Communication, Timing, and Common Learning*

Jakub Steiner[†] Colin Stewart[‡]

August 3, 2010

Abstract

We study the effect of stochastically delayed communication on common knowledge acquisition (common learning). If messages do not report dispatch times, communication prevents common learning under general conditions even if common knowledge is acquired without communication. If messages report dispatch times, communication can destroy common learning under more restrictive conditions. The failure of common learning in the two cases is based on different infection arguments. Communication can destroy common learning even if it ends in finite time, or if agents communicate all of their information. We also identify conditions under which common learning is preserved in the presence of communication.

JEL: D82, D83

Keywords: Common knowledge, learning, communication

*We thank Aviad Heifetz, George Mailath, Friederike Mengel, Stephen Morris, József Sákovics, Larry Samuelson, and audience members at Caltech, McGill, Princeton, Rochester, UBC, the SAET 2009, EEA-ESEM 2009, CETC 2010 conferences, and the Second Spring Mini-Conference at Brown for useful comments. We are especially grateful to the associate editor and several anonymous referees for detailed comments that significantly improved the paper. Steiner benefited from the grant RES-061-25-0171 provided by ESRC and from a collaboration on the research project No. Z70850503 of the Economics Institute of the Academy of Sciences of the Czech Republic. Stewart thanks SSHRC for financial support through a Standard Research Grant.

[†]Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208-2009; email: j-steiner@kellogg.northwestern.edu

[‡]Department of Economics, University of Toronto, 150 St. George St., Toronto, ON, Canada M5S 3G7; email: colin.stewart@utoronto.ca

1 Introduction

It is well known that common knowledge can be a crucial determinant of equilibrium behavior in settings ranging from coordination games to repeated games with imperfect monitoring (see, e.g., Lewis [7], Rubinstein [12], and Mailath and Morris [8]). When players learn prior to or during the play of the game, whether common learning of a parameter occurs—that is, approximate common knowledge of the parameter is acquired—depends on the nature of the learning process (Cripps, Ely, Mailath, and Samuelson [2]; henceforth CEMS). We focus on the role of private communication in common learning. In a simple setting where players commonly learn in the absence of communication, we find general conditions under which communication destroys or preserves common learning as a result of stochastic delays.

Communicated messages can have a direct influence on the evolution of higher order beliefs. Furthermore, if communication is expected, even *not* receiving a message can have profound effects on higher order beliefs. A key feature of many forms of communication is that the sender of a message does not observe the exact time at which the message is received. This feature generates asymmetric information that can lead to persistent higher order uncertainty.

We consider two agents learning the value of some parameter θ. Agent 2 observes θ at time 0, but agent 1 observes θ at a random time. Without communication, the agents commonly learn θ. We study the effect of communication according to protocols in which agent 1 sends a message to agent 2 upon observing θ. This message is received after a stochastic delay. Depending on the protocol, communication can consist of a single message, or it may continue with an exchange of confirmation messages, all subject to delay.

If messages act only as confirmations and carry no additional information, communication destroys common learning whenever communication is not too slow relative to the speed at which agent 1 learns θ (in a sense made precise below). Even if each message reports all of the sender's information at the time of sending, communication that never ceases can destroy common learning for some delay distributions. However, when messages report the date at which they are sent, communication preserves common learning if it almost surely ends in finite time.

Our negative results are based on two different infection arguments, corresponding to distinct channels through which communication can generate higher order uncertainty. First, if many confirmation messages are sent, higher order uncertainty can arise from a persistent belief that the last message has not yet been received (along the lines of Rubinstein [12]), even if messages report all of the sender's information. Second, when messages are undated, higher order uncertainty can arise much more generally from beliefs that a message was delayed. This latter effect is powerful enough to destroy common learning even if a single message is sent that is almost surely received within two periods (see Section 2). The effect persists indefinitely after all communication has ended.

Our negative results also apply in a stronger form to a setting in which agent 2 only learns the value of θ from agent 1's first message (with the structure and timing otherwise identical). Communication is essential for common learning in this setting since agent 2 trivially fails to learn θ without communication. If the agents communicate using undated messages, they fail to commonly learn θ even with a fixed finite number of messages that are never lost. In this case, persistent uncertainty about the timing of agent 1's observation of θ renders common learning of θ impossible.

In order to focus on the effects of timing, we take communication as exogenous and ignore agents' incentives. In light of our negative results, one may wonder why agents would choose to communicate in settings where doing so destroys common learning. There are several possible reasons. First, it may be that at least one agent prefers that common knowledge *not* be acquired. Second, it is not the choice to communicate that destroys common learning but rather agents' expectations of communication. The fact that one agent expects to receive a message can destroy common learning regardless of whether that message is actually sent. Moreover, in some settings, communication can occur in every equilibrium even if it destroys common learning and common knowledge makes agents strictly better off (see Steiner and Stewart [14] for an example).[1]

