument is to first show that $b_\pm$ are both superharmonic, hence b is is superharmonic and hence it must be identically zero on X by the maximum principle. However, this argument completely fails in our situation because knowing that $b_\pm$ are superharmonic *does not* imply that b is superharmonic too as the Laplace operator is not known to be linear yet - we are trying to prove that it is.

Our proof of the splitting theorem goes along very different lines. It relies on the Flat Strip Theorem for $CAT(0)$ spaces to conclude that $b\equiv 0$ and to get the splitting.

By [GMR15] "tangents of tangents are tangents" a.e., i.e. there is a set $A\subset X$ of full measure such that for *every* point $p\in A$ for *any* tangent cone (T_pX,d_p,m_p) and *any* point $y\in T_pX$ any tangent cone $T_y(T_pX)$ is a tangent cone at p. Using the splitting theorem this easily implies that there exists a tangent cone at p isometric to $\mathbb{R}^k$ for some $k\leq n$.

Now infinitesimal Hilbertianness of X easily follows by an application of Cheeger's celebrated generalization of Rademacher's theorem to doubling metric measure spaces which satisfy the Poincaré inequality [Che99].

The second major part in the proof of the main theorem is showing that it holds if X is $RCD(K,n)$, $CAT(\kappa)$ and $m=\mathcal{H}_n$.

The obvious proof which works for Riemannian manifolds does not easily generalize as there is no notion of curvature or Ricci tensors on X. Let us describe an argument that does generalize. Let (M^n,g) be a complete Riemannian manifold with $\sec\leq\kappa, \mathrm{Ric}\geq K$. Fix any $\hat\kappa<K-\kappa(n-2)$. To verify that $\sec_M\geq\hat\kappa$ it's enough to show that for any $p\in M$ the distance function to p is more concave than the distance function in the simply connected space form of constant curvature $\hat{k}$. For points q near p this is equivalent to checking that

$$\mathrm{Hess}(d_p|_q)(V,V)\leq \cot_{\hat\kappa}(d_p(q)) \quad \text{for any unit } V\in T_qM \text{ orthogonal to } \nabla d_p \tag{1}$$

where $\cot_k(t)$ is the generalized cotangent function (see section 2.7 for the definition).

The condition that $\sec_M\leq\kappa$ implies that

$$\mathrm{Hess}(d_p)(V,V)\geq \cot_{\kappa}(d_p(q)) \quad \text{for any unit } V\in T_qM \text{ orthogonal to } \nabla d_p \tag{2}$$

On the other hand, since $\mathrm{Ric}_M\geq K$, by Laplace comparison we have that

$$\Delta d_p(q) = \sum_i \mathrm{Hess}(d_p)(V_i, V_i) \leq (n-1)\cot_{K/(n-1)}(d_p(q)) \tag{3}$$

where $V_1, \ldots, V_{n-1}$ is an orthonormal basis of $\nabla d_p^{\perp} \subset T_q M$.

Combining the above inequalities gives that $\kappa(n-1) \geq K$ and that for any $i = 1, \ldots n-1$

$$\mathrm{Hess}(d_p)(V_i, V_i) \leq (n-1)\cot_{K/(n-1)}(d_p(q)) - (n-2)\cot_{\kappa}(d_p(q)) \leq \cot_{\hat{\kappa}}(d_p(q))$$

when $d(p,q)$ is sufficiently small. Hence $\sec M \geq \hat{\kappa}$. Since $\hat{\kappa} < K - \kappa(n-2)$ was arbitrary this shows that $\sec M \geq K - \kappa(n-2)$.

There are a number of technical challenges in generalizing this argument to the setting of Theorem 1.1. The first one is to get a *lower* laplacian bound on the distance functions using the upper curvature bound. To do this we first show that the set of regular points X_{reg} is open, *convex* and is a topological n-manifold. A crucial point in showing convexity of X_{reg} is proving that the density function is semiconcave on X (Lemma 5.4). This uses the CAT property of X and need not be true for general noncollapsed $RCD(K,n)$ spaces.

By a homological argument the fact that X_{reg} is a manifold implies that geodesics on X_{reg} are locally extendible. Once this has been established it follows from contraction properties of the inverse gradient flow of d_p that Δd_p is bounded below on X_{reg}. $RCD(K,n)$ condition implies that it's bounded above which implies that distance functions locally lie in the domain of the laplacian. This allows us to apply to the distance functions analytic tools we develop in Section 4 which relate convexity properties of functions in the domain of the laplacian on RCD spaces to bounds on their Hessians. Using the calculus of tangent modules developed by Gigli [Gig14] and a result of Han [Han14] that for a sufficiently regular function f on a noncollapsed $RCD(K,n)$ space $\Delta f = \mathrm{tr}\,\mathrm{Hess}\, f$, we are able to carry out the Riemannian argument that was outlined earlier to obtain the same concavity properties of distance functions locally on X_{reg}. By a globalization result of Petrunin this implies that X is Alexandrov.

The paper is structured as follows. In Section 2 we provide preliminaries on synthetic Ricci curvature bounds, calculus for metric measure spaces and curvature bounds for metric spaces in the sense of Alexandrov.

In Section 3 we prove a lower Laplace bound for distance functions in the context of metric spaces which are topological manifolds and satisfy RCD and CBA bounds.

In Section 4 we establish a result that gives a characterization of local κ-convexity of Lipschitz functions that are in the domain of the Laplace operator, in terms of almost everywhere lower bounds for the Hessian.

In Section 5 we prove the main theorem under an extra assumption that X is infinitesimally Hilbertian making use of several tools and results for the Laplace operator and the tangent module of metric measure spaces.

Finally, in Section 6 we prove that a space satisfying $CD^*(K,n)$ and $CBA(\kappa)$ for finite n must be infinitesimally Hilbertian (Theorem 6.2). Combined with the results of Section 5 this finishes the proof of the main theorem.

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2. Preliminaries

2.1. **Curvature-dimension condition for metric measure spaces.**

Definition 2.1. Let $[a,b]\subset\mathbb{R}$ be an interval. We say a lower semi-continuous function $u:[a,b]\to(-\infty,\infty]$ is (K,N)-convex for $K\in\mathbb{R}$ and $N\in(0,\infty]$ if u is absolutely continuous and

$$u''\geq K+\frac{1}{N}(u')^2$$

holds in the distributional sense where $\frac{1}{\infty}=:0$. We say that u is K-convex if u is (K,∞)-convex, and we say that u is K-concave if $-u$ is $-K$-convex.

If $N<\infty$, we define $U_N(t)=e^{-\frac{1}{N}t}$. Then, $u:[a,b]\to(-\infty,\infty]$ is (K,N)-convex if and only if $U_N(u)=:v:[a,b]\to[0,\infty)$ satisfies $v''\leq -K/Nv$ on $[a,b]$ in the distributional sense.

Let (X,d) be a metric space. If $A\subset X$, the induced metric on A is denoted by d_A. We say a rectifiable constant speed curve $\gamma:[a,b]\to X$ is a *minimizing geodesic* or just *geodesic* if $\mathrm{L}(\gamma)=d(\gamma(a),\gamma(b))$. We say (X,d) is a *geodesic metric space* if for any pair $x,y\in X$ there exists a geodesic between x and y. For a geodesic γ between points $x,y\in X$ we will also use the notation $[x,y]$ where in this case we think of the geodesic as its image in X. Similarly $]x,y[=[x,y]\setminus\{x,y\}$.

Definition 2.2. Let (X,d) be a metric space, and let $V:X\to(-\infty,\infty]$ be a lower semi-continuous function. We set $\operatorname{Dom}V:=\{x\in X: V(x)<\infty\}$. Let $K\in\mathbb{R}$ and $N\in(0,\infty]$.

- (i) We say that V is weakly (K,N)-convex if for every pair $x,y\in\operatorname{Dom}V$ there exists a unit speed geodesic $\gamma:[0,d(x,y)]\to X$ between x and y such that $V\circ\gamma:[0,d(x,y)]\to(-\infty,\infty]$ is (K,N)-convex.
- (ii) If (X,d) is a geodesic metric space we say V is (K,N)-convex if $V\circ\gamma$ is (K,N)-convex for any unit speed geodesic $\gamma:[0,L]\to\operatorname{Dom}V$.
- (iii) We say $V:X\to(-\infty,\infty]$ is semi-convex if for any $x\in X$ there exists a neighborhoud U of x and $K\in\mathbb{R}$ such that $V|_U$ is K-convex.

$\mathcal{P}^2(X)$ denotes the set of Borel probability measures μ on (X,d) such that $\int_X d(x_0,x)^2d\mu(x)<\infty$ for some $x_0\in X$. For any pair $\mu_0,\mu_1\in\mathcal{P}^2(X)$ we denote with $W_2(\mu_0,\mu_1)$ the L^2-*Wasserstein distance* that is finite and defined by

$$W_2(\mu_1,\mu_2)^2:=\inf_{\pi\in\mathrm{Cpl}(\mu_1,\mu_2)}\int_{X^2}d^2(x,y)d\pi(x,y), \tag{4}$$

where $\mathrm{Cpl}(\mu_1,\mu_2)$ is the set of all couplings between μ_1 and μ_2, i.e. of all the probability measures $\pi\in\mathcal{P}(X^2)$ such that $(P_i)_\sharp\pi=\mu_i$, $i=1,2$, P_1,P_2 being the projection maps. $(\mathcal{P}^2(X),W_2)$ becomes a separable metric space that is a geodesic metric space provided X is a geodesic metric space. A coupling $\pi\in\mathrm{Cpl}(\mu_1,\mu_2)$ is optimal if it is a minimizer for (4). Optimal couplings always exist We call the metric space $(\mathcal{P}^2(X),W_2)$ the L^2-*Wasserstein space* of (X,d). The subspace of probability measures with bounded support is denoted with $\mathcal{P}^2_b(X)$.

Definition 2.3. A *metric measure space* is a triple $(X,d,m)=:X$ where (X,d) is a complete and separable metric space and m is a locally finite measure.

The space of m-absolutely continuous probability measures in $\mathcal{P}^2(X)$ is denoted by $\mathcal{P}^2(X,m)$. The *Shanon-Boltzmann entropy* of a metric measure space (X,d,m) is defined as

$$\mathrm{Ent}_m:\mathcal{P}^2_b(X)\to(-\infty,\infty],\quad \mathrm{Ent}_m(\mu)=\int\log\rho d\mu \text{ if } \mu=\rho m \text{ and } (\rho\log\rho)_+ \text{ is } m\text{-integrable},$$

and ∞ otherwise. Note $\operatorname{Dom}\mathrm{Ent}_m\subset\mathcal{P}^2(X,m)$, and $\mathrm{Ent}_m:\mathcal{P}^2_b(X)\to(-\infty,\infty]$ is lower semi-continuous. By Jensen's inequality one has $\mathrm{Ent}_m\mu\geq-\log m(\operatorname{supp}\mu)$ if $m(\operatorname{supp}\mu)<\infty$.

Definition 2.4 ([Stu06a, LV09, EKS15])**.** A metric measure space (X,d,m) satisfies the *curvature-dimension condition* $CD(K,\infty)$ for $K\in\mathbb{R}$ if Ent_m is weakly K-convex.

A metric measure space (X,d,m) satisfies the *entropic curvature-dimension condition* $CD^e(K,N)$ for $K\in\mathbb{R}$ and $N\in(0,\infty)$ if Ent_m is weakly (K,N)-convex.

The N-*Renyi entropy* is defined as

$$S_N(\cdot|m) : \mathcal{P}^2_b(X) \to (-\infty, 0], \quad S_N(\mu|m) = -\int \rho^{1-\frac{1}{N}} dm \ \text{ if } \mu = \rho m, \text{ and } 0 \text{ otherwise.}$$

Note that $\mu = \rho m \in \mathcal{P}(X, m)$ implies $\rho \in L^{1-\frac{1}{N}}(m)$, and therefore S_N is well-defined. S_N is lower semi-continuous, and $S_N(\mu) \geq -m(\operatorname{supp}\mu)^{\frac{1}{N}}$ by Jensen's inequality.

Definition 2.5. For $\kappa \in \mathbb{R}$ we define $\cos_\kappa : [0, \infty) \to \mathbb{R}$ as the solution of

$$v'' + \kappa v = 0 \quad v(0) = 1 \ \ \& \ \ v'(0) = 0.$$

$\sin_\kappa$ is defined as solution of the same ODE with initial value $v(0) = 0$ & $v'(0) = 1$. That is

$$\cos_\kappa(x) = \begin{cases} \cosh(\sqrt{|\kappa|}x) & \text{if } \kappa < 0 \\ 1 & \text{if } \kappa = 0 \\ \cos(\sqrt{\kappa}x) & \text{if } \kappa > 0 \end{cases} \quad \sin_\kappa(x) = \begin{cases} \frac{\sinh(\sqrt{|\kappa|}x)}{\sqrt{|\kappa|}} & \text{if } \kappa < 0 \\ x & \text{if } \kappa = 0 \\ \frac{\sin(\sqrt{\kappa}x)}{\sqrt{\kappa}} & \text{if } \kappa > 0 \end{cases}$$

Let π_κ be the diameter of a simply connected space form $\mathbb{S}^2_\kappa$ of constant curvature κ, i.e.

$$\pi_\kappa = \begin{cases} \infty & \text{if } \kappa \leq 0 \\ \frac{\pi}{\sqrt{\kappa}} & \text{if } \kappa > 0 \end{cases}$$

For $K \in \mathbb{R}$, $N \in (0, \infty)$ and $\theta \geq 0$ we define the *distortion coefficient* as

$$t \in [0,1] \mapsto \sigma^{(t)}_{K,N}(\theta) = \begin{cases} \frac{\sin_{K/N}(t\theta)}{\sin_{K/N}(\theta)} & \text{if } \theta \in [0, \pi_{K/N}), \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\sigma^{(t)}_{K,N}(0) = t$. Moreover, for $K \in \mathbb{R}$, $N \in [1, \infty)$ and $\theta \geq 0$ the *modified distortion coefficient* is defined as

$$t \in [0,1] \mapsto \tau^{(t)}_{K,N}(\theta) = \begin{cases} \theta \cdot \infty & \text{if } K > 0 \text{ and } N = 1, \\ t^{\frac{1}{N}} \left[\sigma^{(t)}_{K,N-1}(\theta)\right]^{1-\frac{1}{N}} & \text{otherwise.} \end{cases}$$

Definition 2.6 ([Stu06b, LV09, BS10])**.** We say (X, d, m) satisfies the *curvature-dimension condition $CD(K, N)$* for $K \in \mathbb{R}$ and $N \in [1, \infty)$ if for every $\mu_0, \mu_1 \in \mathcal{P}^2_b(X, m)$ there exists an L^2-Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ and an optimal coupling π between μ_0 and μ_1 such that

$$S_N(\mu_t|m) \leq -\int \left[\tau^{(1-t)}_{K,N}(d(x,y))\rho_0(x)^{-\frac{1}{N}} + \tau^{(t)}_{K,N}(d(x,y))\rho_1(y)^{-\frac{1}{N}}\right] d\pi(x,y)$$

where $\mu_i = \rho_i dm$, $i = 0, 1$.

We say (X, d, m) satisfies the *reduced curvature-dimension condition* $CD^*(K, N)$ for $K \in \mathbb{R}$ and $N \in (0, \infty)$ if we replace in the previous definition the modified distortion coefficients $\tau^{(t)}_{K,N}(\theta)$ by the usual distortion coefficients $\sigma^{(t)}_{K,N}(\theta)$.

If $K = 0$, the condition $CD(K, N)$ coincides with the condition $CD^*(K, N)$ and is simply convexity of the N-Renyi entropy functional.

Remark 2.7. We note that if a metric measure space (X, d, m) satisfies a curvature dimension condition $CD(K, N)$, $CD^*(K, N)$ or $CD^e(K, N)$ for $N < \infty$, the support $\operatorname{supp} m$ of m with the induced metric $d_{\operatorname{supp} m}$ becomes a geodesic space. This follows since $(\operatorname{supp} m, d_{\operatorname{supp} m})$ is complete and a curvature-dimension condition yields that $\operatorname{supp} m$ is a length space, and is locally compact by Bishop-Gromov-type comparison that holds in any case [Stu06b, LV09, BS10, EKS15]. In this paper we will always assume that $\operatorname{supp} m = X$.

Theorem 2.8 ([Stu06a, Stu06b, LV09, GMS15, EKS15])**.** *All of the previous curvature-dimension conditions $CD(K,N)$, $CD^*(K,N)$ and $CD^e(K,N)$ are stable under pointed measured Gromov-Hausdorff convergence, and yield Brunn-Minkowski-type inequalities if N is finite. In the case $K=0$ the latter is the same statement for every curvature-dimension condition: For each pair of measurable subsets $A_0, A_1 \subset X$ it holds that*

$$m(A_t)^{\frac{1}{N}} \geq (1-t)m(A_0)^{\frac{1}{N}} + tm(A_1)^{\frac{1}{N}} \tag{5}$$

where A_t is the set of t-midpoints of geodesics with endpoints in A_0 and A_1 respectively.

The next fact, the next lemma and the next theorem collect a number of important properties of spaces that satisfy a curvature-dimension condition.

Fact 2.9 ([Stu06b, BS10, EKS15])**.** *(X,d,m) satisfies a condition $CD(K,N)$ for $K \in \mathbb{R}$ and $N \in (0,\infty)$. Then*

(i) *$(\operatorname{supp} m, d_{\operatorname{supp} m})$ is locally compact, a geodesic space and satisfies a Bishop-Gromov-type comparison and a doubling property.*

(ii) *$(X, \alpha d, \beta m)$ satisfies the condition $CD(\alpha^{-2}K, N)$ for every $\alpha, \beta > 0$.*

(iii) *If $U \subset X$ is geodesically convex and closed, $(U, d_U, m|_U)$ satisfies the condition $CD(K,N)$.*

(iv) *If (X,d,m) satisfies a condition $CD(K,N)$ for $N<\infty$, then it satisfies $CD(K,\infty)$.*

Each of the previous statements holds for $CD^*(K,N)$ and $CD^e(K,N)$ as well.

Lemma 2.10 ([EKS15])**.** *Let (X,d,m) be a metric measure space satisfying a condition $CD(K,\infty)$, $CD(K,N)$, $CD^*(K,N)$ or $CD^e(K,N)$ for some $K \in \mathbb{R}$ and $N \in (0,\infty)$, and let $f : X \to \mathbb{R}$ be κ-convex for $\alpha \in \mathbb{R}$ and bounded from below. Then the metric measure space $(X, d, e^{-f}m)$ satisfies the condition $CD(K+\kappa, \infty)$.*

Theorem 2.11 ([Raj12b, Raj12a])**.** *A metric measure space (X,d,m) that satisfies $CD(K,N)$, $CD^*(K,N)$ or $CD^e(K,N)$ for $K \in \mathbb{R}$, $N \in (0,\infty)$, admits a weak local 1-1 Poincaré inequality.*

[Raj12b, Raj12a] proves this for $CD(K,N)$ and $CD^*(K,N)$ but it's easy to see that the proof also works for $CD^e(K,N)$.

2.2. **Cheeger energy and calculus for metric measure spaces.** In the following we present the framework for calculus on metric measure spaces by Ambrosio, Gigli and Savaré [AGS13, AGS14a, AGS14b, Gig15]. Let (X,d,m) be a metric measure space, and $\operatorname{Lip}(X)$ and $\operatorname{Lip}_b(X)$ be the space of Lipschitz functions, and bounded Lipschitz functions respectively. For $f \in \operatorname{Lip}(X)$ its local slope is

$$\operatorname{Lip}(f)(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x,y)}, \quad x \in X.$$

If $f \in L^2(m)$ a function $g \in L^2(m)$ is called *relaxed gradient* if there exists sequence of Lipschitz functions f_n which L^2-converges to f, and there exists $\widetilde{g}$ such that $\operatorname{Lip} f_n$ weakly converges to $\widetilde{g}$ in $L^2(m)$ and $\widetilde{g} \leq g$ m-a.e. . We say g is the *minimal relaxed gradient* of f if it is a relaxed gradient and minimal w.r.t. to the L^2-norm amongst all relaxed gradients.

