The Heintze-Karcher inequality for metric measure spaces

Christian Ketterer

University of Toronto

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where $S_\epsilon^+ := \{\exp_x(tN^+(x)) : t \in (0,\epsilon), x \in S\}$, N^+ *is one of two unit normal fields on* S, H *is the mean curvature of* S *and the Jacobian*

$$J_{H,K,n}(t) = \left(\cos(t\sqrt{K/(n-1)}) + \tfrac{H}{n-1}\sin(t\sqrt{K/(n-1)})\right)_+^n.$$

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Also

$$\mathrm{vol}_M(M) \leq \int \int J_{H(p),K,n}(t)dtd\,\mathrm{vol}_S(p)$$

with "=" iff $M = \mathbb{S}^n$ *and* S *has constant mean curvature.*

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Definition (Lott-Sturm-Villani)

(X, d, m) satisfies the curvature-dimension condition $CD(0, N)$ if $\forall \mu_0, \mu_1 \in \mathcal{P}(X)$ there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ such that

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The curvature-dimension condition $CD(K, N)$ for $K \in \mathbb{R}$ is defined similarly using the notion of "(K, N)-convexity."

Properties of *CD* spaces

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- M^n a Riemannian manifold s.t. $M\backslash\partial M$ is geodesically convex and $e^{-f}\,\mathrm{vol}_M =: \mathrm{m}$, $f \in C^\infty(M)$. $K \in \mathbb{R}$, $N \geq n$. Then

$$(M^n, d_M, \mathrm{m}) \text{ satisfies } CD(K,N)$$

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- $[a,b] \subset \mathbb{R}$. $K \in \mathbb{R}$ and $N > 1$.

$$([a,b], |\cdot|_2, m) \text{ satisfies } CD(K,N)$$
$$\Longleftrightarrow$$
$$m = h d\mathcal{L}^1 \text{ with } h \text{ continuous } \ \& \ \frac{d^2}{dt^t} h^{\frac{1}{N-1}} + \frac{K}{N-1} h^{\frac{1}{N-1}} \leq 0.$$

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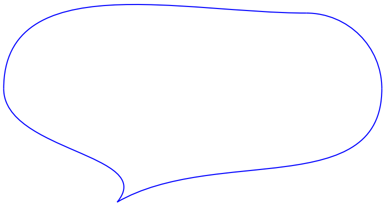

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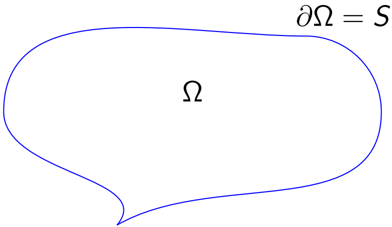

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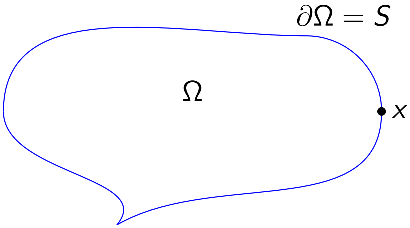

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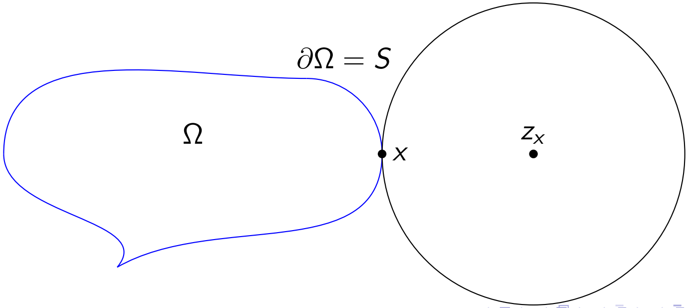

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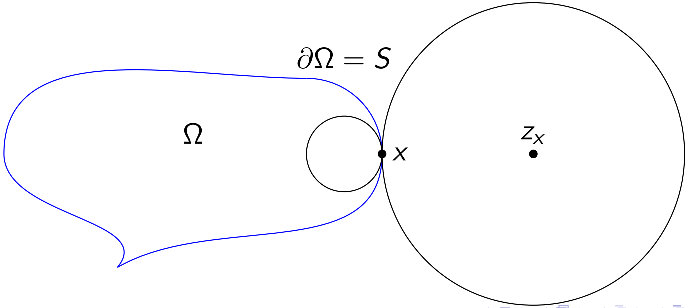