The *Cheeger energy* $\operatorname{Ch}^X : L^2(m) \to [0,\infty]$ is defined as

$$2\operatorname{Ch}^X(f) = \liminf_{f_n \in \operatorname{Lip}(X) \overset{L^2(m)}{\longrightarrow} f} \int_X \operatorname{Lip}(f_n)^2 dm.$$

The space of L^2-*Sobolev functions* is then

$$W^{1,2}(X) := D(\operatorname{Ch}^X) := \left\{ f \in L^2(m) : \operatorname{Ch}^X(f) < \infty \right\}.$$

For any $f \in W^{1,2}(X)$ there exists a minimal relaxed that is denoted with $|\nabla f|$ and unique up to set of measure 0. One also calls $|\nabla f|$ the *minimal weak upper gradient* of f. Then, one has

$$\mathrm{Ch}^X(f) = \frac{1}{2}\int |\nabla f|^2 dm.$$

The space $W^{1,2}(X)$ equipped with the norm $\|f\|^2_{W^{1,2}(X)} = \|f\|^2_{L^2} + \||\nabla f|\|^2_{L^2}$ is a Banach space. If $W^{1,2}(X)$ is a Hilbert space, we say (X,d,m) is *infinitesimally Hilbertian.*

Remark 2.12. Note that in general $|\nabla u| \neq \mathrm{Lip}\, u$ for a Lipschitz function u unless (X,d,m) satisfies a Poincaré inequality and a doubling property [Che99, Theorem 5.1]. By [Raj12b, Raj12a] spaces satisfying $CD(K,N)$, $CD^*(K,N)$ or $CD^e(K,N)$ with $N<\infty$ do satisfy a 1-1 Poincaré inequality (Theorem 2.11). Hence for such spaces [Che99] applies and $|\nabla u| = \mathrm{Lip}\, u$ a.e. for any Lipschitz u.

If (X,d,m) is infinitesimally Hilbertian, by polarization of $|\nabla f|^2$ we can define

$$(f,g)\in W^{1,2}(X)^2 \mapsto \langle \nabla f, \nabla g\rangle := \frac{1}{4}|\nabla(f+g)|^2 - \frac{1}{4}|\nabla(f-g)|^2 \in L^1(m).$$

We say that $f\in W^{1,2}(X)$ is in the domain of the *Laplace operator* Δ if there exists a function $g =: \Delta f \in L^2(m)$ such that for every $h\in W^{1,2}(X)$

$$\int \langle \nabla f, \nabla h\rangle dm = -\int h\Delta f dm.$$

In this case we say that $f\in D(\Delta)$. The vector space $D(\Delta)$ is equipped with the operator norm

$$\|f\|^2_{D(\Delta)} = \|f\|^2_{L^2} + \|\Delta f\|^2_{L^2}.$$

Convergence in $D(\Delta)$ implies convergence in $W^{1,2}(X)$. If $\mathbb{V}$ is any subspace of $L^2(m)$ and we have that $\Delta f\in\mathbb{V}$, we write $f\in D_{\mathbb{V}}(\Delta)$. $(P_t)_{t\in(0,\infty)}$ denotes the heat semi-group associated to Δ.

More generally, – assuming X is locally compact – if U is an open subset of X, we say $f\in W^{1,2}(X)$ is in the domain $D(\boldsymbol{\Delta}, U)$ of the *distributional Laplace operator* $\boldsymbol{\Delta}$ on U if there exists and element $T\in C_c(U)'$ such that

$$\int \langle \nabla g, \nabla f\rangle dm = -T(g) =: -\boldsymbol{\Delta} f(g).$$

It's well known that any elements of $C_c(U)'$ are exactly differences of two Radon measures. Therefore there exist Radon measures μ^+ and μ^- such that for every Lipschitz function g with bounded support in U we have

$$\int \langle \nabla g, \nabla f\rangle dm = -\int g d\mu^+ + \int g d\mu^- =: -\boldsymbol{\Delta} f(g).$$

If $U = X$, we write $D(\boldsymbol{\Delta}, U) = D(\boldsymbol{\Delta})$.

The elements of $C_c(U)'$ are also called Radon functionals (for instance see [CM18]). We however will stick to the notation above for the rest of the article.

If at most one of the measures μ^+ or μ^- above attains the value ∞, then $\boldsymbol{\Delta} f$ is a signed Radon measure and we write $\mu^+ - \mu^- = \boldsymbol{\Delta} f$, $\boldsymbol{\Delta} f\in\mathcal{M}$ and $\boldsymbol{\Delta} f(g) = \int g d\Delta f$. From the definition it is clear that $D(\boldsymbol{\Delta}, V)\subset D(\boldsymbol{\Delta}, U)$ whenever $V\subset U$, and $\boldsymbol{\Delta}^V(g) = \boldsymbol{\Delta}^U(g)$ whenever g is a Lipschitz function with bounded support in V. In particular, if $\bar V\subset U$ and compact, $\boldsymbol{\Delta}^V$ is a signed Radon measure.

Proposition 2.13 ([Gig15](Proposition 4.11))**.** *Let (X,d,m) be an infinitesimally Hilbertian metric measure space, let $f\in D(\boldsymbol{\Delta}, U)\cap \mathrm{Lip}(X)$ for an open subset $U\subset X$, let $I\subset\mathbb{R}$ be an open subset, assume $m(f^{-1}(\mathbb{R}\backslash I)) = 0$, and let $\varphi\in C^2(I)$. Then $\varphi\circ f\in D(\boldsymbol{\Delta}, U)$ and*

$$\boldsymbol{\Delta}(\varphi\circ f) = \varphi'(f)\boldsymbol{\Delta} f + \varphi''(f)|\nabla f|^2 m \quad \text{on } U$$

in the distributional sense.

Definition 2.14. A metric measure space (X, d, m) satisfies the Riemannian curvature-dimension condition $RCD(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty]$ if it satisfies the curvature-dimension condition $CD(K, N)$, and if it is infinitesimally Hilbertian.

Remark 2.15. We make a few historical comments on the origin of the previous definition.

For $N = \infty$ the condition $CD(K, \infty)$ together with Ch^X beeing a quadratic form (and hence $W^{1,2}(X)$ beeing a Hilbert space) first appears in [AGS14b] as one of three equivalent conditions that were used in the same paper to define the class of $RCD(K, \infty)$ spaces [AGS14b, Theorem 5.1]. In particular, they show stability w.r.t. measured Gromov-Hausdorff convergence. In [Gig15] Gigli introduced the notion infinitesimally Hilbertianity for $W^{1,2}(X)$ beeing a Hilbert space [Gig15, Definition 4.19].

In [Gig15, Remark 4.20] Gigli then suggested that $CD(K, N)$ plus beeing infinitesimally Hilbertian might also be the correct definition for $N < \infty$ since he observed that this two properties together are stable w.r.t. measured Gromov-Hausdorff convergence [Gig15, Remark 4.20] (while beeing infinitesimal Hilbertian by itself is in general not). Moreover, this is also the framework where Gigli proved the sharp Laplace comparison theorems (for $K \in \mathbb{R}$) and his Splitting theorem $(K = 0)$ [Gig13]. However, at this point it was not clear to experts wether the condition $CD(K, N)$ for $K \neq 0$ would admit a local-to-global property, even under non-branching assumptions, or that $CD(K, N)$ for any K would imply the expectected properties for the heat flow.

The latter problem was resolved in [EKS15] where the authors introduce the entropic curvature-dimension condition $CD^e(K, N)$. They use this new condition to define so-called $RCD^*(K, N)$ spaces as combination of $CD^*(K, N)$ and infinitesimally Hilbertianity [EKS15, Definition 3.16 and Theorem 3.17]. Erbar, Kuwada and Sturm show stability under measured GH convergence and also derive several equivalent properties, in particular the celebrated finite dimensional Bochner inequality [EKS15, Theorem 7] (Theorem 2.30 below, see also [AMS15] for an alternative proof). Moreover, Gigli's Splitting theorem naturally fit into their framework.

Finally, Cavalletti and Milman [CM16] were able to show the local-to-global property of the condition $CD(K, N)$ under the essentially non-branching assumption. As consequence of this globalization theorem, and by results in [EKS15], [AGS14b] and [RS14] it is equivalent to require the condition $CD^*(K, N)$ or the condition $CD^e(K, N)$ in the definition of $RCD(K, N)$, confirming Gigli's prediction in [Gig15].

More precisely, since the condition $RCD^e(K, N)$ implies $CD(K, \infty)$ (for instance, see [EKS15, Lemma 3.2]), (X, d, m) satisfies the condition $RCD(K, \infty)$ in the sense of [AGS14b]. Therefore, the Boltzmann-Shanon entropy is strongly K-convex, and hence the corresponding metric measure space (X, d, m) is essentially non-branching by [RS14, Corollary 1.3]. By this, first, we know that $CD^e(K, N)$ is equivalent to $CD^*(K, N)$ by [EKS15, Theorem 3.12]. Second, from the globalization result in [CM16] we have that $CD^*(K, N)$ is equivalent to $CD(K, N)$.

Definition 2.16. We define $\mathrm{md}_\kappa : [0, \infty) \to [0, \infty)$ as the solution of

$$v'' + \kappa v = 1 \quad v(0) = 0 \ \ \& \ \ v'(0) = 0.$$

More explicitly

$$\mathrm{md}_\kappa(x) = \begin{cases} \frac{1}{\kappa}\left(1 - \cos_\kappa x\right) & \text{if } \kappa \neq 0, \\ \frac{1}{2}x^2 & \text{if } \kappa = 0. \end{cases}$$

Theorem 2.17 ([Gig15])**.** *Assume (X, d, m) satisfies the condition $RCD(K, N)$ for $N < \infty$, and for $x \in X$ we define $d_x : X \to [0, \infty)$ via $d_x(y) = d(x, y)$. Then*

$$\frac{1}{2}d_y^2 \in D(\mathbf{\Delta}) \quad \textit{and} \quad \mathbf{\Delta}\frac{1}{2}d_y^2 \leq \left[1 + (N-1)d_y \cot_{K/(N-1)} d_y\right] m$$

where $\cot_k(x) = \frac{\cos_k(x)}{\sin_k(x)}$. In particular, for $K = 0$ the estimate is precisely $\mathbf{\Delta}\frac{1}{2}d_y^2 \leq Nm$.

Moreover, if X is compact $\mathbf{\Delta}d_y^2/2$ is a signed Radon measure.

Remark 2.18. The Theorem is proven by Gigli under the condition $CD(K,N)$ and it is sharp in this context. Moreover, for the sharp comparison result of $\frac{1}{2}d_y^2$ only the weaker $MCP(K,N)$-condition is needed that follows from the reduced curvature-dimension condition $CD^*(K,N)$ by [CS12] provided the space is essentially non-branching. In [CM18] the authors give a refined version of Theorem 2.17 under MCP and essentially non-branching where they exactly locate the singular parts of $\boldsymbol{\Delta} d_y^2$.

Corollary 2.19. *Assume (X,d,m) satisfies the condition $RCD(K,N)$ for $N<\infty$. Then for any $y\in X$ we have*

$$\mathrm{md}_\kappa\, d_y \in D(\boldsymbol{\Delta}) \quad \textit{for any } \kappa\in\mathbb{R} \textit{ and } \quad \boldsymbol{\Delta}\, \mathrm{md}_{K/(N-1)}\, d_y + \frac{NK}{N-1}\mathrm{md}_{K/(N-1)}\, d_y \le Nm.$$

Proof. Indeed, By the chain rule Theorem 2.17 implies (see also [Gig15, Corollary 5.15] and [CM18]) that $d_y\in D(\boldsymbol{\Delta}, X\backslash\{y\})$ and

$$\boldsymbol{\Delta} d_y \le [(N-1)\cot_{K/(N-1)} d_y] m \text{ on } X\backslash\{y\}$$

Applying the chain rule one more time this yields

(6)
$$\mathrm{md}_{K/(N-1)}\, d_y \in D(\boldsymbol{\Delta}, X\backslash\{y\}) \text{ and } \boldsymbol{\Delta}\,\mathrm{md}_{K/(N-1)}\, d_y + \frac{NK}{N-1}\mathrm{md}_{K/(N-1)}\, d_y \le Nm \text{ on } X\backslash\{y\}.$$

Further note that

$$\mathrm{md}_\kappa(x) = \frac{1}{2}x^2 - \frac{\kappa}{24}x^4 + \dots\ .$$

and hence $\mathrm{md}_\kappa(x) = \varphi_\kappa(x^2/2)$ where φ_κ is a smooth function on $\mathbb{R}$ given by

$$\varphi_\kappa(x) = x - \frac{\kappa}{6}x^2 + \dots$$

Therefore $\mathrm{md}_\kappa\, d_y = \varphi_k(\frac{d_y^2}{2}) \in D(\boldsymbol{\Delta})$ for any $\kappa\in\mathbb{R}$ and (6) can be improved to

$$\boldsymbol{\Delta}\,\mathrm{md}_{K/(N-1)}\, d_y + \frac{NK}{N-1}\mathrm{md}_{K/(N-1)}\, d_y \le Nm \text{ on all of } X.$$

□

2.3. **Tangent modules of metric measure spaces.** In this section we present a general construction of *tangent spaces* for metric measure spaces due to Gigli (inspired by an idea of Weaver [Wea99]).

Definition 2.20 ([Gig14])**.** Let (X,d,m) be a metric measure space, and let $\mathcal{M}$ be a Banach space. $\mathcal{M}$ is called an $L^2(m)$-normed $L^\infty(m)$-module provided it is endowed with a bilinear map $L^\infty(m)\times\mathcal{M} : (f,v)\mapsto fv\in\mathcal{M}$, and a function $|\cdot| : \mathcal{M}\to L^2(m)^+$ which satisfy the following properties:

(i) $f(gv) = (fg)v$ for all $f,g\in L^\infty(m)$ and $v\in\mathcal{M}$,

(ii) $\mathbf{1}v = v$ for any $v\in\mathcal{M}$ where $\mathbf{1}\in L^\infty(m)$ is the function equal 1,

(iii) $\||v|\|_{L^2} = \|v\|_{\mathcal{M}}$ for any $v\in\mathcal{M}$,

(iv) $|fv| = |f||v|$ m-a.e. for any $f\in L^\infty(m)$ and $v\in\mathcal{M}$.

Consider a Borel measurable set $A\subset X$. The restriction $\mathcal{M}|_A$ of $\mathcal{M}$ to A is defined as

$$\mathcal{M}|_A = \{v\in\mathcal{M} : 1_{A^c}v = 0 \ m\text{-a.e.}\}\,.$$

$\mathcal{M}|_A$ inherits the structure of $L^2(m)$-normed $L^\infty(m)$-module.

Given two $L^2(m)$-normed $L^\infty(m)$-modules $\mathcal{M}$ and $\mathcal{N}$ we say that a map $T:\mathcal{M}\to\mathcal{N}$ is *module morphism* provided T is linear, continuous and it satisfies

$$T(fv) = fT(v) \text{ for every } f\in L^\infty(m) \text{ and every } v\in\mathcal{M}.$$

Definition 2.21. Given an $L^2(m)$-normed $L^\infty(m)$-module $\mathcal{M}$ we define the dual module $\mathcal{M}^*$ as the space of all maps T from $\mathcal{M}$ to $L^1(m)$ that are L^∞-linear (that is additive and L^∞-homogenuous) and continuous. $\mathcal{M}^*$ again has a natural structure of an $L^2(m)$-normed $L^\infty(m)$-module. For details we refer to [Gig14].

Definition 2.22. We call an $L^\infty(m)$-module $\mathcal{M}$ an *Hilbert module* if $\mathcal{M}$ is an Hilbert space. In this case $\mathcal{M}$ becomes an $L^2(m)$-normed $L^\infty(m)$-module, and the pointwise norm $|\cdot|$ satisfies

$$|v+w|^2+|v-w|^2 = 2|v|^2+2|w|^2 \ \text{ for any } v,w\in\mathcal{M}.$$

We define the pointwise inner product $\mathcal{M}^2\to L^1(m)$ via $4\langle v,w\rangle = |v+w|^2-|v-w|^2$. Following [Gig14] we also note that $\mathcal{M}$ and $\mathcal{M}^*$ are canonical isomorphic as Hilbert modules.

Definition 2.23. Let $\mathcal{M}$ be an $L^2(m)$-normed $L^\infty(m)$-module, and let $A\subset X$ be a Borel set such that $m(A)>0$. We say that

(i) $v_1,\dots,v_n\in\mathcal{M}$ are *independent* on A if for $f_1,\dots,f_n\in L^\infty(m)$ it holds that
$$1_A\sum_{i=1}^n f_iv_i=0 \quad\Rightarrow\quad f_1,\dots,f_n=0 \ m\text{-a.e. in } A,$$

(ii) $S\subset\mathcal{M}$ generates $\mathcal{M}|_A$ if $\mathcal{M}|_A$ is the L^2-closure of elements v in $\mathcal{M}|_A$ such that there is a decomposition of $\{A_i\}_{i\in\mathbb{N}}$ of A, vectors $v_{i,1},\dots,v_{i,m_i}\in S$, and functions $f_{i,1},\dots,f_{i,m_i}\in L^\infty(m)$ which satisfy
$$1_{A_i}v=\sum_{k=1}^{m_i} f_{i,k}v_{i,k} \ m\text{-a.e. for each } i\in\mathbb{N},$$

(iii) $v_1,\dots,v_n\in\mathcal{M}$ is a *(module) basis* on A if they are independent on A and generate $\mathcal{M}|_A$. If A admits a basis of finite cardinality $n\in\mathbb{N}$, we say that A has *local dimension* n. If A admits no basis of finite cardinality, we say A has infinite local dimension.

Remark 2.24. It is easy to see that in (ii) one only needs to require that $\mathcal{M}|_A$ is the L^2-closure of finite L^∞-linear combinations of elements in S where an L^∞-linear combination is defined by

$$\sum_{l=1}^m f_lv_l \ \text{ where } f_l\in L^\infty(m) \text{ and } v_l\in S.$$

The more general statement (ii) - that also appears in [Gig14] - is to deal with L^∞-modules that are not necessarily $L^p(m)$-normed.

Proposition 2.25 (Proposition 1.4.4. [Gig14])**.** *Local dimension is well-defined: If both $v_1,\dots,v_n$ and $w_1,\dots,w_n$ are bases of an $L^2(m)$-normed $L^\infty(m)$-module $\mathcal{M}$ on A for $n,m\in\mathbb{N}$, then $m=n$.*

Proposition 2.26 (Proposition 1.4.5. [Gig14])**.** *There is a unique partition $\{E_k\}_{k\in\mathbb{N}\cup\{\infty\}}$ of X such that for any $k\in\mathbb{N}$ with $m(E_k)>0$ E_k has local dimension k, and any $E\subset E_\infty$ with $m(E)>0$ has infinite local dimension.*

Proposition 2.27 (Proof of Theorem 1.4.11. [Gig14])**.** *Let $\mathcal{M}$ be an Hilbert module. Then, for every $n\in\mathbb{N}$ and any Borel set $B\subset X$ that has local dimension n and finite measure, there exists a unit orthogonal basis $e_1,\dots,e_n\in\mathcal{M}$ on B. That is $\langle e_i,e_j\rangle=\delta_{i,j}$ m-almost everywhere.*

Remark 2.28. If $(e_i)_{i=1,\dots,n}$ is a unit orthogonal basis on B, and $v1_A=\sum_{i=1}^n f_ie_i\in\mathcal{M}$ for $f_i\in L^\infty$ and A a Borel subset in B, then

$$|v1_A|^2=\sum_{i=1}^n|f_i|^2 \ \text{ and } \ \|v1_A\|^2_{\mathcal{M}}=\sum_{i=1}^n\|f_i\|^2_{L^2(m)}.$$

Theorem 2.29 ([Gig14])**.** *Let (X,d,m) be metric measure space. There exists a unique (up to module isomorphisms) couple $(\mathcal{M},d)$ where $\mathcal{M}$ is an $L^2(m)$-normed $L^\infty(m)$-module and d is a linear map $W^{1,2}(m)\to\mathcal{M}$ such that*

(i) $|df| = |\nabla f|$ *holds* m*-a.e. on* X*, and*

(ii) $\{df \in \mathcal{M} : f \in W^{1,2}(m)\}$ *generates* $\mathcal{M}$ *on* X.

If two couples $(\mathcal{M}, d)$ *and* $(\mathcal{M}', d')$ *satisfy the properties above then there exists a unique module isomorphism* $\Phi : \mathcal{M} \to \mathcal{M}'$ *such that* $\Phi \circ d = d'$.

The unique module above is called the cotangent module of (X, d, m)*, and it is denoted with* $L^2(T^*X)$*. Its dual module is called the tangent module and denoted with* $L^2(TX)$*. Elements in* $L^2(T^*X)$ *are called 1-forms, and elements in* $L^2(TX)$ *are called vector fields. The map d is called differential.*

If (X, d, m) *is infinitesimally Hilbertian, then* $L^2(T^*X)$ *is a Hilbert module, and we have* $\Phi : L^2(T^*X) \equiv L^2(TX)$ *for the Hilbert module isomorphism* $\Phi(X) = \langle X, \cdot \rangle : L^2(TX) \to L^1(m)$. $\Phi^{-1} \circ d = \nabla$ *is called gradient.*

2.4. **Bakry-Emery condition.** The following was introduced in [AGS15]. Let (X, d, m) be a metric measure space that is infinitesimally Hilbertian but does not necessarily satisfy a curvature-dimension condition. For $f \in D_{W^{1,2}(X)}(\Delta)$ and $\varphi \in D_{L^\infty}(\Delta) \cap L^\infty(m)$ we define the *carré du champ operator* as

$$\Gamma_2(f; \varphi) = \int \frac{1}{2} |\nabla f|^2 \Delta \varphi dm - \int \langle \nabla f, \nabla \Delta f \rangle \varphi dm.$$

We say that (X, d, m) satisfies the *Bakry-Emery condition* $BE(K, N)$ for $K \in \mathbb{R}$ and $N \in (0, \infty]$ if it satisfies the weak Bochner inequality

$$\Gamma_2(f; \varphi) \geq \frac{1}{N} \int (\Delta f)^2 \varphi dm + K \int |\nabla f|^2 \varphi dm.$$

for any $f \in D_{W^{1,2}(X)}(\Delta)$ and $\varphi \in D_{L^\infty}(\Delta) \cap L^\infty(m)$, $\varphi \geq 0$.

We say a metric measure space satisfies the *Sobolev-to-Lipschitz* property if

$$\{f \in W^{1,2}(X) : |\nabla f| \in L^\infty(m)\} = \mathrm{Lip}(X)$$

More precisely, for any Sobolev function with bounded minimal weak upper gradient there exist a Lipschitz function $\bar{f}$ that coincides m-almost everywhere with f such that $\mathrm{Lip}(\bar{f}) \leq |\nabla f|$.

For RCD spaces the Sobolev-to-Lipschitz property was proved in [AGS14b, Theorem 6.2].

Theorem 2.30 ([EKS15, AGS15])**.** *Let* (X, d, m) *be a metric measure space. Then, the condition* $RCD(K, N)$ *for* $K \in \mathbb{R}$ *and* $N > 1$ *holds if and only if* (X, d, m) *is infinitesimally Hilbertian, it satisfies the Sobolev-to-Lipschitz property and it satisfies the Bakry-Emery condition* $BE(K, N)$.

Remark 2.31. The case $N = \infty$ was proved in [AGS15], the case $N < \infty$ in [EKS15]. Shortly after [EKS15] an alternative proof for the finite dimensional case - following a different strategy - was established in [AMS15].

2.5. **Rectifiability.** Following [GP16] we say a family $\{A_i\}_{i \in \mathbb{N}}$ is an m-partition of $E \subset X$ if it is a partition of some Borel set $F \subset X$ such that $m(E \backslash F) = 0$.

Definition 2.32. A metric measure space (X, d, m) is strongly m-rectifiable if there exists a m-partition $\{A_k\}_{k \in \mathbb{N}}$ of X into measurable sets A_k such that for each $k \in \mathbb{N}$ and every $\epsilon > 0$ there exists an m-partition $\{U_i\}_{i \in \mathbb{N}}$ of A_k and measurable maps $\varphi_i : U_i \to \mathbb{R}^k$ such that for every $i \in \mathbb{N}$

$$\varphi_i : U_i \to \varphi(U_i) \ \text{ is } \ (1+\epsilon)\text{-biLipschitz } \ \& \ \ (\varphi_i)_\star(m|_{U_i}) \ll \mathcal{L}^k.$$

The partition $\{A_k\}_{k \in \mathbb{N}}$ that is unique up to a m-negligible set is called dimensional partition of X.

Theorem 2.33 ([MN14, KM16, GP16])**.** *Let* (X, d, m) *be a metric measure space that satisfies the curvature-dimension condition* $RCD(K, N)$ *for* $N \in (0, \infty)$*. Then* (X, d, m) *is strongly* m*-rectifiable, and* $m(A_k) = 0$ *for* $k > N$.

Theorem 2.34 ([GP16])**.** *Let* (X, d, m) *be a metric measure space that satisfies the condition* $RCD(K, N)$ *for* $N \in (0, \infty)$*, and let* A_k *its dimensional decomposition. Then, the local dimension of* A_k *is* $k \in \mathbb{N} \cup \{\infty\}$*. Hence* $A_k = E_k$ *for any* $k \in \mathbb{N}$ *where* E_k *is as in Proposition 2.26.*

Definition 2.35. If (X, d, m) satisfies the condition $RCD(K, N)$, we say $x_0 \in A_k$ is a regular point if the Gromov-Hausdorff tangent cone at x_0 is $\mathbb{R}^k$ where $\{A_k\}_{k\in\mathbb{N}}$ is the dimensional decomposition of X. We denote the set of all regular points X_{reg}.

2.6. **Hessian operator.** Recall from [Gig14] that $L^2((T^*X)^{\otimes 2})$ is the subset of elements A in the L^0-module-tensor product $L^0((T^*X)^{\otimes 2})$ such that

$$\||A|_{HS}\|^2_{L^2(m)} = \int |A|^2_{HS} dm < \infty.$$

$|A|_{HS}$ is the *pointwise Hilbert-Schmidt norm* whose construction can be found in [Gig14]. Similar, one constructs $L^2((TX)^{\otimes 2})$, and if $L^2(T^*X)$ and $L^2(TX)$ are Hilbert modules, $L^2(T^*X^{\otimes 2})$ and $L^2(TX^{\otimes 2})$ are isomorphic Hilbert modules as well. $L^2(T^*X^{\otimes 2})$ can be seen as the space of all continuous bilinear forms $A : L^2(TX)^2 \to L^0(m)$ such that $\||A|_{HS}\|_{L^2(m)} < \infty$. L^0-continuity corresponds for m finite and normalized to *convergence in probability*. Recall also that for $A \in L^2(T^*X^{\otimes 2})$

$$A(X, Y) \le |A|_{HS}|X||Y| \ m\text{-a.e.}$$

when $X, Y \in L^2(TX)$. In particular, $A(X, Y) \notin L^1(m)$ in general. In this sense, $A(\cdot, X)$ and $A(Y, \cdot)$ are well-defined objects in $L^2(T^*X)$ if $X, Y \in L^\infty(TX)$. In particular, $A(X, Y)$ is L^∞-homogeneous in X and Y.

Definition 2.36. The space of test functions is

$$\mathbb{D}_\infty = D_{W^{1,2}(X)}(\Delta) \cap L^\infty(m) \cap \left\{f \in W^{1,2}(X) : |\nabla f| \in L^\infty(m)\right\}.$$

If (X, d, m) satisfies a Riemannian curvature-dimension condition, then $P_t L^\infty(m)$ is a subset of $\mathbb{D}_\infty$, it is dense in $W^{1,2}(X)$ and in $D_{L^2(m)}(\Delta)$ w.r.t. the Sobolev norm and the graph norm of the operator Δ respectively, and by the Sobolev-to-Lipschitz property we have $\mathbb{D}_\infty \subset \mathrm{Lip}_b(X)$. Moreover, the co-tangent module $L^2(T^*X)$ is generated by

$$T\mathbb{D}_\infty = \left\{\sum_{i=1}^n g_i df_i : n \in \mathbb{N},\ g_i, f_i \in \mathbb{D}_\infty, i = 1, \dots, n\right\}.$$

Definition 2.37 ([Gig14])**.** The space $W^{2,2}(X) \subset W^{1,2}(X)$ is the set of all functions $f \in W^{1,2}(X)$ for which there exists $A \in L^2(T^*X \otimes T^*X)$ such that for all $h, g_1, g_2 \in \mathbb{D}_\infty$ we have

$$\begin{aligned} &2\int hA(dg_1, dg_2)dm \\ &= -\int h\langle\nabla f, \nabla\langle\nabla g_1, \nabla g_2\rangle\rangle dm - \int \langle\nabla f, \nabla g_1\rangle \operatorname{div}(h\nabla g_2)dm - \int \langle\nabla f, \nabla g_2\rangle \operatorname{div}(h\nabla g_1)dm \end{aligned} \tag{7}$$

where $\operatorname{div}(h\nabla g) = \langle\nabla h, \nabla g\rangle + h\Delta g$. In this case the operator A will be called the *Hessian* of f and will be denoted with $\operatorname{Hess} f$. $W^{2,2}(X)$ is equipped with the norm

$$\|f\|^2_{W^{2,2}(X)} = \|f\|^2_{L^2} + \||\operatorname{Hess} f|_{HS}\|^2_{L^2}.$$

We say that $\operatorname{Hess} f \ge \kappa$ on a measurable subset B for $\kappa \in \mathbb{R}$ if for any $u \in \mathbb{D}_\infty$

$$\operatorname{Hess} f(1_B\nabla u, \nabla u) \ge \kappa|1_B\nabla u|^2 = \kappa 1_B|\nabla u|^2 \ m\text{-almost everywhere.}$$

Remark 2.38. We note that the density property of $\mathbb{D}_\infty$ and the fact that $\mathbb{D}_\infty$ is an algebra ensure that $\operatorname{Hess} f \in L^2(T^*X \otimes T^*X)$ is uniquely determined by (7). Then it is clear that $\operatorname{Hess} f$ depends linearly on f, and $W^{2,2}(X)$ therefore becomes a vector space.

Lemma 2.39. *Consider $f \in W^{2,2}(X)$, and assume* $\operatorname{Hess} f \ge \kappa$ *m-a.e. on B. Then* $\operatorname{Hess} f(1_B V, 1_B V) \ge \kappa|V|^2 1_B$ *m-a.e. for every $V \in L^2(TX)$.*

Proof. Let $V \in L^2(TX)$. It is enough to consider the case $B = X$. Since $T\mathbb{D}_\infty$ generates $L^2(TX)$, Remark 2.24 we find functions $f_{k,i}, g_{k,i} \in \mathbb{D}_\infty$ with $i = 1, \dots, n_k \in \mathbb{N}$ and $k \in \mathbb{N}$ such that

$$V_k = \sum_{i=1}^{n_k} f_{k,i} \nabla g_{k,i} \to V \quad \text{in } L^2(TX).$$

Then, since every $f_{k,i}$ can be approximated in L^2-sense by measurable functions that take only finitely many values in $\mathbb{R}$, and since ∇ is linear, we see that X is approximated in L^2-sense by vector fields W_k of the form

$$W_k = \sum_{i=1}^{n_k} 1_{B_{k,i}} \nabla h_{i,k}$$

for measurable decompositions $\{B_{k,i}\}_{i\in\mathbb{N}}$ of X, and $h_{k,i} \in \mathbb{D}_\infty$. Hence, we have

$$\begin{aligned}\operatorname{Hess} f(W_k, W_k) &= \sum_{i,j=1}^{n_k} \operatorname{Hess} f(1_{B_{k,i}} \nabla h_{i,k}, 1_{B_{k,j}} \nabla h_{j,k}) \\ &= \sum_{i=1}^{n_k} 1_{B_{k,i}} \operatorname{Hess} f(\nabla h_{k,i}, \nabla h_{k,i}) \geq \sum_{i=1}^{n_k} \kappa |\nabla h_{k,i}|^2 = \kappa |W_k|^2 \quad m\text{-a.e. on } B.\end{aligned}$$

The second equality is the L^∞-homogeneity of $\operatorname{Hess} f(\cdot,\cdot)$. Hence, by L^2-convergence of W_k in $L^2(TX)$ the right hand side converges m-a.e. to $|V|$ after taking a subsequence. Moreover, by continuity of the bilinear for $\operatorname{Hess} f : L^2(TX)^2 \to L^0(m)$ the left hand side converges m-a.e. to $\operatorname{Hess} f(V, V)$ after taking another subsequence. Then, the claim follows. □

Theorem 2.40 ([Gig14, Sav14])**.** *Let $f \in D_{L^2}(\Delta)$. Then $f \in W^{2,2}(X)$, and*

$$\int |\operatorname{Hess} f|^2_{HS} dm \leq \int \left[(\Delta f)^2 - K|\nabla f|^2 \right] dm.$$

In particular, $\mathbb{D}_\infty \subset W^{2,2}(X)$.

Proposition 2.41 ([Gig14])**.** *Let $f \in W^{2,2}(X) \cap \mathrm{Lip}(X)$ and let $\varphi \in C^2(\mathbb{R})$ with bounded first and second derivative. Then $\varphi \circ f \in W^{2,2}(X)$ and the following formula holds*

$$\operatorname{Hess}(\varphi \circ f)(\nabla u, \nabla u) = \varphi' \circ f \operatorname{Hess} f(\nabla u, \nabla u) + \varphi'' \circ f \langle \nabla f, \nabla u \rangle \quad \forall u \in W^{1,2}(X).$$

Definition 2.42. $H^{2,2}(X)$ is defined as the closure of $\mathbb{D}_\infty$ in $W^{2,2}(X)$.

Remark 2.43. $H^{2,2}(X)$ actually coincides with the $W^{2,2}$-closure of $D(\Delta)$ (Proposition 3.3.18 in [Gig14]). In particular, any $f \in D(\Delta)$ is in $H^{2,2}(X)$ and admits a Hessian.

Proposition 2.44 ([Gig14], Proposition 3.3.22)**.** *Let $f \in W^{2,2}(X) \cap \mathrm{Lip}(X)$ and $g_1, g_2 \in H^{2,2}(X) \cap \mathrm{Lip}(X)$. Then, $\langle \nabla f, \nabla g_i \rangle \in W^{1,2}(X), i = 1, 2$ and*

$$2 \operatorname{Hess} f(\nabla g_1, \nabla g_2) = \langle \nabla g_1, \nabla \langle \nabla f, \nabla g_2 \rangle \rangle + \langle \nabla g_2, \nabla \langle \nabla f, \nabla g_1 \rangle \rangle - \langle \nabla f, \nabla \langle \nabla g_1, \nabla g_2 \rangle \rangle.$$

Remark 2.45. In the case when $f \in D(\Delta)$ is Lipschitz $\operatorname{Hess} f(\nabla g_1, \nabla g_2)$ for $g_1, g_2 \in H^{2.2}(X)$ can be computed explicitly by the previous Proposition since $D(\Delta) \subset H^{2,2}(X)$ by the previous remark.

Theorem 2.46 ([Gig15], [GT17])**.** *Let (X, d, m) be a metric measure space that satisfies the condition $RCD(K, N)$ for $N < \infty$. Let $\mu_0, \mu_1 \in \mathcal{P}^1(X)$ such $\mu_i = \rho_i m \leq Cm$ for $C > 0$ and $i = 0, 1$, and let $(\mu_t)_{t\in[0,1]}$ be the unique L^2-Wasserstein geodesic.*

(i) *First variation formula. Let $f \in W^{1,2}(X)$. Then, the map $t \in [0, 1] \to \int f d\mu_t$ belongs to $C^1([0, 1])$ and for every $t \in [0, 1]$ it holds*

$$\frac{d}{dt} \int f d\mu_t = \int \langle \nabla f, \nabla \varphi_t \rangle d\mu_t.$$

(ii) *Second variation formula. Moreover, let $f \in H^{2,2}(X)$. Then, the map $t \in [0,1] \to \int f d\mu_t$ belongs to $C^2([0,1])$ and for every $t \in [0,1]$ it holds*

$$\frac{d^2}{dt^2} \int f d\mu_t = \int \operatorname{Hess} f(\nabla \varphi_t, \nabla \varphi_t) d\mu_t$$

where φ_t is the function such that for some $t \neq s \in [0,1]$ the function $-(s-t)\varphi_t$ is a Kantorovich potential between μ_t and μ_s.

Theorem 2.47 ([Sav14, Gig14])**.** *If (X,d,m) satisfies the condition $RCD(K,\infty)$, and $f \in \mathbb{D}_\infty \subset W^{2,2}(X)$, then $|\nabla f|^2 \in W^{1,2}(X) \cap D(\mathbf{\Delta})$ and an improved Bochner formula holds in the sense of measures involving the Hilbert-Schmidt norm of the Hessian of f:*

$$\mathbf{\Gamma}_2(f) := \frac{1}{2}\mathbf{\Delta}|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle m \geq \left[K|\nabla f|^2 + |\operatorname{Hess} f|^2_{HS}\right] m$$

where $\mathbf{\Delta}$ is the distributional Laplace operator, and $\mathbf{\Gamma}_2$ is called measure valued Γ_2-operator. In particular, the singular part of the left hand side is non-negative.

In the context of $RCD(K,N)$-spaces with finite N the previous theorem was improved by Han [Han14], and in particular, he obtains the following.

Theorem 2.48 ([Han14])**.** *Let (X,d,m) be a metric measure space that satisfies the condition $RCD(K,N)$ with $N < \infty$, and let $\{A_k\}_{k \in \mathbb{N} \cup \{\infty\}}$ be its dimensional decomposition. If $A_N \neq \emptyset$ and therefore $N \in \mathbb{N}$, then for any $f \in \mathbb{D}_\infty$ we have that $\Delta f = \operatorname{tr} \operatorname{Hess} f$ m-a.e. in A_N. More precisely, if $B \subset A_N$ is a set of finite measure and $(e_i)_{i=1,\dots,N}$ is a unit orthogonal basis on B, then*

$$\Delta f|_B = \sum_{i=1}^N \operatorname{Hess} f(e_i 1_B, e_i 1_B) = \sum_{i=1}^N \operatorname{Hess} f(e_i, e_i) 1_B =: [\operatorname{tr} \operatorname{Hess} f]|_B.$$

Corollary 2.49. *Let (X,d,m) be a metric measure space that satisfies the condition $RCD(K,n)$ with $m = \mathcal{H}^n$ and $n \in \mathbb{N}$. Then $A_k = \emptyset$ for $k \neq n$, and for any $f \in D_{L^2}(\Delta)$ we have that $\Delta f = \operatorname{tr} \operatorname{Hess} f$ m-almost everywhere in the sense of the previous theorem.*

Proof. The first claim is clear from the assumptions. Let $f \in D_{L^2}(\Delta)$, and consider a sequence $\varphi_j \in \mathbb{D}_\infty$ that approximates f in $D_{L^2}(\Delta)$. Moreover, let $B \subset A_n$ be a Borel set of finite measure and let $(e_i)_{i=1,\dots,n}$ be a unit orthogonal basis and tr be the corresponding trace. It follows

$$\begin{aligned}\int 1_B(\Delta f - \operatorname{tr} \operatorname{Hess} f|_B)^2 dm &= \int 1_B(\Delta f - \Delta \varphi_j + \operatorname{tr} \operatorname{Hess} \varphi_j|_B - \operatorname{tr} \operatorname{Hess} f|_B)^2 dm \\ &\leq 2 \int 1_B \left(\Delta f - \Delta \varphi_j\right)^2 dm + 2 \int 1_B \left(\operatorname{tr}[\operatorname{Hess}(\varphi_j - f)]|_B\right)^2 dm \\ &\leq 2 \left\|\Delta f - \Delta \varphi_j\right\|^2_{L^2(m)} + n \left\||\operatorname{Hess}(f - \varphi_j)\right\|^2_{HS} \leq \epsilon\end{aligned}$$

where the last inequality holds for arbitrary $\epsilon > 0$ provided j is sufficiently large. We obtain that $\Delta f|_B = \operatorname{tr} \operatorname{Hess} f|_B$ m-a.e. and therefore the claim. ☐

2.7. **Upper and lower sectional curvature bounds for metric spaces.** We recall the following notions of spaces with curvature bounded below (above) for geodesic metric spaces

Definition 2.50. We say that a complete geodesic metric space (X,d) is $CBB(\kappa)$ or has curvature bounded below by $\kappa \in \mathbb{R}$ (respectively is $CAT(k)$) if for any triple of points $x,y,z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2\pi_\kappa$ the following condition holds.