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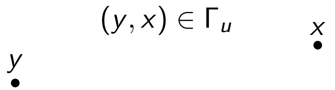

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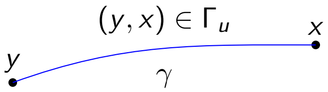

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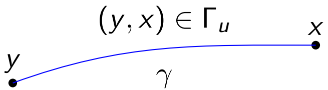

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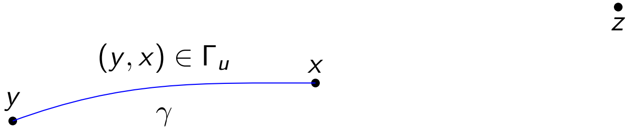

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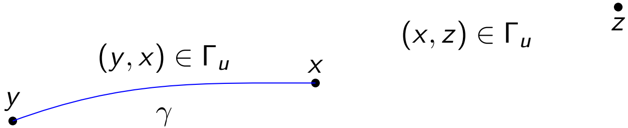

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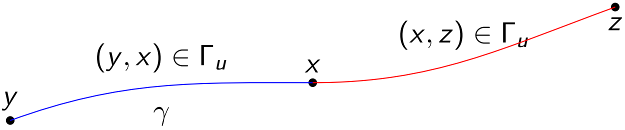

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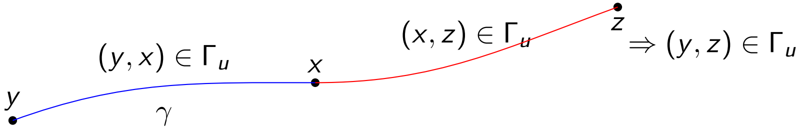

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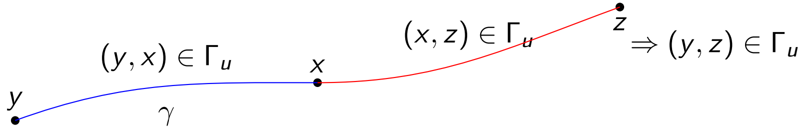

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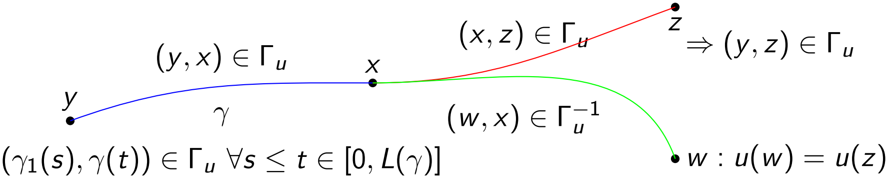

$1D$ localisation method (Cavalletti-Mondino)

Let u be 1-Lipschitz. Define

$$\Gamma_u = \{(x, y) \in X^2 : u(y) - u(x) = d(x, y)\}$$

If $\gamma : [a, b] \to X$ is a (minimal) geodesic and $(\gamma(a), \gamma(b)) \in \Gamma_u$, then

$$(\gamma(s), \gamma(t)) \in \Gamma_u \; \forall s \le t \in [a, b].$$

Γ_u is transitive but not symmetric.

$\Gamma_u^{-1} = \{(x, y) : (y, x) \in \Gamma_u\}$. Define *transport relation*

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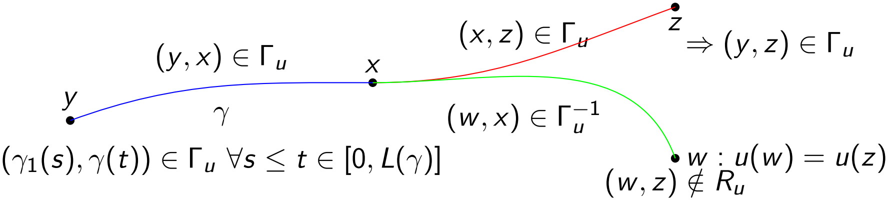