For any geodesic $[xz]$ and $q \in]xz[$, we have

$$d(y,q) \geq d(\bar{y},\bar{q}) \,(\text{ respectively } d(y,q) \leq d(\bar{y},\bar{q})) \tag{8}$$

where $\triangle(\bar{x},\bar{y},\bar{z})_{\mathbb{S}^2_\kappa}$ is a comparison triangle in $\mathbb{S}^2_\kappa$ with $d(\bar{x},\bar{y}) = d(x,y), d(\bar{x},\bar{z}) = d(x,z), d(\bar{z},\bar{y}) = d(z,y)$ and $\bar{q} \in]\bar{x}\bar{z}[$ satisfies $d(\bar{q},\bar{x}) = d(q,x)$.

If $\kappa > 0$ and a X is a 1-dimensional manifold with possibly nonempty boundary for X to be $CBB(\kappa)$ we additionally require that $\operatorname{diam} X \le \pi_\kappa$.

Property (8) is equivalent to saying that for any unit speed geodesic $\gamma : [0,l] \to X$ such that

$$d(y,\gamma(0)) + l + d(\gamma(l), y) < 2\pi_k, \tag{9}$$

it holds that

$$[\mathrm{md}_\kappa(d_y \circ \gamma)]'' + \mathrm{md}_\kappa(d_y \circ \gamma) \le 1 (\text{ respectively } \ge 1) \tag{10}$$

A reformulation of this inequality is - taking into account the definition of md_κ via $\cos_\kappa$ -

$$\left[\frac{1}{\kappa}\cos_\kappa(d_y \circ \gamma)\right]'' \ge -\cos_\kappa(d_y \circ \gamma) \;\; (\text{respectively } \le \cos_\kappa(d_y \circ \gamma)) \;\text{ if }\; \kappa \neq 0 \;\;\&$$

$$\left[\frac{1}{2}d_y^2 \circ \gamma\right]'' \le 1 \;\; (\ge 1) \quad \text{if } \; \kappa = 0.$$

In particular, (X,d) has curv ≥ 0 (is $CAT(0)$) if and only if for any $y \in X$ the function $\frac{1}{2}d_y^2$ is 1-concave (1-convex).

We will refer to CBB version of inequality (8) as $(8)_{(CBB)}$ and to the CAT version of it as $(8)_{(CAT)}$. We will employ the same convention for (10).

Remark 2.51. It's immediate from the definition that in a $CAT(\kappa)$ space X geodesics of length $< \pi_\kappa$ are unique and depend continuously on their endpoints. Also, local geodesic of length $< \pi_\kappa$ are *distance minimizing*, i.e. are geodesics.

Definition 2.52. We say that a complete geodesic space (X,d) is $CBA(\kappa)$ (has curvature bounded above by κ) if for every point $p \in X$ there is $r_p > 0$ such that $(8)_{(CAT)}$ holds for any $x, y, z \in B_{r_p}(p)$.

Remark 2.53. In the above definition of $CBA(\kappa)$ we do not require that the geodesics $[xy], [xz], [yz]$ lie in $B_{r_p}(p)$. However, it immediately follows from $(8)_{(CAT)}$ that $B_r(p)$ is convex for any $r < \min(r_p, \pi_k/2)$. If X is $CAT(\kappa)$ this gives that $B_r(p)$ is convex and $\bar{B}_r(p)$ is $CAT(\kappa)$ for any $r < \pi_\kappa/2$.

Example 2.54. The standard sphere $\mathbb{S}^n$ and any geodesically convex subsets of $\mathbb{S}^n$ are $CAT(1)$. On the other hand, for $n \ge 2$ any non simply connected n-manifold of $\sec \equiv 1$ (e.g. $\mathbb{PR}^n$) is $CBA(1)$ but not $CAT(1)$.

Remark 2.55. It follows directly from the definition of $CAT(\kappa)$ and from the corresponding computations in $\mathbb{S}^2_\kappa$ that if X is $CAT(\kappa)$ then d_y, d_y^2, and $\mathrm{md}_\kappa(d_y)$ are *convex* in $B_{\pi_k/2}(y)$ for any y in X.

Remark 2.56. Similarly to the definition of $CBA(\kappa)$ one can define locally $CBB(\kappa)$ spaces by requiring that they satisfy $(8)_{(CBB)}$ locally. However, it turns out that this is equivalent to requiring that they satisfy $(8)_{(CBB)}$ *globally* by the Globalization Theorem [BGP92].

If (X,d) is $CAT(\kappa_1)$ $(CBB(\kappa_1))$ and $\kappa_1 \le \kappa_2$ ($\kappa_1 \ge \kappa_2$) then (X,d) is $CAT(\kappa_2)$ $(CBB(\kappa_2))$.

If (X,d) is $CAT(\kappa)$ $(CBB(\kappa))$ then $(X,\lambda d)$ is $CAT(\kappa/\lambda^2)$ $(CBB(\kappa/\lambda^2))$. Therefore after appropriate rescaling any $CAT(\kappa)$ $(CBB(\kappa))$ becomes $CAT(1)$ $(CBB(-1))$.

Theorem 2.57 ([BBI01] Theorem 4.7.1)**.** *Let (X,d) be a complete geodesic space. Then the euclidean cone $C(X)$ over (X,d) is $CAT(0)$ $(CBB(0))$ if and only if (X,d) is $CAT(1)$ $(CBB(1))$.*

Let X be $CAT(\kappa)$ and let $y \in X$. For $0 \le t \le 1$ let Φ_t^y be the $(1-t)$-homothety map centered at y defined on the ball $B_{\pi_\kappa}(y)$. That is, if $x \in B_{\pi_k}(y)$ and $\gamma\colon [0,1] \to X$ is the constant speed geodesic with $\gamma(0) = x, \gamma(1) = y$ then

$$\Phi_t^y(x) = \gamma(t) \;\text{ for any } t \in [0,1]. \tag{11}$$

Then Φ_t is 1-Lipschitz by the $CAT(\kappa)$ condition since the same holds true in $\mathbb{S}^2_\kappa$.

Lemma 2.58. *Let X be $CAT(\kappa)$ with* $\operatorname{diam} X < \pi_\kappa/2$. *Let $y \in X$.*

Let $y \in X$ and let Φ_t be as above. Then for any $n \geq 1, 0 \leq t \leq 1$ we have

(1) *For any Borel set $A \subset X$ it holds that $\mathcal{H}^n(\Phi_t(A)) \leq \mathcal{H}^n(A)$.*
(2) *For any Borel set $A \subset \Phi_t(X)$*

$$(\Phi_t)_\star \mathcal{H}^n(A) \geq \mathcal{H}^n(A) = (\Phi_0)_\star \mathcal{H}^n(A).$$

Definition 2.59. Given a point p in a $CAT(\kappa)$ space X we say that two unit speed geodesics starting at p define the same direction if the angle between them is zero. This is an equivalence relation by the triangle inequality for angles and the angle induces a metric on the set $S_p^g(X)$ of equivalence classes. The metric completion $\Sigma_p^g X$ of $S_p^g X$ is called the *space of geodesic directions* at p. The Euclidean cone $C(\Sigma_p^g X)$ is called the *geodesic tangent cone* at p and is denoted by $T_p^g X$.

The following theorem is due to Nikolaev [BH99, Theorem 3.19]:

Theorem 2.60. *$T_p^g X$ is $CAT(0)$ and $\Sigma_p^g X$ is $CAT(1)$.*

Note that this theorem in particular implies that $T_p^g X$ is a geodesic metric space which is not obvious from the definition. Note further that $\Sigma_p^g X$ need not be path connected. In this case the above theorem means that each path component of $\Sigma_p^g X$ is $CAT(1)$ and the distance between points in different components is π.

2.8. **Spaces with two sided sectional Alexandrov curvature bounds.** Spaces with two sides Alexandrov bounds (i.e. spaces satisfying $CBA(\kappa_1), CBB(\kappa_2)$ for some $\kappa_1, \kappa_2 \in \mathbb{R}$) have been studied by Alexandrov, Nikolaev and Berestovsky. The following structure theorem holds:

Theorem 2.61 ([BN93]). *Let (X, d) be a complete finite dimensional geodesic metric space which is $CBA(\kappa_1), CBB(\kappa_2)$ for some $\kappa_1, \kappa_2 \in \mathbb{R}$.*

Then $\kappa_2 \leq \kappa_1$ and X is an n-dimensional topological manifold (possibly with boundary) for some $n \geq 1$. Moreover, $\operatorname{Int} X$ *possesses a canonical $C^{3,\alpha}$-atlas for $\alpha \in (0,1)$ of harmonic coordinate charts such that in each chart d is induced by a Riemannian tensor g whose coefficients $g_{i,j}$ w.r.t. this chart are in the class $W^{2,p} \cap C^{1,\alpha}$ for any $1 \leq p < \infty, 0 < \alpha < 1$.*

Remark 2.62. If X has nonempty boundary then the boundary need not be smooth. E.g. if X is a closed convex body in $\mathbb{R}^n$ then it's $CBB(0)$ and $CBA(0)$.

The following lemma is elementary and is left to the reader as an exercise.

Lemma 2.63. *Let (X, d) be an n-dimensional space which is $CAT(\kappa)$ and $CBB(\kappa)$. Then X is isometric to a convex subset of $\mathbb{S}^n_\kappa$.*

3. Lower bounds for the distributional Laplace operator

The following lemma is well-known (see e.g. [BH99]) but we include the proof for completeness.

Lemma 3.1. *Let X be $CAT(\kappa)$ and let $p \in X$. Suppose $B_r(p)$ is a topological n-manifold for some $r < \pi_\kappa/2$. Then every geodesic $[xy] \subset B_r(p)$ can be extended to a geodesic with end points on $S_r(p)$.*

Proof. By completeness of X it's enough to show that geodesics can not terminate at points in $B_r(p)$. Suppose to the contrary that a geodesic $[xy] \subset B_r(p)$ can not be extended past y. By possibly changing x we can assume that $\bar{B}_{2l}(x) \subset U \subset B_r(p)$ where $l = d(x, y)$ and U is homeomorphic to $\mathbb{R}^n$. Since $H_{n-1}(U \backslash \{y\}) \cong \mathbb{Z} \neq 0$, the inclusion $i : \bar{B}_{2l}(y) \backslash \{y\} \to U \backslash \{y\}$ is not homotopic to a point. On the other hand, since $[xy]$ can not be extended past y, the "straight line" homotopy (which is continuous by remark 2.51) along geodesics emanating from x gives a homotopy of i and the constant map $\bar{B}_{2l}(y) \backslash \{y\} \to \{x\}$. This is a contradiction and hence all geodesics in $B_r(p)$ can be extended till they hit the sphere $S_r(p)$. □

Recall that if (X, d, m) is RCD, then by Theorem 2.17 and Corollary 2.19 $\operatorname{md}_\kappa(d_y), d_y^2/2 \in D(\mathbf{\Delta})$ and if X is compact, $\mathbf{\Delta} \operatorname{md}_\kappa(d_y), \mathbf{\Delta} d_y^2/2$ are signed Radon measures.

Theorem 3.2. *Let (X,d) be a metric space that is $CAT(\kappa)$ and $\operatorname{diam}_X < \pi_\kappa/2$. Assume $(X,d,\mathcal{H}^n)$ is a metric measure space satisfying the condition $RCD(K,n)$ for $n\in\mathbb{N}$. Let $x_0\in X$ be a point such that there is an open neighbourhood U of x_0 that is homeomorphic to an n-manifold. Then, there exists $\epsilon>0$ such that for any $y\in X$ we have*

$$[\mathbf{\Delta}\operatorname{md}_\kappa(d_y)]|_{B_\epsilon(x_0)} \geq -C(\kappa)\epsilon^2 \quad \& \quad [\mathbf{\Delta} d_y^2/2]|_{B_\epsilon(x_0)} \geq 0$$

in the distributional sense, where $C(\kappa)>0$ for $\kappa>0$ and 0 otherwise. In particular, $[\mathbf{\Delta}\operatorname{md}_\kappa(d_y)]|_{B_\epsilon(x_0)}$ and $[\mathbf{\Delta} d_y^2/2]|_{B_\epsilon(x_0)}$ are $\mathcal{H}^n$-absolutely continuous signed Radon measures.

Proof of Theorem 3.2. We first give a proof for $d_y^2/2$.

1. Assume w.l.o.g. that $B_{8\epsilon}(x_0)\subset U$ and U is homeomorphic to $\mathbb{R}^n$.

By the assumptions on the diameter of X the ball $\overline{B_{4\epsilon}(x_0)}$ is geodesically convex and geodesics in it are unique. Let $(Y,d_Y,m)=(\overline{B_{4\epsilon}(x_0)},d,\mathcal{H}_n)$

In particular, Y again satisfies $RCD(K,n)$ and is $CAT(\kappa)$.

Then by Lemma 3.1 there is $\delta=\delta(\epsilon,\kappa)>0$ such that any unit speed geodesic $\gamma:[0,L]\to\overline{B_{4\epsilon}(x_0)}$ we have that

(12) γ can be extended to a geodesic $\hat\gamma:[-\delta,L+\delta]\to B_{8\epsilon}(x_0)$ with $\hat\gamma|_{[0,L]}=\gamma$.

2. Now, let $y\in X$ and $B_{4\epsilon}(x_0)$ as before.

Let Φ_t^y be the homothety map defined in (11).

From (12) it easily follows that $B_\epsilon(x_0)\subset\Phi_t^y(B_{3\epsilon}(x_0))$ for all sufficiently small t.

Since by Lemma 2.58 $(\Phi_t)_\star\mathcal{H}^n(A)\geq\mathcal{H}^n(A)$ for any subset $A\subset\Phi_t(X)$, we obtain for any Lipschitz function g with compact support in $B_\epsilon(x_0)$

$$\int g(\Phi_t)_\star\mathcal{H}^n\geq\int gd\mathcal{H}^n.$$

3. On the other, if we define a probability measure $\mu_0=\mathcal{H}^n(B_{2\epsilon}(x_0))^{-1}\mathcal{H}^n|_{B_{2\epsilon}(x_0)}$, then $\mu_t=(\Phi_t)_\star\mu_0$, $t\in[0,1]$ is precisely the W_2-geodesic between μ_0 and δ_y in $\mathcal{P}^2(X)$ where μ_t is $\mathcal{H}^n$-absolutely continuous for $t\in[0,1)$ since X is $RCD(K,n)$ and hence $MCP(K,n)$.

Fix an $s\in(0,1)$. Then $\nu_t=\mu_{st}$ with $t\in[0,1]$ is constant speed geodesic between μ_0 and μ_s Moreover, $\varphi=sd_y^2/2$ is clearly a Kantorovich potential for μ_0 and $\mu_s=\nu_1$. Hence, we can apply (i) in Theorem 2.46 to ν_t.

This yields that for any $g\in W^{1,2}(X)$

$$-\int\langle\nabla g,\nabla sd_y^2/2\rangle d\mu_0=\lim_{t\to0}\frac{1}{t}\left[\int gd\mu_{st}-\int gd\mu_0\right]=\lim_{t\to0}\frac{1}{t}\left[\int g\circ\Phi_{st}d\mu_0-\int gd\mu_0\right].$$

Note the sign in front of the equality.

By definition of the distributional Laplace operator, we compute

$$\begin{aligned}\mathcal{H}^n(B_{2\epsilon}(x_0))^{-1}\mathbf{\Delta} d_y^2/2(g)&=-\int\frac{1}{s}\langle\nabla g,\nabla sd_y^2/2\rangle d\mu_0\\&=\lim_{t\to0}\frac{1}{st}\left[\int g\circ\Phi_{st}d\mu_0-\int gd\mu_0\right]\\&=\lim_{t\to0}\frac{1}{\mathcal{H}^n(B_{2\epsilon}(x_0))st}\left[\int gd(\Phi_{st})_\star\mathcal{H}^n-\int gd\mathcal{H}^n\right]\geq0\end{aligned}$$

for any $g\in\operatorname{Lip}_c(B_\epsilon(x_0))$, $g\geq0$. Hence $\mathbf{\Delta} d_y^2/2|_{B_\epsilon(x_0)}\geq0$ for any $y\in X$.

4. Recall from the proof of Corollary 2.19 that $\operatorname{md}_\kappa(d_y)=\varphi(d_y^2/2)$ for a smooth function $\varphi:\mathbb{R}\to\mathbb{R}$ such that $\varphi'(t)\geq0$ and $\varphi''(t)\geq-C(\kappa)$ for $|t|\leq\pi_\kappa$.

Hence, by the chain rule for the distributional Laplacian (Proposition 2.13) we obtain

$$\mathbf{\Delta}\operatorname{md}_\kappa(d_y)=\varphi'(d_y^2/2)\mathbf{\Delta} d_y^2/2+\varphi''(d_y^2/2)d_y^2\geq-C(\kappa)\epsilon^2.$$

$\square$

Corollary 3.3. *Let (X,d,m), d_y, $x_0 \in X$ and $U \subset X$ be as in Theorem 3.2. Then there exists $\epsilon > 0$ such that for any cutoff function $\chi \in \mathbb{D}^X_\infty$ with $\Delta\chi \in L^\infty(\mathcal{H}^n)$, $\operatorname{supp}\chi \subset B_\epsilon(x_0)$ and $\chi|_{B_{\epsilon/2}(x_0)} = 1$ it holds that $\chi \cdot \mathrm{md}_\kappa(d_y) \in D_{L^\infty}(\Delta) \cap L^\infty(\mathcal{H}^n) \cap \mathrm{Lip}(X)$.*

Proof. We choose $\epsilon > 0$ as in the previous theorem, and a corresponding cutoff function χ. Clearly it holds that $\mathrm{md}_\kappa(\chi \cdot d_y) \in L^\infty(\mathcal{H}^n) \cap \mathrm{Lip}(X)$. By Corollary 2.19 we also have $\mathrm{md}_\kappa(d_y) \in D(\mathbf{\Delta})$.

Hence, the Leibniz rule for the distributional Laplacian [Gig14, Theorem 4.12] yields

$$\mathbf{\Delta}(\chi \,\mathrm{md}_\kappa(d_y)) = \chi \mathbf{\Delta}\,\mathrm{md}_\kappa(d_y)|_{B_\epsilon(x_0)} + \mathrm{md}_\kappa(d_y)\Delta\chi + 2\langle \nabla\,\mathrm{md}_\kappa(d_y), \nabla\chi\rangle.$$

By Theorem 3.2 and again by Theorem 2.17 we know that $\chi\mathbf{\Delta}\,\mathrm{md}_\kappa(d_y) \in L^\infty(\mathcal{H}^n)$. It follows that $\mathbf{\Delta}(\chi \cdot \mathrm{md}_\kappa(d_y)) \in L^\infty(\mathcal{H}^n)$. Since $\chi \cdot \mathrm{md}_\kappa(d_y)$ is compactly supported in $B_\epsilon(x_0)$, we also get that $\chi \cdot \mathrm{md}_\kappa(d_y) \in D(\Delta)$ and therefore $\mathbf{\Delta}(\chi \cdot \mathrm{md}_\kappa(d_y)) = \Delta(\chi \cdot \mathrm{md}_\kappa(d_y))$. □

4. On the relation between convexity and the Hessian

In this section we explore the relation between convexity and almost everywhere lower bounds for the Hessian of a function f that is in a sufficiently regular subspace of $W^{2,2}(X)$. This relation has already been studied in previous publications [Ket15, GKKO17, Han17, GT17]. A novelty of our situation is that we give a localized statement that is needed in the course of the paper. Moreover, we will show that κ-convexity implies a lower κ-bound for the Hessian. By the second variation formula this lower bound holds if the Hessian is evaluated on gradients of Kantorovich potentials. However, we require the estimate for the Hessian evaluated on gradients of test functions.