Forward and backward branching points:

$$A_+ = \{x \in \mathcal{T}_u : \exists y, z \in \mathcal{T}_u \text{ s.t. } (x,y),(x,z) \in \Gamma_u, (y,z) \notin R_u\}$$
$$A_- = \left\{x \in \mathcal{T}_u : \exists y, z \in \mathcal{T}_u \text{ s.t. } (x,y),(x,z) \in \Gamma_u^{-1}, (y,z) \notin R_u\right\}$$

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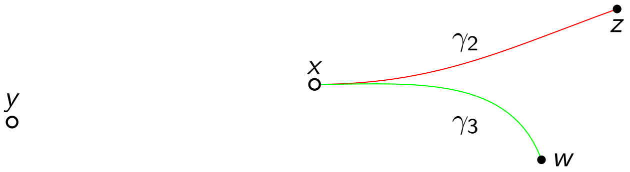

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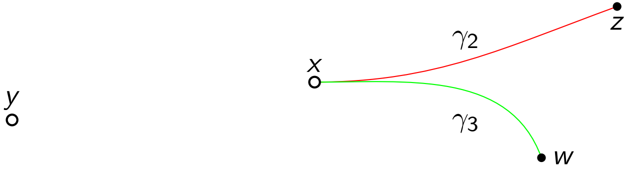

R_u restricted to $\mathcal{T}_u^b$ is an equivalence relation with quotient space Q,

$\mathfrak{Q} : \mathcal{T}_u^b \to Q$ quotient map.

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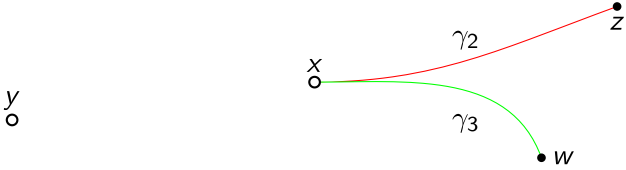

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Each equivalence class is given by the image of a distance preserving map $\gamma : I_\gamma \subset \mathbb{R} \to X$. $\left(\mathcal{T}_u^b = \dot{\bigcup}_{\gamma \in Q} \mathrm{Im}(\gamma)\right)$

Disintegration formula:

$$m|_{\mathcal{T}_u^b} = \int \mathrm{m}_\gamma \, d\mathfrak{q}(\gamma)$$

where $\mathfrak{q} = \mathfrak{Q}_\# m$ and the measures m_γ are concentrated on $\mathrm{Im}(\gamma)$.

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Theorem (Cavalletti-Mondino)

Let (X, d, m) be an essentially non-branching $CD(K, N)$-space. Then

- $m(A_+ \cup A_-) = 0$,
- *For $\mathfrak{q}$-a.e. γ the metric measure space $(\overline{Im(\gamma)}, d, \mathrm{m}_\gamma)$ is $CD(K, N)$.*

Remark: $\mathrm{m}_\gamma = \gamma_\# \left(h_\gamma d\mathcal{L}^1|_{I_\gamma}\right)$ for $h_\gamma : \overline{I_\gamma} \to [0, \infty)$ continuous such that

$$\frac{d^2}{dt} h_\gamma^{\frac{1}{N-1}} + \frac{K}{N-1} h_\gamma^{\frac{1}{N-1}} \leq 0 \text{ in distrib. sense.}$$

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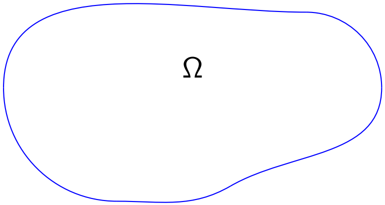

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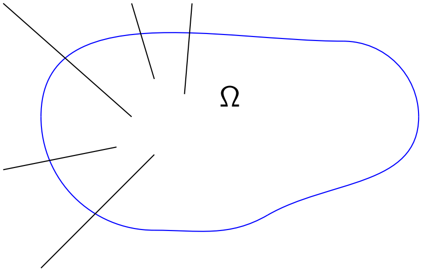