Throughout this section let (X, d_X, m_X) be a compact metric measure space satisfying the condition $RCD(K,N)$, and let Z be a closed subset of X such that $m_X(\mathrm{Int}\, Z) > 0$, $m(\partial Z) = 0$ and (Z, d_Z, m_Z) is a metric measure space that also satisfies the condition $RCD(K,N)$. We denote by Δ^X, Γ_2^X ect. and Δ^Z, Γ_2^Z ect. the Laplace operator, the Γ_2-operator ect. of X and Z, respectively. In particular, (Z, d_Z) is geodesically convex and compact as well.

Let $f \in D(\Delta^X) \cap L^\infty(m_X) \cap \mathrm{Lip}(X)$ with $\|f\|_{L^\infty}, \||\nabla f|\|_{L^\infty} \le C$ for $C \in (0,\infty)$. In particular, $f \in H^{2,2}(X)$. We introduce the transformed measures

$$\widetilde{m}_Z := \left[e^{-f} m_X\right]|_Z \tag{13}$$

and consider the metric measure space $(Z, d_Z, \widetilde{m}_Z) = \widetilde{Z}$.. We remark that $f|_Z \in L^\infty(m_Z) \cap \mathrm{Lip}(Z)$ with $\|f\|_{L^\infty}, \||\nabla f|\|_{L^\infty} \le C$ but $f \notin D(\Delta^Z)$. We observe that, for $p \in [1,\infty]$,

$$e^{-C/p}\|u\|_{L^p(m_Z)} \le \|u\|_{L^p(\widetilde{m}_Z)} \le e^{C/p}\|u\|_{L^p(m_Z)}$$

for all $u \in L^p(m_Z) = L^p(\widetilde{m}_Z)$, and

$$e^{-C/p}\big\||\nabla u|\big\|_{L^p(m_Z)} \le \big\||\nabla u|\big\|_{L^p(\widetilde{m}_Z)} \le e^{C/p}\big\||\nabla u|\big\|_{L^p(m_Z)}$$

for all $u \in W^{1,2}(Z) = W^{1,2}(\widetilde{Z})$. In addition, the minimal weak upper gradient of $u \in W^{1,2}(\widetilde{Z})$ induced by $\widetilde{m}_Z$ coincides with $|\nabla u|$ (see [AGS14a, Lemma 4.11]).

Lemma 4.1. *Let f and $(Z, d_Z, \widetilde{m}_Z)$ be as above. Then we have $D(\Delta^{\widetilde{Z}}) = D(\Delta^Z)$ and, for any $u \in D(\Delta^{\widetilde{Z}})$,*

(i) $\Delta^{\widetilde{Z}} u = \Delta^Z u - \langle \nabla f, \nabla u\rangle$,

(ii) $\|\Delta^{\widetilde{Z}} u\|^2_{L^2(\widetilde{m}_Z)} \le 2e^{C/2}\left(\|\Delta^Z u\|^2_{L^2(m_Z)} + \||\nabla f|\|^2_{L^\infty}\big\||\nabla u|\big\|^2_{L^2(m_Z)}\right)$,

(iii) $\|\Delta^Z u\|^2_{L^2(m_Z)} \le 2e^{C/2}\left(\|\Delta^{\widetilde{Z}} u\|^2_{L^2(\widetilde{m}_Z)} + \||\nabla f|\|^2_{L^\infty}\big\||\nabla u|\big\|^2_{L^2(\widetilde{m}_Z)}\right)$.

In particular, if $u \in D(\Delta^{\widetilde{Z}})$, then $P^Z_t u \in D(\Delta^{\widetilde{Z}})$ and $P^Z_t u \to u$ in $D(\Delta^{\widetilde{Z}})$ as $t \to 0$.

Proof. The lemma can be found in [GKKO17, Lemma 3.4] where it is assumed that $f \in \mathbb{D}^Z_\infty$. However, one can easily check that the proof works for $f \in L^\infty(m) \cap \mathrm{Lip}(Z)$. □

Proposition 4.2. *Let (X,d,m), (Z,d_Z,m_Z), f and $\widetilde{Z}$ be as above. Assume $f|_Z$ is κ-convex on $(Z,d_Z,m|_Z)$ for $\kappa\in\mathbb{R}$. Then $\widetilde{Z}$ satisfies the condition $RCD(K+\kappa,\infty)$, and for $u\in\mathbb{D}^Z_\infty$ with* $\operatorname{supp} u\subset \operatorname{Int} Z$, $\varphi\in \operatorname{Lip}(Z)$, $\varphi\geq 0$ *and* $\widetilde{\varphi}:=e^f\varphi$ *we have* $u\in\mathbb{D}^{\widetilde{Z}}_\infty$ *and*

$$\begin{aligned}\int_Z(\kappa+K)|\nabla u|^2\varphi dm &\leq \int \widetilde{\varphi}d\mathbf{\Gamma}_2^{\widetilde{Z}}(u)\\ &= \int \varphi d\mathbf{\Gamma}_2^{Z}(u)+\int_Z \operatorname{Hess} f(\nabla u,\nabla u)\varphi dm. \end{aligned}\tag{14}$$

Remark 4.3. The conditions $RCD(K,N)$ and $m(Z)>0$ for $(Z,d_Z,m|_Z)$ imply that (Z,d_Z) is a complete, compact, *geodesic* metric space. Therefore, it makes sense to consider functions f on Z that are κ-convex in the sense of Definition 2.2.

Proof. **1.** That $\widetilde{Z}$ satisfies the condition $RCD(\kappa+K,\infty)$, follows from Fact 2.9 (iii), Lemma 2.10 and from the fact that $\widetilde{Z}$ is again infinitesimally Hilbertian.

2. We show that $u\in\mathbb{D}^{\widetilde{Z}}_\infty$. Since $u\in\mathbb{D}^Z_\infty$, we have by definition that $u\in D_{W^{1,2}}(\Delta^Z)\cap \operatorname{Lip}(Z)\cap L^\infty(m_Z)$. From Lemma 4.1 we know that $D(\Delta^Z)=D(\Delta^{\widetilde{Z}})$ and $\Delta^{\widetilde{Z}}u=\Delta^Z u-\langle\nabla u,\nabla f\rangle$. Since $u\in D_{W^{1,2}}(\Delta^Z)$ we already know that $\Delta^Z u\in W^{1,2}(Z)=W^{1,2}(\widetilde{Z})$. Moreover, since $\operatorname{supp} u\in \operatorname{Int} Z$, it easily follows that $u\in\mathbb{D}^X_\infty$. Then, since $f\in W^{2,2}(X)$ and since u and f are Lipschitz, it follows by Proposition 2.44 that $\langle\nabla u,\nabla f\rangle\in W^{1,2}(X)$. Hence, $\langle\nabla u,\nabla f\rangle\in W^{1,2}(Z)$. Consequently, $\langle\nabla u,\nabla f\rangle\in W^{1,2}(\widetilde{Z})$ and therefore $\Delta^{\widetilde{Z}}u\in W^{1,2}(\widetilde{Z})$. Moreover, since $u\in\mathbb{D}^X_\infty$ $\operatorname{Hess} f(\nabla u,\nabla u)\in L^2(m_X)$ is well-defined.

3. Since $\widetilde{Z}$ satisfies the condition $RCD(\kappa+K,\infty)$, the improved Bochner inequality yields for $u\in\mathbb{D}^{\widetilde{Z}}_\infty$

$$\mathbf{\Gamma}_2^{\widetilde{Z}}(u)=\frac{1}{2}\mathbf{\Delta}^{\widetilde{Z}}|\nabla u|^2-\langle\nabla u,\nabla\Delta u\rangle\left[e^{-f}m\right]|_Z\geq(\kappa+K)|\nabla u|^2\left[e^{-f}m\right]|_Z.$$

Recall that $|\nabla u|^2\in D(\mathbf{\Delta}^{\widetilde{Z}})$ if $u\in\mathbb{D}^{\widetilde{Z}}_\infty$. If we integrate $\widetilde{\varphi}=\varphi e^f\in\operatorname{Lip}(Z)$ w.r.t. the previous measures, the definition of the distributional Laplacian yields for the left hand side

$$\begin{aligned}\int\widetilde{\varphi}d\mathbf{\Gamma}_2^{\widetilde{Z}}(u)&=-\frac{1}{2}\int_Z\varphi e^f d\mathbf{\Delta}^{\widetilde{Z}}|\nabla u|^2-\int_Z\langle\nabla u,\nabla\Delta^{\widetilde{Z}}u\rangle\varphi dm\\ &=-\frac{1}{2}\int_Z\langle\nabla|\nabla u|^2,\nabla\varphi e^f\rangle e^{-f}dm-\int_Z\langle\nabla u,\nabla\Delta^{\widetilde{Z}}u\rangle\varphi dm\\ &=-\frac{1}{2}\int_Z\langle\nabla|\nabla u|^2,\nabla\varphi\rangle dm-\frac{1}{2}\int_Z\langle\nabla|\nabla u|^2,\nabla f\rangle\varphi dm\\ &\qquad\qquad-\int_Z\langle\nabla u,\nabla\Delta^Z u\rangle\varphi dm+\int_Z\langle\nabla u,\nabla\langle\nabla u,\nabla f\rangle\rangle\varphi dm\\ &=\int\varphi d\mathbf{\Gamma}_2^Z(u)+\int_Z\operatorname{Hess} f(\nabla u,\nabla u)\varphi dm.\end{aligned}$$

For the last equality also recall that for every $g\in W^{1,2}(X)$ we have $g\in W^{1,2}(Z)=W^{1,2}(\widetilde{Z})$ and $|\nabla^Z g|=|\nabla^{\widetilde{Z}}g|=|\nabla^X g||_Z$. This completes the proof of the proposition. $\square$

Let us first recall the following lemma from [AMS16, Lemma 6.7].

Lemma 4.4. *Let (X,d,m) be a metric measure space satisfying a RCD-condition. Then for all $E\subset X$ compact and all $G\subset X$ open such that $E\subset G$ there exists a Lipschitz function $\chi:X\to[0,1]$ with*

(i) $\chi=1$ *on* $E_h=\{x\in X:\exists y\in E: d(x,y)<h\}$ *and* $\operatorname{supp}\chi\subset G$,

(ii) $\mathbf{\Delta}\chi\in L^\infty(m)$ *and* $|\nabla\chi|^2\in W^{1,2}(X)$.

Remark 4.5. Following the proof of this Lemma in [AMS16] we see that one can choose χ to be in $\mathbb{D}^X_\infty$.

Corollary 4.6. *Let X, Z and f be as in Proposition 4.2. Then*

$$\operatorname{Hess} f(1_Z\nabla u, 1_Z\nabla u) = 1_Z \operatorname{Hess} f(\nabla u, \nabla u) \geq \left[\kappa|\nabla u|^2\right] 1_Z \quad m\text{-a.e.} \tag{15}$$

for every $u \in \mathbb{D}^X_\infty$.

Proof. **1.** The first equality in (15) is the L^∞-homogeneity of the Hessian.

Let $\alpha > 0$ and define $f/\alpha = f_\alpha$. Then f_α is $\frac{\kappa}{\alpha}$-convex. Let $u \in \mathbb{D}^X_\infty$ and choose a cut-off function $\chi \in \mathbb{D}^X_\infty$ such that $\chi = 1$ on $A \subset \operatorname{Int} Z$ for a closed set A and $\operatorname{supp} u \subset \operatorname{Int} Z$. Then $\chi \cdot u \in \mathbb{D}^Z_\infty$ with $\operatorname{supp} \chi \cdot u \subset \operatorname{Int} Z$. Therefore by Proposition 4.2

$$\int_Z (\kappa/\alpha + K)|\nabla(\chi\cdot u)|^2 \varphi dm \leq \int \varphi d\mathbf{\Gamma}^Z_2(\chi\cdot u) + \int_Z \operatorname{Hess}(f/\alpha)(\nabla(\chi\cdot u), \nabla(\chi\cdot u))\varphi dm. \tag{16}$$

Hence, multiplying with $\alpha > 0$ and letting $\alpha \to 0$ this yields

$$\kappa \int |\nabla(\chi\cdot u)|^2 \varphi dm \leq \int_Z \operatorname{Hess} f(\nabla(\chi\cdot u), \nabla(\chi\cdot u))\varphi dm$$

for every nonnegative $\varphi \in \operatorname{Lip}(X)$. By standard approximation the same holds for any nonnegative $\varphi \in C_b(Z)$.

2. We choose a sequence of nonnegative $\varphi_k \in C_b(X)$, $k \in \mathbb{N}$, compactly supported in $\operatorname{Int} Z$ such that $\varphi_k \uparrow 1$ pointwise m-a.e. . Moreover, we choose cut-off functions χ_k as in **1.** with $A = \operatorname{supp} \varphi_k$. Then

$$\kappa \int |\nabla u|^2 \varphi_k dm \leq \int \operatorname{Hess} f(\nabla u, \nabla u)\varphi_k dm$$

for every $k \in \mathbb{N}$ and $u \in \mathbb{D}^X_\infty$. Since $m(\partial Z) = 0$, letting $k \to \infty$ yields the claim. □

Theorem 4.7. *Let (X, d, m) be a compact metric measure space that satisfies the condition $RCD(K, N)$ for $K \in \mathbb{R}$ and $N > 0$, and $Z \subset X$ be a closed subset such that $m(\operatorname{Int} Z) > 0$, $m(\partial Z) = 0$ and $(Z, d_Z, m|_Z)$ satisfies the condition $RCD(K, N)$ as well. Let $f \in D_{L^2}(\Delta) \cap \operatorname{Lip}(X) \cap L^\infty$. Let $\kappa \in \mathbb{R}$. Then the following statements are equivalent*

(i) *f is κ-convex on (Z, d_Z),*

(ii) *$\operatorname{Hess} f(1_Z\nabla u, 1_Z\nabla u) \geq \kappa|\nabla u|^2 1_Z$ m-a.e. for $u \in \mathbb{D}^X_\infty$.*

Proof. The implication (i) ⇒ (ii) is precisely the content of the previous corollary.

For (ii) ⇒ (i) assume $\operatorname{Hess} f(\nabla u, \nabla u) \geq \kappa|\nabla u|^2$ m-a.e. on Z for any $u \in W^{1,2}(X)$. Then the second variation formula in Theorem 2.46 implies that $t \in [0, 1] \mapsto \mathcal{F}(\mu_t) = \int f d\mu_t$ is in $C^2([0, 1])$ for a L^2-Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ with $\mu_t \leq Cm$ for some constant $C > 0$, and

$$\frac{d^2}{dt^2}\mathcal{F}(\mu_t) = \int \operatorname{Hess}(\nabla\varphi_t, \nabla\varphi_t) d\mu_t \geq \kappa \int |\nabla\varphi_t|^2 d\mu_t = \kappa W_2(\mu_0, \mu_1)^2.$$

Hence, $t \in [0, 1] \mapsto \mathcal{F}(\mu_t)$ is κ-convex, and we obtain that

$$\mathcal{F}(\mu_t) \leq (1 - t)\mathcal{F}(\mu_0) + t\mathcal{F}(\mu_1) - \frac{1}{2}K(1 - t)tW_2(\mu_0, \mu_1)^2 \tag{17}$$

Now, we know that for every point $x_0 \in X$ and m-a.e. point $x_1 \in X$ there exists a unique geodesic γ (Corollary 1.4 in [GRS16]). We pick two such points in Z, and sequences of m-absolutely continuous probability measures $(\mu^k_0)_{k\in\mathbb{N}}$ and $(\mu^k_1)_{k\in\mathbb{N}}$ (for instance $\mu^k_i = m(B_{1/k}(x_i))^{-1} m|_{B_{1/k}(x_i)}$) such that $\mu^k_i \to \delta_{x_i}$ weakly for every $k \in \mathbb{N}$. Assume moreover that μ^k_i, $i = 0, 1$, satisfies $\mu^k_i \leq C(k)m$. Then the Wasserstein geodesic $(\mu^k_t)_{t\in[0,1]}$ between μ^k_0 and μ^k_1 satisfies $\mu^k_t \leq \widetilde{C}(k, C(k), K, N)m$ by [Raj12a]. Hence, (17) holds for $(\mu^k_t)_{t\in[0,1]}$. We can extract a subsequence such that $(\mu^k_t)_{k\in\mathbb{N}}$ converges for $t \in [0, 1] \cap \mathbb{Q}$ to ν_t for a geodesic $(\nu_t)_{t\in[0,1]}$ between δ_{x_0} and δ_{x_1}. Since x_0 and x_1

are chosen such that there is only one geodesic γ in X between them, we must have $\nu_t = \delta_{\gamma(t)}$. Moreover, since f is continuous, by weak convergence of μ_t^k, $\mathcal{F}(\mu_t^k) \to f(\gamma(t))$ for every $t \in [0,1] \cap \mathbb{Q}$. Hence, $\forall t \in [0,1] \cap \mathbb{Q}$, we have

$$f(\gamma_t) \leq (1-t) f(\gamma_0) + t f(\gamma_1) - \frac{1}{2} K (1-t) t d^2(\gamma_0, \gamma_1) \tag{18}$$

and by continuity this holds for every $t \in [0,1]$. Now, one can easily see that this implies (i) for $f : Z \to \mathbb{R}$ in the weak sense. That is for any pair of points $x_0, x_1 \in Z$ we can find a geodesic γ such that the inequality holds. Indeed, if x_0 and x_1 are arbitrary, we can find a point $\widetilde{x}_1^k$ such that $d(x_1, \widetilde{x}_1^k) \to 0$ if $k \to \infty$ and such that the geodesic $\widetilde{\gamma}^k = [x_0, \widetilde{x}_k]$ is unique. Then (18) holds for $\widetilde{\gamma}^k$. Since X is locally compact, by passing to a subsequence we can assume that $[x_0, \widetilde{x}_k] \to [x_0, x_1]$. Since f and d are continuous by passing to the limit we obtain that (18) holds for $[x_0, x_1]$ as well.

Finally, we recall the following theorem by Sturm.

Theorem 4.8 ([Stu14])**.** *Let (X, d, m) be a metric measure space that is locally compact and satisfies the condition $RCD(K, N)$ for $K \in \mathbb{R}$ and $N \in (0, \infty]$. Let $V : X \to (-\infty, \infty]$ be a function that is continuous and satisfies $V(x) \geq -C_2 - C_1 d(x_0, x)^2$ for constants $C_1, C_2 > 0$ and $x_0 \in X$. Let $\kappa \in \mathbb{R}$. Then, the following properties are equivalent:*

(i) *V is weakly κ-convex,*
(ii) *V is κ-convex,*
(iii) *For any $x_0 \in X'$ there exists a curve $(x_t)_{t \geq 0}$ in X' such that for all $z \in X'$ and every $t > 0$ we have*

$$\frac{1}{2} \frac{d}{dt} d(x_t, z)^2 + \frac{\kappa}{2} d(x_t, z)^2 \leq V(z) - V(x_t).$$

where X' is the closure of $\operatorname{Dom} V$ in X. We say $(x_t)_{t \geq 0}$ is an EVI_κ gradient flow curve.

This finishes the proof. $\square$

Remark 4.9. An EVI_κ gradient flow curve $(x_t)_{t \geq 0}$ of V comes with the parametrization such that

$$\frac{d}{dt} V(x_t) = -|\nabla^- V|^2(x_t) = -|\dot{x}_t|^2,$$

where $|\nabla^- V|(x) = \limsup_{y \to x} \frac{(V(y) - V(x))^-}{d(x,y)}$ is the descending slope of V that is general different from the minimal weak upper gradient that was defined before. Hence, an EVI_κ gradient flow is actually an inverse gradient flow in the standard sense.

5. RCD+CAT implies Alexandrov

The goal of this section is to prove the following Theorem which implies Theorem 1.1 by globalization under the extra assumption that X is infinitesimally Hilbertian. In section 6 we will show that the infinitesimal hilbertianness assumption can be dropped which will finish the proof of Theorem 1.1 in full generality.

Theorem 5.1. *Let $2 \leq n \in \mathbb{N}$ and (X, d, m) be a metric measure space satisfying $RCD(K, n)$ for $K \in \mathbb{R}$ with $m = \mathcal{H}^n$, and assume (X, d) is also $CAT(\kappa)$. Then*

(1) *$\kappa(n-1) \geq K$ and (X, d) is an Alexandrov space of curvature bounded below by $K - \kappa(n-2)$.*

(2) *If $\kappa(n-1) = K$, then (X, d) is isometric to a geodesically convex subset of the simply connected space form $\mathbb{S}^n_\kappa$ of constant curvature κ.*

In the proof of Theorem 5.1 we will need the following elementary lemma

Lemma 5.2. *Let $\kappa, K \in R, n \geq 2$. Let $\hat{\kappa} < K - (n-2)\kappa$. There is $\nu = \nu(n, K, \kappa, \hat{\kappa}) > 0$ such that*

$$(n-1) \cot_{K/(n-1)}(t) - (n-2) \cot_\kappa(t) < \cot_{\hat{\kappa}}(t) \text{ for all } 0 < t < \nu. \tag{19}$$

Proof. For any real k we have the following Taylor expansions at 0

$$\sin_k(t) = t - \frac{kt^3}{6} + \dots, \quad \cos_k(t) = 1 - \frac{kt^2}{2} + \dots$$

Hence

$$\cot_k(t) = \frac{1 - \frac{kt^2}{2} + \dots}{t - \frac{kt^3}{6} + \dots} = \frac{1}{t}(1 - \frac{kt^2}{2} + \dots)(1 + \frac{kt^2}{6} + \dots) = \frac{1}{t} - \frac{kt}{3} + \dots$$

Applying this to both sides of (19) yields the Lemma. □

Proof of Theorem 5.1. We will prove the theorem via induction w.r.t. $n \in \mathbb{N}$.

1. Let $n \geq 2$ and suppose Theorem 5.1 is true for $n-1$ if $n > 2$. Let $(X, d, \mathcal{H}_n)$ be $RCD(K, n)$ and $CAT(\kappa)$. The base of induction $n = 2$ will be handled in the same way as the general induction step with one small difference which we'll explicitly indicate.

Following Gigli and Philippis [DPG17] for any $x \in X$ we consider the monotone quantity $\frac{m(B_r(x))}{v_{k,n}(r)}$ which is non increasing in r by the Bishop-Gromov volume comparison. Let $\theta_{n,r}(x) = \frac{m(B_r(x))}{\omega_n r^n}$. Consider the density function $\theta_n(x) = \lim_{r\to 0} \theta_{n,r}(x) = \lim_{r\to 0} \frac{m(B_r(x))}{\omega_n r^n}$.

Since n is fixed throughout the proof we will drop the subscripts n and from now on use the notations $\theta(x)$ and $\theta_r(x)$ for $\theta_n(x)$ and $\theta_{n,r}(x)$ respectively.

Note that $\theta(x) = 1$ a.e. by [DPG17] but it still makes sense and is well defined pointwise. Also, by [DPG17] θ is lower semicontinuous and hence $0 < \theta(x) \leq 1$ for any $x \in X$.

Let $x \in X$ be arbitrary. Let $(T_xX, d_x, m_x, o) = \lim_{r_i \to 0}(X, \frac{1}{r_i}d, \frac{1}{r_i^n}m, x)$ be a tangent cone at x. Note that under this normalization along the sequence the unit balls around x *do not* have measure 1 but the measure $\frac{1}{r_i^n}m$ is equal to $\mathcal{H}_n$ with respect to the rescaled metric $\frac{1}{r_i}d$. Obviously, T_xX is $CAT(0)$ and $RCD(0, n)$. By [DPG17] it is also noncollapsed i.e. $m_x = \mathcal{H}_n$. Therefore, by the defintion of θ we have that $\theta(x) = \frac{\mathcal{H}_n(B(o,1))}{\omega_n}$ where o is the apex of T_xX. Note that this is true even if the tangent cone (T_xX, d_x, m_x, o) is not unique.

Further, (T_xX, d_x, m_x, o) is a volume metric cone by [DPG17, Proposition 2.7]. Therefore, by the *volume-cone-implies-metric-cone* theorem in [DPG16], $T_xX = C(\Sigma)$ where $(\Sigma, d_\Sigma, m_\Sigma)$ is both $CAT(1)$ and $RCD(n-2, n-1)$ and $m_\Sigma = \mathcal{H}_{n-1}$.

Claim: Σ is isometric to $\mathbb{S}^{n-1}$ or to a convex subset of $\mathbb{S}^{n-1}$ with nonempty interior and nonempty boundary.

Indeed, if $n = 2$ then $(\Sigma, d_\Sigma, m_\Sigma)$ is a noncollapsed compact $RCD(0, 1)$ space which is also $CAT(1)$. Hence by [KL16] it's isometric to either a circle $\mathbb{S}^1_R$ of some radius $R > 0$ or to a closed interval I.

Suppose $\Sigma \cong \mathbb{S}^1_R$. Since Σ is $CAT(1)$ we have that $R \geq 1$. On the other hand, since $C(\Sigma)$ is $RCD(0, 2)$ we must have $R \leq 1$. Hence $R = 1$ and $\Sigma \cong \mathbb{S}^1$.

If $\Sigma = I$ then length of I is at most π since otherwise $C(\Sigma)$ is not $RCD(0, 2)$ by the splitting theorem.

If $n > 2$ then the Claim follows by the induction assumption. This is the only place in the proof of the induction step where the induction assumption is used and where the induction step differs from the proof of the base of induction $n = 2$.

Since a proper convex subset of $\mathbb{S}^{n-1}$ is contained in a hemisphere the above Claim implies that we have the following gap phenomena:

$$\text{A point } x \in X \text{ is either regular, in which case } \theta(x) = 1, \\ \text{or } x \in X \text{ is singular, in which case } \theta(x) \leq \tfrac{1}{2} \text{ and } \partial\Sigma \neq \emptyset. \tag{20}$$

Next we prove the following lemma.

Lemma 5.3. *The set of regular points X_{reg} is open and convex in X.*

Since θ is lower semicontinuous, property (20) immediately implies that X_{reg} is open. It remains to verify that it is convex.

Let $f_r(x) = \theta_r(x)^{1/n}$ and $f(x) = \theta(x)^{1/n}$

Lemma 5.4. *$f(x)$ is semiconcave on X.*

Proof. To prove semiconcavity we need to verify that there is a constant C such that every point in X has a neighborhood U such that for any constant speed geodesic $\gamma\colon [0,1] \to U$ and any $0 \le t \le 1$ it holds that

$$f(\gamma(t)) \ge (1-t)f(0) + tf(1) - \frac{C}{2}t(1-t)d(\gamma(0),\gamma(1))^2 \tag{21}$$

To simplify the exposition we will only treat the case $t = 1/2$, i.e. we will verify that

$$f(\gamma(1/2)) \ge \frac{1}{2}f(0) + \frac{1}{2}f(1) - \frac{C}{8}d(\gamma(0),\gamma(1))^2 \tag{22}$$

The proof below easily adapts to the case of general t.

Let $x = \gamma(0), y = \gamma(1), z = \gamma(1/2)$ and $l = d(x, y)$.

By rescaling we can assume that X is $CAT(1)$ and $RCD(-n, n)$.

We will need the following general lemma.

Lemma 5.5. *Let X be a $CAT(\kappa)$ space. Let $\gamma_1, \gamma_2\colon [0,1] \to X$ be constant speed geodesics with length of γ_1 equal to $l < \pi_\kappa/100$ and suppose $d(\gamma_1(0), \gamma_2(0)) \le \delta, d(\gamma_1(1), \gamma_2(1)) \le \delta$ with $\delta < l/100$.*

Then $d(\gamma_1(1/2), \gamma_2(1/2)) \le \delta(1 + C(\kappa)l^2)$ for some universal $C(\kappa) \ge 0$.

Proof. It's well known that when $\kappa \le 0$ one can take $C(\kappa) = 0$ since in this case $t \mapsto d(\gamma_1(t), \gamma_2(t))$ is convex. We will therefore restrict our attention to the case $\kappa > 0$. By rescaling we can assume that $\kappa = 1$. Let X be $CAT(1)$.

Fix a point $\bar{p}$ in the unit round sphere $\mathbb{S}^2$. Let $0 \le t \le 1$. Consider the "t-homothety" map $\varphi_t^{\bar{p}}\colon B_{\pi/100}(\bar{p}) \to B_{\pi/100}(\bar{p})$ sending any point x to the point y on the unique geodesic connecting $\bar{p}$ to x with $d(\bar{p}, y) = td(\bar{p}, x)$. A direct Jacobi field computation shows that the Lipschitz constant of $\varphi_t^{\bar{p}}$ at x with $|\bar{p}x| = l$ is $\frac{\sin(tl)}{\sin(l)} = \sigma_{1,1}^t(l)$.

Taylor expanding in l we get:

$$\begin{aligned}\sigma_{1,1}^t(l) = \frac{\sin(tl)}{\sin(l)} = \frac{tl - (tl)^3/6 + \dots}{l - (l)^3/6 + \dots} = t(1 - t^2l^2/6 + \dots)(1 + l^2/6 + \dots) = \\ = t(1 + l^2(1-t^2)/6 + \dots) \le t(1 + l^2/3) \text{ if } l < \pi/100\end{aligned}$$

which immediately gives that if $d(x, y) < d(\bar{p}, x)/10$ then $d(\varphi_t^{\bar{p}}(x), \varphi_t^{\bar{p}}(y)) \le t(1 + d(\bar{p}, x)^2)d(x, y)$.

The definition of a $CAT(1)$ space immediately gives that the same inequality holds for a similarly defined map φ_t^p for any $p \in X$.

$$d(\varphi_t^p(x), \varphi_t^p(y)) \le t(1 + d(p, x)^2)d(x, y) \text{ if } d(p, x) < \pi/100,\ d(x, y) < d(p, x)/100. \tag{23}$$

Let γ_1, γ_2 be as in the lemma. Let $x = \gamma_1(1/2) = \varphi_{1/2}^{\gamma_1(0)}(\gamma_1(1)) = \varphi_{1/2}^{\gamma_1(1)}(\gamma_1(0))$, $y = \gamma_2(1/2) = \varphi_{1/2}^{\gamma_2(0)}(\gamma_2(1)) = \varphi_{1/2}^{\gamma_2(1)}(\gamma_2(0))$. Let z be the midpoint between $\gamma_1(0)$ and $\gamma_2(1)$.

Then by (23) we have that $d(z, x) \le \frac{1}{2}(1 + l^2)d(\gamma_1(1), \gamma_2(1)) \le \frac{\delta}{2}(1 + 2l^2)$ and $d(z, y) \le \frac{1}{2}(1 + (l + \delta)^2)d(\gamma_1(0), \gamma_2(0)) \le \frac{\delta}{2}(1 + 2l^2)$.

By the triangle inequality this gives that $d(x, y) \le d(x, z) + d(z, y) \le \frac{\delta}{2}(1 + 2l^2) + \frac{\delta}{2}(1 + 2l^2) = \delta(1 + 2l^2)$ which finishes the proof of Lemma 5.5 with $C(1) = 2$. $\square$

We are now ready to continue with the proof of Lemma 5.4.

Let A be the Minkowski sum $\frac{1}{2}B_r(x) + \frac{1}{2}B_r(y)$. By the Brunn-Minkowski inequality [CM15] we have that

$$\begin{aligned}m(A)^{1/n} &\ge \sigma_{-n,n}^{1/2}(l + 2r)m(B_r(x))^{1/n} + \sigma_{-n,n}^{1/2}(l + 2r)m(B_r(y))^{1/n} \\ &= \frac{\sinh\frac{l+2r}{2}}{\sinh(l + 2r)}m(B_r(x))^{1/n} + \frac{\sinh\frac{l+2r}{2}}{\sinh(l + 2r)}m(B_r(y))^{1/n}\end{aligned}$$

$$= \frac{1}{2\cosh\frac{l+2r}{2}} m(B_r(x))^{1/n} + \frac{1}{2\cosh\frac{l+2r}{2}} m(B_r(y))^{1/n}$$

Thus

$$(1+c_1l^2)m(A)^{1/n} \geq \cosh\frac{l+2r}{2} m(A)^{1/n} \geq \frac{1}{2} m(B_r(x))^{1/n} + \frac{1}{2} m(B_r(y))^{1/n} \tag{24}$$

where the first inequality holds when $l < 1/100$ and $r \ll l$.

By Lemma 5.5 we have that $A \subset B_{r(1+c_2l^2)}(z)$. Therefore

$$(1+c_1l^2)m(B_{r(1+c_2l^2)}(z))^{1/n} \geq \frac{1}{2} m(B_r(x))^{1/n} + \frac{1}{2} m(B_r(y))^{1/n}$$

Dividing by $\omega_n^{1/n} r$ this gives

$$(1+c_1l^2)(1+c_2l^2)\frac{m(B_{r(1+c_2l^2)}(z))^{1/n}}{\omega_n^{1/n} r(1+c_2l^2)} \geq \frac{1}{2r\omega_n^{1/n}} m(B_r(x))^{1/n} + \frac{1}{2r\omega_n^{1/n}} m(B_r(y))^{1/n}$$

or

$$(1+c_3l^2)f_{r(1+c_2l^2)}(z) \geq \frac{1}{2} f_r(x) + \frac{1}{2} f_r(y)$$

Taking the limit as $r \to 0$ this gives

$$(1+c_3l^2)f(z) \geq \frac{1}{2} f(x) + \frac{1}{2} f(y)$$

Taking into the account that $0 \leq f(z) \leq 1$ this gives

$$f(z) \geq \frac{1}{2} f(x) + \frac{1}{2} f(y) - c_3l^2$$

which finishes the proof of (22) and hence of Lemma 5.4. □

Since a bounded semiconcave function on a closed interval $I \subset \mathbb{R}$ is continuous on the interior of I, the openness of X_{reg} together with the gap property (20) immediately imply that f must be equal to 1 along any geodesic with endpoints in X_{reg}. Again using (20) we conclude that X_{reg} is convex.

This finishes the proof of Lemma 5.3. □

2.

Claim: X_{reg} is a topological n-manifold.

Proof. By Lemma 5.3 X_{reg} is open. Therefore it is an n-manifold by Reifenberg's principle [CC97, Theorem A.1.1]. □

3.

Fix an arbitrary $\hat\kappa < K-(n-2)\kappa$ and let $\nu = \nu(n, K, \kappa, \hat\kappa)$ be provided by the Lemma 5.2.

We pick $x_0 \in X_{reg}$ and a positive $\epsilon < \min\{\nu/2, \pi_\kappa/2\}$ such that $\overline{B}_\epsilon(x_0) =: Y$ is contained in X_{reg}.

Then Y is geodesically convex, uniquely geodesic, $(Y, d_Y, m|_Y)$ satisfies $RCD(K,n)$ and (Y, d_Y) is $CAT(\kappa)$.

Let $y \in B_\epsilon(x_0)$ and consider $d_y : Y \to [0,\infty)$. We pick any point $z \in B_\epsilon(x_0)\backslash\{y\}$ and $0 < \delta < \min(\frac{\epsilon-d(x_0,z)}{2}, \frac{d(z,y)}{2})$. Then $\overline{B_\delta(z)}$ is convex, it is contained in $B_\epsilon(x_0)$ and $y \notin \overline{B_\delta(z)}$. We also can pick a cutoff function $\chi \in \mathbb{D}^Y_\infty$ (see Lemma 4.4) that $\Delta\chi \in L^\infty(\mathcal{H}^n)$, $\operatorname{supp}\chi \subset B_\delta(z)$ and $\chi|_{B_{\delta/2}(z)} = 1$.

Then, by Corollary 3.3 $\chi \cdot d_y \in D_{L^\infty}(\Delta) \cap L^\infty(\mathcal{H}^n) \cap \operatorname{Lip}(X)$ and therefore $\chi \cdot d_y \in H^{2,2}(X)$ by Remark 2.43.

In particular, $\chi \cdot d_y$ induces an element $\nabla(\chi \cdot d_y) =: u \in L^2(TY)$ such that $|u| = 1 \neq 0$ m-a.e. on $B_{\delta/2}(z)$. We consider the submodule $\mathcal{N} \subset L^2(TY)$ that is generated by u. The orthogonal submodule is defined as

$$\mathcal{N}^\perp = \left\{ v \in L^2(TY) : \langle v, u \rangle = 0 \ m\text{-a.e.} \right\}.$$

It is not hard to check that $\mathcal{N}^\perp$ is an L^∞-premodule in the sense of Definition 1.2.1 in [Gig14]. $\mathcal{N}$ and $\mathcal{N}^\perp$ are Hilbert spaces and hence L^2-normed L^∞-modules (compare with Proposition 1.2.21 in [Gig14]). Moreover $\mathcal{N}^\perp$ is the orthogonal complement of $\mathcal{N}$ in the sense of Hilbert spaces, and $\mathcal{N} \oplus \mathcal{N}^\perp = L^2(TY)$.

According to Proposition 2.26 $\mathcal{N}^\perp$ yields a partition $\{B_k\}_{k\in\mathbb{N}}$ of Y such that the local dimension (in the sense of Definition 2.23) of $\mathcal{N}^\perp|_{B_k}$ is $k \in \mathbb{N}$. Note that $m(B_\infty) = 0$ since the local dimension of $L^2(TY)$ is finite on X, and therefore $\mathcal{N}^\perp$ is finitely generated m-a.e. as well. Hence, if $k \in \mathbb{N}$ such that $m(B_k) > 0$, any subset B of B_k with finite measure admits a unit orthogonal module basis $v_1, \dots, v_k$, and for any $v \in \mathcal{N}^\perp$ we have that $v1_B$ is the L^2-limit of finite L^∞-linear combinations in $\mathcal{N}^\perp$ of the form

$$w = \sum_{j=1}^{k} f_j v_j \quad \text{for } f_j \in L^\infty(m), \ j = 1, \dots, k.$$

At the same time we have that any $w \in L^2(TY)$ can be written as a sum $\alpha u + \beta v$ with $\alpha, \beta \in \mathbb{R}$ and $v \in \mathcal{N}^\perp$. Hence $u, v_1, \dots, v_k$ generates $L^2(TY)|_B = (\mathcal{N} \oplus \mathcal{N}^\perp)|_B$ in the sense of modules. Moreover, since $\mathcal{N}$ and $\mathcal{N}^\perp$ are orthogonal w.r.t. the pointwise inner product $\langle \cdot, \cdot \rangle$ and since $|u| = 1$ m-a.e. on $B_{\delta/2}(z)$, it is easy to check that $u, v_1, \dots, v_k$ are linearly independent on $B_{\delta/2}(z) \cap B$ (again in the sense of Definition 2.23), and hence form a module basis of $L^2(TY)$ on $B_{\delta/2}(z) \cap B$. Since the local dimension of $L^2(TY)$ is n this implies $k = n-1$ whenever $B_{\delta/2}(z) \cap B \neq \emptyset$, and $u =: E_1, v_2 =: E_2, \dots, v_{n-1} =: E_n$ is a unit orthogonal basis of $L^2(TY)$ on $B_{\delta/2}(z) \cap B$. In particular, for the decomposition $\{B_k\}_{k\in\mathbb{N}}$ we have $m(B_k) = 0$ if $k \neq n-1$, and we can choose B as the ball $B_{\delta/2}(z)$ itself.