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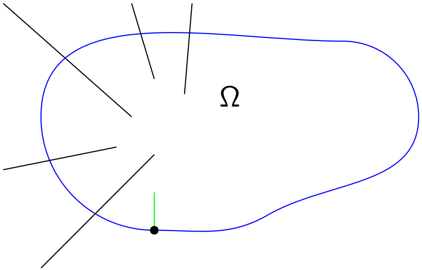

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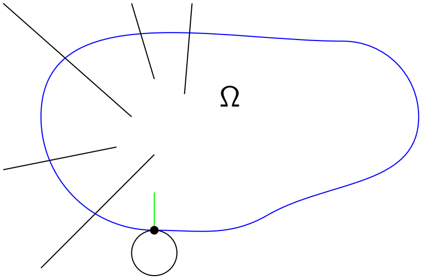

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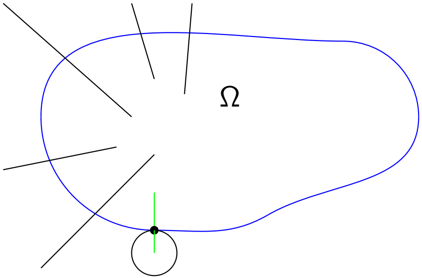

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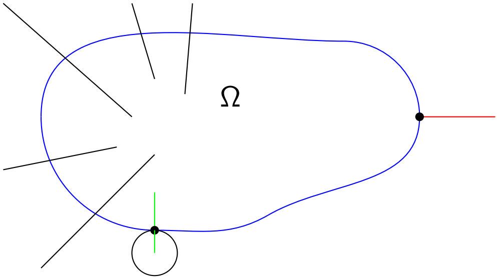

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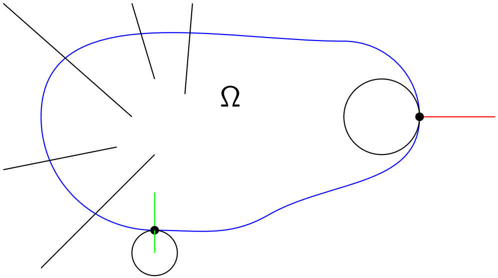

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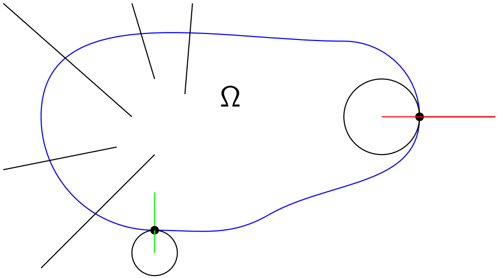

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Define the mean curvature of S as

$$H(p) := \max\left\{\frac{d^+}{dt}\log h_\gamma(0), \frac{d^-}{dt}\log h_\gamma(0)\right\},\ p = \gamma(0)$$

Heintze-Karcher inequality for metric measure spaces

Theorem (K. 2019)

Let (X, d, m) *be an essentially nonbranching* $CD(K, N)$ *space, and let* S *be as before.* $S_\epsilon^+ = B_\epsilon(\Omega)\backslash\overline{\Omega}$. *Then*

$$\mathrm{m}(S_\epsilon^+) \leq \int_S \int_0^\epsilon J_{H(p),K,N}(t)dtd\,\mathrm{m}_S(p)$$

where

$$J_{H,K,N}(t) = \left(\cos(t\sqrt{K/(N-1)}) + \tfrac{H}{N-1}\sin(t\sqrt{K/(N-1)})\right)_+^N.$$

Also

$$\mathrm{m}(M) \leq \int \int J_{H(p),K,N}(t)dtd\,\mathrm{m}_S(p).$$

For X *satisfying* $RCD(K, N)$ *"="* *if and only if there exists a* $RCD(K, N-1)$ *space* Y *such that* X *is an* $N-1$*-suspension over* Y.

Proof of the first inequality

$$\mathrm{m}(S_\epsilon^+)$$

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\mathsf{m}(S_\epsilon^+) &= \int \mathsf{m}_\gamma(B_\epsilon(\Omega)\backslash\Omega)\,d\mathfrak{q}(\gamma)\\
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Thank you!