4. Again from Corollary 3.3 and we have that $\mathrm{md}_{K/(n-1)}(\chi \cdot d_y) \in D_{L^\infty}(\Delta) \cap L^\infty(\mathcal{H}^n) \cap \mathrm{Lip}(X)$. Then the precise estimate in the Laplace operator comparison statement (Theorem 2.17) for $\mathrm{md}_{K/(n-1)} d_y$ on Y yields

$$\Delta(\mathrm{md}_{K/(n-1)}(\chi \cdot d_y))|_{B_{\delta/2}(z)} = \boldsymbol{\Delta} \, \mathrm{md}_{K/(n-1)}(d_y)|_{B_{\delta/2}(z)} \leq \frac{Kn}{n-1} \mathrm{md}_{K/(n-1)}(d_y)|_{B_{\delta/2}(z)} \ m\text{-a.e.}$$

where we used the locality property of $\boldsymbol{\Delta}$. Applying the chain rule for the Laplacian yields

$$\Delta(\chi \cdot d_y)|_{B_{\delta/2}(z)} \leq (n-1) \cot_{K/(n-1)}(d_y)|_{B_{\delta/2}(z)}. \tag{25}$$

On the other hand the condition $CAT(\kappa)$ on Y implies the following. First, by continuity reasons for any $\vartheta > 0$ there exists $\eta > 0$ as above such that for any η-ball $|\mathrm{md}_\kappa d_y(x) - \mathrm{md}_\kappa d_y(z)| \leq \vartheta$ for $x \in B_\eta(z)$. Therefore, if we choose $\delta/2 \leq \eta$

$$(\mathrm{md}_\kappa(\chi d_y) \circ \gamma)'' \geq 1 - \mathrm{md}_\kappa d(z,y) - \vartheta =: \lambda(\kappa, \vartheta, z) \ \text{ for any unit speed geodesic } \gamma \text{ in } B_{\delta/2}(z).$$

Hence, $\mathrm{md}_\kappa(\chi d_y)$ is $\lambda(\kappa, \vartheta, z)$-convex on $B_{\delta/2}(z)$. Now, since $\mathrm{md}_\kappa(\chi d_y) \in D_{L^\infty}(\Delta) \cap L^\infty(\mathcal{H}^n) \cap \mathrm{Lip}(X)$ - and again in particular $\mathrm{md}_\kappa(\chi \cdot d_z) \in H^{2,2}(X)$ - , we can apply Theorem 4.7 where $f = \mathrm{md}_\kappa(\chi d_y)$, $X = Y$ and $Z = B_{\delta/2}(z)$. We obtain for $V \in L^2(TY)$

$$\mathrm{Hess}(\mathrm{md}_\kappa(\chi d_y))(V,V) \geq \lambda(\kappa, \vartheta, z)|V|^2 \geq (1 - \mathrm{md}_\kappa d_y - 2\vartheta)|V|^2 \ m\text{-a.e. on } B_{\delta/2}(z). \tag{26}$$

Now, we also can cover $B_{\delta/2}(z)$ with η-balls as above. Since any of these balls is geodesically convex by Remark 2.53 and our choice of $\vartheta > 0$, the estimate (26) holds with $\delta/2$ replaced by η. Hence, the estimate holds m-a.e. on $B_{\delta/2}(z)$ with arbitrary small $\vartheta > 0$, so we actually have

$$\mathrm{Hess}(\mathrm{md}_\kappa(\chi d_y))(V,V) + \mathrm{md}_\kappa d_y |V|^2 \geq |V|^2 \ m\text{-a.e. on } B_{\delta/2}(z). \tag{27}$$

Applying another time the chain rule for the Hessian (Proposition 2.41) in particular yields

(28)

$$\operatorname{Hess}(\chi d_y)(\nabla\chi d_y,\nabla\chi d_y)|_{B_{\delta/2}(z)}=0\ \ \&\ \ \operatorname{Hess}(\chi d_y)(E_i,E_i)|_{B_{\delta/2}(z)}\geq \cot_\kappa d_y|_{B_{\delta/2}(z)}\ \text{for}\ \ i=2,\dots,n$$

where the first identity follows – for instance – from the claim below, and the second one follows from (27) after applying the chain rule for the Hessian.

Then, since $\chi\cdot d_y\in D(\Delta)$, Corollary 2.49 and the fact that we have the unit orthogonal module basis $(E_i)_{i=1,\dots,n}$ from **3.** together with (25) and (28) immediately gives us

$$(n-1)\cot_{K/(n-1)} d_y\geq \Delta(\chi\cdot d_y)=\sum_{i=2}^n \operatorname{Hess}(\chi d_y)(E_i,E_i)\geq (n-1)\cot_\kappa d_y\ m\text{-a.e. on } B_{\delta/2}(z).$$

Since $k\mapsto \cot_k$ is monotone decreasing this implies that $\kappa(n-1)\geq K$ and

$$\operatorname{Hess}(\chi d_y)(E_i,E_i)\leq (n-1)\cot_{K/(n-1)} d_y-(n-2)\cot_\kappa d_y\ m\text{-a.e. on } B_{\delta/2}(z). \tag{29}$$

Therefore, by Lemma 5.2

$$\operatorname{Hess}(\chi d_y)(E_i,E_i)\leq \cot_{\hat\kappa} d_y\ \ \text{on } B_{\delta/2}(z)\ \text{for}\ \ i=2,\dots,n. \tag{30}$$

5. *Claim:* Let $h,\varphi_k\in H^{2,2}(X)$ with $|\nabla h|=1$ m-a.e. on $B_{\delta/2}(z)$, and $V=\sum_{k=1}^m f_k\nabla\varphi_k$, then

$$\operatorname{Hess} h(\nabla h,V)|_{B_{\delta/2}(z)}=0\ m\text{-a.e. }.$$

Proof of the claim. Since $h,\varphi_k\in H^{2,2}(X)$, $k=1,\dots,m$, using Proposition 2.44 we can compute

$$\begin{aligned}2\operatorname{Hess} h(\nabla h,V)&=2\sum_{k=1}^m f_k \operatorname{Hess} h(\nabla h,\nabla\varphi_k)\\&=\sum_{k=1}^m f_k\left(\langle h,\nabla\langle\nabla\varphi_k,\nabla h\rangle+\langle\nabla\varphi_k,\nabla|\nabla h|^2\rangle-\langle h,\nabla\langle\nabla h,\nabla\varphi_k\rangle\right)=\langle V,\nabla|\nabla h|^2\rangle\ m\text{-a.e. }.\end{aligned}$$

By locality of $|\nabla\cdot|$ this yields $2\operatorname{Hess} h(\nabla h,V)|_{B_{\delta/2}(z)}=\langle X,\nabla 1\rangle|_{B_{\delta/2}(z)}=0$ m-a.e. . □

The claim in particular applies for $\chi\cdot d_y=h$ and $V=\sum_{i=1}^l f_i\nabla\varphi_k$ with $f_k\in L^\infty(m)$ and $\varphi_k\in\mathbb{D}^Y_\infty$. Now any E_i, $i=1,\dots,n$ can be approximated by vector fields of this form. Therefore, by $L^2(TY)-L^0$-continuity of the Hessian

$$\operatorname{Hess}(\chi d_y)(\nabla(\chi d_y),E_i)=\operatorname{Hess}(\chi d_y)(E_1,E_i)=0\ m\text{-a.e. on } B_{\delta/2}(z). \tag{31}$$

Claim: Let $V\in\mathcal{N}^\perp$ be arbitrary. Then

$$\operatorname{Hess}(\chi d_y)(V,V)\leq \cot_{\hat\kappa} d_y|V|^2\ m\text{-a.e. on } B_{\delta/2}(z). \tag{32}$$

Proof of the claim. Let $E_2,\dots,E_n$ be the unit orthogonal basis of $\mathcal{N}^\perp$ on $B_{\delta/2}(z)=:B$ as in **3.**. By definition of a module basis and Remark 2.24, for every $k\in\mathbb{N}$ we find functions $f_i^k\in L^\infty$, $i=2,\dots,n$, such that

$$\left\|V-V^k\right\|_{L^2(TY)|_B}<\frac{1}{k}\ \ \text{where } V^k\in L^2(TY)|_B \text{ such that }\ V^k=\sum_{i=2}^n f_i^kE_i.$$

We can approximate every f_i^k in $L^2(\mathcal{H}^n)$ by step functions that take only finitely many values. Therefore, it is sufficient to assume that $f_i^k=\alpha_i^k\in\mathbb{R}$. Moreover, since B has finite $\mathcal{H}^n$-measure we can assume that $|\alpha_i^k|\in[\frac{1}{nk},\frac{k}{n}]$. Hence, it follows that $|V^k|\in[\frac{1}{k},k]$ for every $k\in\mathbb{N}$.

We can define $W^k=|V^k|^{-1}V^k\in L^2(TY)|_B$. Then W^k satisfies $|W^k|=1$ $\mathcal{H}^n$-a.e. on B. Now, we can choose the unit orthogonal basis $E_1,\dots,E_n$ of $L^2(TY)$ on $B_{\delta/2}(z)$ such that $E_2=W^k$. This is achieved in the same way as we were able to choose $E_1=\nabla(\chi\cdot d_y)$ in step **3.** since $|W^k|=1$ $\mathcal{H}^k$-a.e. . Hence, we obtain (32) first for W^k and by L^∞-homogeneity of $\operatorname{Hess}(\chi d_y)$ also for V^k. Finally, since V^k approximates V in L^2-sense, and since $\operatorname{Hess}(\chi d_y)$ is a L^2-L^0-continuous bilinear form on $L^2(TY)$, we obtain the desired estimate for $V \in \mathcal{N}^\perp$. □

Hence, again applying the chain rule together with (31) and (32) yields

$$\operatorname{Hess} \operatorname{md}_{\hat\kappa}(\chi d_y)(V,V) + \hat\kappa \operatorname{md}_{\hat\kappa}(d_y)|V|^2 \le |V|^2 \ m\text{-a.e. on } B_{\delta/2}(z) \ \forall V \in L^2(TY).$$

6. We consider another time the second variation formula (Theorem 2.46). It yields that the function $\mathcal{F}(\mu) = \int \operatorname{md}_{\hat\kappa}(\chi d_y) d\mu$ for $\mu \in \mathcal{P}^2(Y,m)$ satisfies

(33)
$$\int_0^1 \mathcal{F}(\mu_t)\varphi''(t)dt + \hat\kappa \int_0^1 \int \operatorname{md}_{\hat\kappa}(\chi d_y)|\nabla\psi_t|^2 d\mu_t \varphi(t) dt \le \int_0^1 \int |\nabla \psi_t|^2 d\mu_t \varphi(t)dt, \quad \varphi \in C^2((0,1))$$

for Wasserstein geodesics $(\mu_t)_{t\in[0,1]}$ supported in $B_{\delta/2}(z)$, and $\psi_t \equiv \varphi_t$ as in Theorem 2.46. Note that

$$\int |\nabla\psi_t|^2 d\mu_t = \frac{1}{(s-t)^2}\int |\nabla \psi|^2 d\mu_t = \frac{1}{(s-t)^2} W_2(\mu_t,\mu_s)^2 = W_2(\mu_0,\mu_1)^2$$

where ψ is a Kantorovich potential between μ_t and μ_s for some $s \in [0,1]$, $s \neq t$. Furthermore, by the metric Brenier theorem $|\nabla\psi_t|^2(\gamma_t) = \frac{1}{(s-t)^2}|\nabla\psi|^2(\gamma_t) = d(\gamma_t,\gamma_s)^2$ for Π-a.e. every geodesic where Π is the optimal dynamical plan associated to the geodesic $(\mu_t)_{t\in[0,1]}$. Hence

$$\int \operatorname{md}_{\hat\kappa}(\chi d_y)|\nabla\psi_t|^2 d\mu_t = \int \operatorname{md}_{\hat\kappa}(\chi d_y)(\gamma_t) d(\gamma_t,\gamma_s)^2 d\Pi(\gamma).$$

Now, we choose points $x_0, x_1 \in B_{\delta/2}(z)$, and $\mu_0^k, \mu_1^k \in \mathcal{P}^2(X,\mathcal{H}^n)$ with $\operatorname{supp} \mu_i^k \subset B_{\delta/2}(z)$ and $\mu_i^k \to \delta_{x_i}$ weakly $k \to \infty$. Then, by compactness of $\overline{B_{\delta/2}(z)}$ the Wasserstein geodesic $(\mu_t^k)_{t\in[0,1]}$ between μ_0^k and μ_1^k converges to the Wasserstein geodesic $(\nu_t)_{t\in[0,1]}$ between δ_{x_0} and δ_{x_1}. By uniqueness of geodesics between points in $B_{\delta/2}(z)$ – because of the CAT-condition – , $\nu_t = \delta_{\gamma(t)}$ where γ is the geodesic between x_0 and x_1. Moreover, since $\operatorname{md}_{\hat\kappa}(\chi d_y)$ is continuous and bounded, we obtain from the definition of weak convergence

$$\mathcal{F}(\mu_t^k) \to \mathcal{F}(\nu_t) = \int \operatorname{md}_{\hat\kappa}(\chi d_y) d\nu_t = \operatorname{md}_{\hat\kappa} d_y(\gamma(t)), \ \forall t \in (0,1).$$

Similarly, after taking another subsequence the associated dynamical plans Π^k weakly converge as well, and again by uniqueness of geodesics they converge to the measure δ_γ that is supported on the single geodesic between x_0 and x_1. Since $\gamma \mapsto \operatorname{md}_{\hat\kappa}(\chi d_y)(\gamma_t)d(\gamma_t,\gamma_s)^2$ is a continuous and bounded function on the space of geodesics, we get by the CAT-condition that

$$\frac{1}{(s-t)^2}\int \operatorname{md}_{\hat\kappa}(\chi d_y)(\gamma_t) d(\gamma_t,\gamma_s)^2 d\Pi^k(\gamma) \to \operatorname{md}_{\hat\kappa}(d_y(\gamma_t)) d(\gamma_0,\gamma_1)^2, \ \forall t \in (0,1).$$

Hence, by the dominated convergence theorem we obtain the differential inequality (33) for $\operatorname{md}_{\hat\kappa} d_y \circ \gamma$ along geodesics γ in $B_{\delta/2}(z)$.

Since $z \in B_\epsilon(x_0)\backslash\{y\}$ was arbitrary we have that $\operatorname{md}_{\hat\kappa}(d_y)$ satisfies $(10)_{(CBB)}$ for any $\gamma \subset B_\epsilon(x_0)\backslash\{y\}$.

If γ passes through y then the same property holds for trivial reasons. As this holds for any $y \in B_\epsilon(x_0)$ and $B_\epsilon(x_0)$ is convex we get that $CBB(\hat\kappa)$ property $(8)_{(CBB)}$ holds for all triangles with vertices in $B_\epsilon(x_0)$.

Since $x_0 \in X_{reg}$ was arbitrary and X_{reg} is convex in X, by Petrunin's globalization theorem [Pet16] (cf. [Li15]) it follows that X satisfies $CBB(\hat\kappa)$.

Since this holds for arbitrary $\hat\kappa < K-(n-2)\kappa$ we conclude that X satisfies $CBB(K-(n-2)\kappa)$.

This proves part (1) of Theorem 5.1.

Now part (2) follows by Lemma 2.63.

This concludes the proof of the induction step and hence of Theorem 5.1. □

6. CD+CAT to RCD+CAT

In this section we study metric measure spaces (X, d, m) satisfying

$$(X, d, m) \text{ is } CAT(\kappa) \text{ and satisfies any of the conditions } CD(K, N),\ CD^*(K, N) \text{ or } CD^e(K, N) \text{ for } 1 \le N < \infty,\ K, \kappa < \infty. \tag{34}$$

Remark 6.1. Proposition 6.9 at the end of this section shows that a space X satisfying (34) is non-branching, and hence it is essentially non-branching. For essentially non-branching spaces conditions $CD(K, N)$, $CD^*(K, N)$ and $CD^e(K, N)$ are known to be equivalent [EKS15, Theorem 3.12] and [CM16, Theorem 1.1]. Therefore, for $CAT(\kappa)$ spaces conditions $CD(K, N)$, $CD^*(K, N)$ and $CD^e(K, N)$ with $1 \le N < \infty$ are equivalent. However, we will not use this fact in the proof of Theorem 6.2 below.

The main goal of this section is to prove the following theorem.

Theorem 6.2. *Let (X, d, m) satisfy (34). Then X is infinitesimally Hilbertian.*
In particular, (X, d, m) satisfies $RCD(K, N)$.

Together with Theorem 5.1 this immediately gives Theorem 1.1. Let us mention that the proof of Theorem 6.2 can be simplified under various extra regularity assumptions such as requiring X to have extendible geodesics or to have metric measure tangent cones equal to geodesic tangent cones. Note that it's easy to see that for a space (X, d, m) satisfying (34) for any p and *any* tangent cone (T_pX, d_p, m_p) at p there is a canonical distance preserving embedding $T_p^gX \subset T_pX$. However, it is not a priori clear if this embedding is always onto.

Let us give two instructive examples to keep in mind.

Example 6.3. Let X be the union of $\mathbb{R}$ with closed intervals of length 2^k attached at the point 2^k for all integer k. It's easy to see that X is $CAT(0)$. Let $p = 0$ be the base point. Then any two geodesics starting at p to the right have a common beginning and hence the geodesic tangent space T_p^gX is isometric to $\mathbb{R}$. On the other hand, (X, p) is self similar with respect to multiplication by 2 and hence the tangent cone $T_pX = \lim_{k\to\infty}(2^kX, p)$ is isometric to X. Note that this tangent cone is not a metric cone.

Example 6.4. Let Γ be a binary tree, Let ε_n be a sequence of positive numbers such that $R = \sum_n \varepsilon_n < \infty$. Define the metric on Γ by prescribing the length of any edge from level n to the level $n+1$ to be ε_n. Let X be the metric completion of Γ. Then X is $CAT(0)$ of topological dimension 1. Let p be the root of the tree Γ. Then the cut locus of p coincides with the metric sphere of radius R at p. It is a Cantor set and for an appropriately chosen sequence ε_n it can have arbitrary large Hausdorff dimension. Furthermore, for any $q \in S_R(p)$ the geodesic tangent space T_q^gX is still a ray and is different from any tangent cone obtained as a blow up limit.

The key step in the proof of Theorem 6.2 is the following splitting theorem

Proposition 6.5 (Splitting theorem)**.** *Let (X, d) be $CAT(0)$ and $CD(0, N)$ or $CD^e(0, N)$ for some finite n. Suppose X contains a line. Then $(X, d) \cong (Y, d_Y) \times (\mathbb{R}, d_{Eucl})$ for some $CAT(0)$ metric space (Y, d_Y).*

Proof. Let $\gamma\colon (-\infty, \infty) \to X$ be a line in X. Let $\gamma_\pm$ be the rays $\gamma_+(t) = \gamma(t)$ and $\gamma_-(t) = \gamma(-t)$ for $t \ge 0$. Let $b_\pm(x) = \lim_{t\to\infty} d(x, \gamma_\pm(t)) - t$ be the corresponding Busemann functions. Note that $b_\pm$ are both convex and 1-Lipschitz since they are limits of 1-Lipschitz convex functions.

For any $r > 0$ let $f_r(x) = m^{1/N}(B_r(x))$.

By the same proof as in Lemma 5.4, for any fixed r the function $f_r(x)$ is *concave* on X. We recall the argument which is particularly simple in our case because the lower Ricci and the upper curvature bounds are both zero.

It's well-known that geodesics in $CAT(0)$ spaces satisfy the following "fellow travel" property:

If two constant speed geodesics $\sigma_1, \sigma_2\colon [0, 1] \to X$ satisfy $d(\sigma_1(0), \sigma_2(0)) \le r, d(\sigma_1(1), \sigma_2(1)) \le r$ then $d(\sigma_1(t), \sigma_2(t)) \le r$ for all $0 \le t \le 1$.

This immediately follows from the fact that the function $t \mapsto d(\sigma_1(t), \sigma_2(t))$ is convex in t which is an easy consequence of the $CAT(0)$ condition.

Let x, y be any two points in X and let $\sigma\colon [0,1] \to X$ be a constant speed geodesic from x to y. Let $A = \bar{B}_r(x), B = \bar{B}_r(y)$. Let $0 \le t \le 1$ and let $C_t = (1-t)A + tB$ be their t-Minkowski sum. By the "fellow travel" property we have that $C_t \subseteq \bar{B}_r(\sigma(t))$. Also, C_t is clearly closed. By the Brunn-Minkowski inequality (Theorem 2.8) we have that $m^{1/N}(C_t) \ge (1-t)m^{1/N}(A) + tm^{1/N}(B)$. Using that $m(C_t) \le m(\bar{B}_r(\sigma(t)))$ this gives $f_r(\sigma(t)) \ge (1-t)f_r(x) + tf_r(y)$ i.e. f_r is concave.

Thus we have that for any $r > 0$ the map $t \to f_r(\gamma(t))$ is concave and positive on $\mathbb{R}$. This implies that it's constant. Therefore, $m(\bar{B}_r(\gamma(t)))$ is constant in t. This means that in the proof of concavity of f along $\gamma(t)$ all inequalities must be equalities and hence for any t_1, t_2 it holds that the 1/2-Minkowski sum of $\bar{B}_r(\gamma(t_1))$ and $\bar{B}_r(\gamma(t_2))$ is equal to $\bar{B}_r(\gamma(\frac{t_1+t_2}{2}))$. Since the 1/2-Minkowski sum is closed, the open complement in $\bar{B}_r(\gamma(\frac{t_1+t_2}{2}))$ must be empty.

Let $q \in X$ be any point an let $r = d(q, \gamma(0))$. By above, for any $t \ge 0$ there exist $q_t \in B_r(\gamma(t))$ and $q_{-t} \in B_r(\gamma(-t))$ such that q is the midpoint of $[q_t, q_{-t}]$. Moreover, again using the "fellow travel" property we get that the whole geodesic $[q_t, q_{-t}]$ lies in the r-neighbourhood of γ. By letting $t \to \infty$ and passing to the limit along a subsequence we obtain a line $\gamma_q\colon (-\infty, \infty) \to X$ such that $\gamma_q(0) = q$ and the whole γ_q lies in the r-neighbourhood of γ. By the triangle inequality this implies that $d(\gamma(t), \gamma_q(t)) \le 3r$ for any $t \in \mathbb{R}$. By the Flat Strip Theorem [BH99, Theorem 2.13] this implies that the convex hull of $\gamma \cup \gamma_q$ is isometric to the flat strip $[0, D] \times \mathbb{R}$ for some $D \le r$ with γ and γ_q corresponding to $\{0\} \times \mathbb{R}$ and $\{D\} \times \mathbb{R}$ respectively. We will call two lines in X parallel if they bound such flat strip.

The above trivially implies that $b = b_+ + b_- \equiv 0$ on γ_q and since q was arbitrary, $b \equiv 0$ on all of X. Since $b_\pm$ are both convex this implies that they are both affine and hence $\{b_+ = c\}$ is convex in X for any real c. Further, because of the flat strip property above it holds that $b_+(\gamma_q(t)) = b_+(q) - t$ for any $t \ge 0$. Thus, $\gamma_q(t)$ is an (inverse) gradient curve of b_+ and we have a similar property for b_-. This easily implies that γ_q is unique. That is we claim that for every q there is a unique line through q parallel to γ. Indeed, by possibly changing $\gamma(t)$ to $\gamma(t+c)$ we can assume that $b_+(q) = b_-(q) = 0$. By the above $b_+(\gamma_q(t)) = -t$ for any $t \ge 0$. Since b_+ is 1-Lipschitz this means that $\gamma_q(t)$ is the closest point in $\{b_+ \le -t\}$ to q. Since b_+ is a convex function, the set $\{b_+ \le -t\}$ is convex. In $CAT(0)$ spaces nearest point projections to convex subsets are unique (this is immediate from the definition of $CAT(0)$). Hence $\gamma_q(t)$ is uniquely determined by q and $t \ge 0$. The same works for $t \le 0$ using b_-.

Everything we've shown for the line γ applies to the line γ_q as well. In particular any point q' in X is contained in a flat strip containing γ_q with edges γ_q and $\gamma_{q'}$. This easily implies the metric splitting $X \cong Y \times \mathbb{R}$ with $Y = \{b_+ = 0\}$. Y will obviously be $CAT(0)$.

□

Remark 6.6. Since $m(B_r(\gamma_q(t)))$ is constant in t for any $q \in X, r > 0$ it is easy to see that one gets the splitting of the measure in the above proposition as well, that is $(X, d, m) \cong (Y, d_Y, m_Y) \times (\mathbb{R}, d_{Eucl}, \mathcal{H}_1)$ for some measure m_Y on Y. We omit the details since we don't need it for the proof of Theorem 6.2 and the measure splitting will follow by Gigli's splitting theorem [Gig13] anyway once Theorem 6.2 is proved.

Proposition 6.7. *Let (X, d, m) satisfy any of the conditions $CD(K, N)$, $CD^*(K, N)$ or $CD^e(K, N)$ with $N < \infty$. Then X is infinitesimally Hilbertian if and only if for almost all points $p \in X$ some tangent cone T_pX is isometric to $\mathbb{R}^k$ for some $k \le N$.*

Proof. The "only if" direction is well-known and follows from [GMR15] and Remark 2.15.

We observe that the "if" direction easily follows from Cheeger's generalization of Rademacher's theorem to doubling metric-measure spaces satisfying the Poincaré inequality [Che99].

Indeed, suppose (X, d, m) satisfies the assumption of the theorem and for almost every $p \in X$ some tangent cone T_pX is Euclidean as a metric space.

Let f be a Lipschitz function on X. Recall that by Theorem 2.11, if X satisfies any of the conditions $CD(K,N), CD(K,N)^e$ or $CD^*(K,N)$, it admits a weak type 1-1 Poincaré inequality and hence by [Che99, Theorem 5.1] it holds that $\operatorname{Lip} f = |\nabla f|$ a.e. on X.

Further, by [Che99, Theorem 10.2] there is a set of full measure B_f such that for every $p \in B_f$ and every tangent cone T_pX the differential $df_p\colon T_pX \to \mathbb{R}$ (which always exists after possibly passing to a rescaling subsequence) is generalized linear (see [Che99] for the definition). By above we can also assume that $\operatorname{Lip} f(p) = |\nabla f(p)|$ for any $p \in B_f$.

By [Che99, Theorem 8.1] if for $p \in B_f$ it holds that $T_pX \cong \mathbb{R}^k$ as a metric space then (irrespective of the limit measure on T_pX) $df_p\colon \mathbb{R}^k \to \mathbb{R}$ is linear in the ordinary sense and $\operatorname{Lip} df_p = \operatorname{Lip} f(p)$. Given two Lipschitz functions f, g on X, by passing to a subsequence we see that the same works simultaneously for both df_p, dg_p for any $p \in B_f \cap B_g$ which is still a set of full measure in X. Using [Che99, Theorem 5.1] again we can further assume that $\operatorname{Lip}(f \pm g) = |\nabla(f \pm g)|$ everywhere on $B_f \cap B_g$.

Since Lip satisfies the parallelogram rule on the set of linear functions on $\mathbb{R}^k$ and $d(f \pm g)_p = df_p \pm dg_p$ this gives

$$|\nabla(f+g)(p)|^2 + |\nabla(f-g)(p)|^2 = 2|\nabla f(p)|^2 + 2|\nabla g(p)|^2$$

for any $p \in B_f \cap B_g$. Therefore the parallelogram rule holds for the Cheeger energies of f and g:

$$\int_X |\nabla(f+g)|^2 + |\nabla(f-g)|^2 dm = \int_X 2|\nabla f|^2 + 2|\nabla g|^2 dm \tag{35}$$

Since Lipschitz functions are is dense in $W^{1,2}(X)$ this implies that (35) holds for all $f, g \in W^{1,2}(X)$. This means that X is infinitesimally Hilbertian and hence $RCD(K,N)$ by Remark 2.15. This finishes the proof of Proposition 6.7.

□

The above Proposition shows that for $CD(K,N)$-spaces ($CD^*(K,N)$-space, $CD^e(K,N)$-spaces, respectively) with finite N "analytic" infinitesimal Hilbertianness in the sense of the original definition is equivalent to the "geometric" infinitesimal Hilbertianness (i.e. requiring that tangent spaces almost everywhere be Euclidean).

We are now ready to finish the proof of Theorem 6.2.

Proof of Theorem 6.2. First note that by stability of each condition CD, CD^*, CD^e and CAT under measured Gromov-Hausdorff and Gromov-Hausdorff convergence respectively, it follows that tangent cones satisfy $CD(0,N)$, $CD^*(0,N)$, $CD^e(0,N)$ and $CAT(0)$ respectively.

Since m is locally doubling, By [GMR15, Theorem 3.2] there is a set $A \subset X$ of full measure such that for *every* point $p \in A$ for *any* tangent cone (T_pX, d_p, m_p) and *any* point $y \in T_pX$ any tangent cone $(T_y(T_pX), d_y, m_y)$ is a tangent cone at p. Let $p \in A$. Let k be the largest integer such that *some* tangent cone T_pX splits isometrically as $\mathbb{R}^k \times Y$. Clearly $k \leq N$. We claim that $Y = \{pt\}$. If not then take a point $y \in Y$ which is a midpoint on some non-constant geodesic segment. Then $T_{(0,y)}(\mathbb{R}^k \times Y) \cong \mathbb{R}^k \times T_yY$ contains a line l contained in $\{0\} \times T_yY$. Moreover, since any line parallel to l is equidistant from l it easily follows that a line parallel to l and passing through a point in $\{0\} \times T_yY$ is entirely contained in $\{0\} \times T_yY$. The splitting theorem then implies that $\{0\} \times T_yY$ is isometric to $\mathbb{R} \times Z$ for some metric space Z and hence $T_{(0,y)}(\mathbb{R}^k \times Y) \cong \mathbb{R}^k \times T_yY \cong \mathbb{R}^{k+1} \times Z$ But it's a tangent cone at p which contradicts the maximality of k. Hence there is a tangent cone at p isometric to some $\mathbb{R}^k$ with $k \leq N$. Now the result follows by Proposition 6.7.

□

Next we give an example of a space satisfying (34) which is not Alexandrov of $curv \geq \hat{\kappa}$ for any $\hat{\kappa}$.

Example 6.8. Let (Y, d, m) be the closed unit ball $\bar{B}_1(0)$ in $\mathbb{R}^2$ with the standard Euclidean metric and $m = \mathcal{H}_2$. We are going to show that there exist two C^1 functions $\varphi, v\colon Y \to \mathbb{R}$ such that

$X=(\bar B_1(0),e^{\varphi}d,e^{v}m)$ is $RCD(-100,3)$ and $CAT(0)$. The functions φ,v will be C^4 on $B_1(0)$ with the infimum of sectional curvature of $e^{2\varphi}g_{Eucl}$ on $B_1(0)$ equal to $-\infty$. This will obviously imply that X does not satisfy $curv\ge\hat\kappa$ for any $\hat\kappa$.

Recall that given a Riemannian manifold (M^n,g) if we change the Riemannian metric conformally $\widetilde g=e^{2\varphi}g$ then for any smooth function f on M its hessian changes by the formula

$$\widetilde{\mathrm{Hess}}_f(V,V)=\mathrm{Hess}_f(V,V)-2\langle\nabla\varphi,V\rangle\langle\nabla f,V\rangle+|V|^2\cdot\langle\nabla\varphi,\nabla f\rangle \tag{36}$$

here and in what follows $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_\sim$ are the inner products with respect to g and $\widetilde g$ respectively.

Also, recall that when $n=2$ Ricci tensors of g and $\widetilde g$ are related as follows

$$\widetilde{\mathrm{Ric}}(V,V)=\mathrm{Ric}(V,V)-\Delta\varphi|V|^2 \tag{37}$$

Then for any $N>2$ the weighted N-Ricci tensor of $(M^2,\widetilde g,e^{-f}d\,\mathrm{vol}_{\widetilde g})$ is equal to

$$\begin{aligned}\widetilde{\mathrm{Ric}}_f^N(V,V)&=\widetilde{\mathrm{Ric}}(V,V)+\widetilde{\mathrm{Hess}}_f(V,V)-\frac{\langle\widetilde\nabla f,V\rangle_\sim^2}{N-2}=\\ =\mathrm{Ric}(V,V)-\Delta\varphi|V|^2&+\mathrm{Hess}_f(V,V)-2\langle\nabla\varphi,V\rangle\langle\nabla f,V\rangle+|V|^2\cdot\langle\nabla\varphi,\nabla f\rangle-\frac{\langle\nabla f,V\rangle^2}{N-2}\end{aligned} \tag{38}$$

By smoothing out functions of the form $\delta(|(x,y)|-10)$ on $B_{10}(0)\subset\mathbb R^2$ (with $0<\delta\ll1$) it's easy to show that for every $\varepsilon>0,L>0$ there exist smooth functions $f_{\varepsilon,L},\varphi_{\varepsilon,L}\colon B_5(0)\to\mathbb R$ with the following properties

(1) $|f_{\varepsilon,L}|\le\varepsilon,|\varphi_{\varepsilon,L}|\le\varepsilon$;
(2) $f_{\varepsilon,L}$ and $\varphi_{\varepsilon,L}$ are convex and ε-Lipschitz;
(3) $\sup\Delta\varphi_{\varepsilon,L}=L$;
(4) $\mathrm{Hess}\,f_{\varepsilon,L}\ge\Delta\varphi_{\varepsilon,L}$ everywhere on $B_5(0)$;
(5) $|f_{\varepsilon,L}|_{C^4}\le\varepsilon,|\varphi_{\varepsilon,L}|_{C^4}\le\varepsilon$ outside $B_\varepsilon(0)$.

Now let $p_n=(0,1-\frac1n)$ and let $f_n(x)=f_{10^{-n},n}(x-p_n),\varphi_n(x)=\varphi_{10^{-n},n}(x-p_n)$.

Lastly, let $f=\sum_{n=2}^\infty f_n,\varphi=\sum_{n=2}^\infty\varphi_n\colon\bar B_2(0)\to\mathbb R$. Then it's easy to see that f and φ satisfy the following properties

(1) $|f|\le1,|\varphi|\le1$;
(2) f and φ are convex and $1/10$-Lipschitz;
(3) f and φ are C^4 on $B_1(0)$;
(4) $\sup\Delta\varphi|_{B_1(0)}=+\infty$;
(5) $\mathrm{Hess}\,f\ge\Delta\varphi$ everywhere on $B_1(0)$.

We claim that the space $(X,e^{\varphi}d_{Eucl},e^{-f}d\,\mathrm{vol}_{\widetilde g})$ is $RCD(-100,3)$ and $CAT(0)$ where $\widetilde g=e^{2\varphi}g_{Eucl}$ on $B_1(0)$. Indeed, by (37) $\widetilde g$ has $\sec\le0$ since φ is convex and $g=g_{Eucl}$ is flat. In fact, more is true.

Let $u(x,y)=x^2+y^2$. Since f and φ are 1/10-Lipschitz, (36) easily implies that $\widetilde{\mathrm{Hess}}_u\ge0$ on $\bar B_1(0)$, i.e. u is convex with respect to $\widetilde g$ on $\bar B_1(0)$. Therefore, for any $0\le R<1$ the space $X_R=(\bar B_R(0),e^{\varphi}d)$ is $CAT(0)$ since it's locally $CAT(0)$, complete and simply connected. The same holds for $R=1$ because Gromov-Hausdorff limits of $CAT(0)$ spaces are again $CAT(0)$.

Convexity of u with respect to $\widetilde g$ also implies that X_{R_1} is a convex subset of X_{R_2} for any $0<R_1<R_2\le1$. From formula (38) using the properties of f and φ it easily follows that $\widetilde{\mathrm{Ric}}_f^3\ge-100$ on $B_1(0)$. Using that X_{R_1} is a convex subset of X_{R_2} for $R_1\le R_2$ this implies that $(X_R,e^{-f}d\,\mathrm{vol}_{\widetilde g})$ is $RCD(-100,3)$ for any $0<R<1$. By passing to the limit as $R\to1$ we get that $X=(\bar B_1(0),e^{\varphi}d,e^{-f}d\,\mathrm{vol}_{\widetilde g})$ is $RCD(-100,3)$ as well. On the other hand, since $\sup\Delta\varphi|_{B_1(0)}=+\infty$, by (37) the infimum of sectional curvature of $e^{2\varphi}g_{Eucl}$ on $B_1(0)$ is equal to $-\infty$. Therefore X does not satisfy $curv\ge\hat\kappa$ for any real $\hat\kappa$.

Lastly, we show that spaces satisfying (34) are non-branching. This might be somewhat surprising given that branching $CAT(\kappa)$ spaces are quite common (see e.g. Examples 6.3 and 6.4).

Proposition 6.9. *Let X satisfy* (34)*. Then X is non-branching.*

Proof. By rescaling we can assume that X is $CD(-1, N)$ and $CAT(1)$. Suppose $\gamma_1, \gamma_2 \colon [0,1] \to X$ are two branching constant speed geodesics of length $< \pi$. Let $t_0 \in (0,1)$ be the branching point. By uniqueness of geodesics in $CAT(1)$ spaces this means that $\gamma_1|_{[0,t_0]} = \gamma_2|_{[0,t_0]}$ and $\gamma_1(t) \neq \gamma_2(t)$ for any $t > t_0$. Let $x = \gamma_1(0) = \gamma_2(0), y = \gamma_1(1), z = \gamma_2(1), p = \gamma_1(t_0)$. By shortening the geodesics we can assume that $|xp| = |yp| = |zp| = l < \pi/10$. Then $t_0 = 1/2$. Recall that since X is $CAT(1)$ the homothety map Φ^x_s centered at x is 1-Lipschitz on $B_{\pi/2}(x)$ for any $0 \le t \le 1$. Therefore, for any $0 < r < 1/10, 0 < s < 1$ we have that $\Phi^x_s(B_r(\gamma_i(t)) \subset B_r(\gamma_i(st))$. Note that $\Phi^x_s(B_r(\gamma_i(t))$ is the s-Minkowski sum $(1-s)\{x\} + sB_r(\gamma_i(t))$. Therefore $\Phi^x_{\frac{t_0}{t_0+\varepsilon}}(B_r(\gamma_i(t_0+\varepsilon))) \subset B_r(\gamma_i(t_0))$ for $\varepsilon, r < 1/10$.

On the other hand, by the Brunn-Minkowski inequality for $0 < r < \varepsilon$ we have that that $m(\Phi^x_{\frac{t_0}{t_0+\varepsilon}}(B_r(\gamma_i(t+\varepsilon))) \ge (1-\delta(\varepsilon))m(B_r(\gamma_i(t_0+\varepsilon)))$ where $\delta(\varepsilon) = \delta(\varepsilon, l, N) \to 0$ as $\varepsilon \to 0$. (Note that conditions $CD(K,N), CD^*(K,N)$ and $CD^e(K,N)$ give slightly different Brunn-Minkowski inequalities when $K \neq 0$ but all of them trivially imply the existence of $\delta(\varepsilon)$ as above). Therefore,

$$m(B_r(\gamma_i(t_0))) \ge m(\Phi^x_{\frac{t_0}{t_0+\varepsilon}}(B_r(\gamma_i(t_0+\varepsilon))) \ge (1-\delta(\varepsilon))m(B_r(\gamma_i(t_0+\varepsilon)))$$

Applying the same argument to Φ^y_s, Φ^z_s gives that $m(B_r(\gamma_i(t_0+\varepsilon))) \ge (1-\delta(\varepsilon))m(B_r(\gamma_i(t_0)))$. Combining the above we get

$$\frac{1}{1-\delta(\varepsilon)} \ge \frac{m(B_r(p))}{m(B_r(\gamma_i(t_0+\varepsilon)))} \ge 1-\delta(\varepsilon) \quad \text{for } 0 < r < \varepsilon \tag{39}$$

Fix an $0 < \varepsilon < 1/10$ small enough so that $\delta(\varepsilon) < 1/100$. Since $\gamma_1(t_0+\varepsilon) \neq \gamma_2(t_0+\varepsilon)$, for all small $r < \varepsilon$ we have that $B_r(\gamma_1(t_0+\varepsilon)) \cap B_r(\gamma_1(t_0+\varepsilon)) = \emptyset$. Then using (39) and the Brunn-Minkowski inequality for $A = \{x\}$ and $B = B_r(\gamma_1(t_0+\varepsilon)) \cup B_r(\gamma_2(t_0+\varepsilon))$ we get that

$$m(\Phi^x_{\frac{t_0}{t_0+\varepsilon}}(B)) \ge (1-\delta(\varepsilon))m(B) \ge (1-\delta(\varepsilon))2(1-\delta(\varepsilon))m(B_r(p)) \ge \frac{3}{2}m(B_r(p))$$

On the other hand $\Phi^x_{\frac{t_0}{t_0+\varepsilon}}(B) \subset B_r(p)$ and hence

$$m(\Phi^x_{\frac{t_0}{t_0+\varepsilon}}(B)) \le m(B_r(p))$$

This is a contradiction and hence the proposition is proved and X is non-branching. □

Remark 6.10. The above proof only uses the Brunn-Minkowski inequality when one of the sets is a point. Therefore, the proposition remains true if the condition that X be $CD(K,N)$ is replaced by the weaker condition that it satisfies the measure-contracting property $MCP(K,N)$.

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