The Rubinstein [12] email game showed that communication can have a double-edged effect on common knowledge acquisition. In the email game, agent 1 observes a parameter, sends a message informing agent 2 of the parameter, agent 2 sends a confirmation message, and so on. Communication terminates at each step with some small fixed probability. On the one hand, communication enhances knowledge acquisition; without communication, agent 2 never learns the value of the parameter. Furthermore, if communication is restricted to a fixed number of messages, beliefs approach common knowledge with high probability as the likelihood of delivery failure vanishes. On the other hand, when the number of messages is unbounded, approximate common knowledge of the parameter is never acquired. Our framework differs from that of the email game in two significant respects. First, we focus on a setting in which common knowledge is obtained without communication, and identify conditions under which communication only hinders common learning. Second, we model timing of communication explicitly, and show that it plays a crucial role in common knowledge acquisition. Unlike in the email game, communicating with a fixed finite number of messages does not guarantee (approximate) common learning, and protocols in which messages are lost with positive probability do *not* destroy common learning even with an unbounded number of messages.

[1] See Morris [10] and Binmore and Samuelson [1] for analysis and discussion of voluntary communication in Rubinstein's email game.

A number of earlier papers have considered common knowledge acquisition when learning occurs only through communication (except for information that agents possess initially). Geanakoplos and Polemarchakis [3] show that alternating communication of posterior beliefs according to a deterministic protocol leads to common knowledge. Based on an observation of Parikh and Krasucki [11], Heifetz [5] shows that communication according to a stochastic protocol may fail to generate common knowledge. Using an infection argument similar to that underlying the email game of Rubinstein [12], Koessler [6] generalizes this result to show that full common knowledge of an event is never attained under any noisy and non-public communication protocol unless the event was common knowledge initially. Halpern and Moses [4] obtain a similar result when messages have unbounded delivery times. Morris [10] proves that common knowledge is not acquired in a variant of the email game in which, as in our model, messages are delivered at stochastic times.

The explicit inclusion of time in our model has two distinct consequences. On the one hand, timing opens a new infection channel based on asymmetric information about delivery times. This infection channel can lead to the failure of common learning even if only one message is sent. On the other hand, including information about timing in the messages can lead to positive results. Under some conditions, even with infinitely many messages, common learning occurs if agents report the dispatch date in each message they send.

CEMS study a model in which each agent learns about an underlying parameter through an infinite sequence of signals. They prove that if signal spaces are finite, individual learning of the parameter implies common learning regardless of how signals are correlated across agents. Our model of communication does not fit into their framework since they assume independence of signal profiles across time conditional on the parameter. Communication naturally generates correlation of signals across time (and across agents) since messages

	I	N
I	$-p,-p$	$-p,0$
N	$0,-p$	$0,0$

θ_1

	I	N
I	$1-p,1-p$	$-p,0$
N	$0,-p$	$0,0$

θ_2

Figure 1: Payoffs in an investment game, with $p \in \left(\frac{1}{2}, 1\right)$.

received by an agent generally depend on the information possessed by the sender at the time the message was sent. This correlation across time can lead to a failure of common learning even with finite signal and message spaces. Since receipt of messages is positively associated with past learning by another agent, delays in communication can generate persistent higher-order uncertainty (even when communication does not influence first-order beliefs). While correlations across agents in the CEMS framework allow signals to influence interactive beliefs based on contemporaneous information, our negative results are driven by intertemporal effects that are precluded by their temporal independence assumption.

2 Example

The following example, loosely based on an example from CEMS, illustrates how, through its effect on common knowledge, communication can have a large influence on rationalizable behavior.

Two agents share a uniform prior belief about a parameter $\theta \in \{\theta_1, \theta_2\}$. Agent 1 perfectly observes the value of θ at a stochastic time $t_0 \in \mathbb{N}$ distributed geometrically, and receives no additional information. Agent 2 perfectly observes θ at time 0. At some exogenously determined time t, the two agents play the simultaneous move game depicted in Figure 1. The agents make no other choices at any time. Note that the profile (I, I) is payoff-dominant when $\theta = \theta_2$, but each agent prefers to choose action N if either $\theta = \theta_1$

or the other agent chooses N. We consider whether, for a given value of p, there exists a rationalizable action profile in which the efficient outcome (I, I) occurs with positive probability in state θ_2 when t is sufficiently large.

Consider first under what conditions on beliefs coordination on the payoff-dominant outcome at θ_2 is rationalizable. Mutual knowledge that $\theta = \theta_2$ is insufficient for coordination on I. Action I is a best response only for a type that p-believes—that is, assigns probability at least p to—the joint event that $\theta = \theta_2$ and the other agent chooses I. Therefore, letting S_t denote the set of histories at time t after which the agents coordinate on I, both agents must p-believe S_t at every history in S_t; in other words, S_t must be p-evident. In addition, both agents must p-believe θ_2 on S_t. As Monderer and Samet [9] show, these two conditions imply that θ_2 is common p-belief on S_t. Thus common p-belief of θ_2 is a necessary condition for the profile (I, I) to be rationalizable. Conversely, there exists a rationalizable strategy profile specifying (I, I) at precisely those histories where θ_2 is common p-belief since common p-belief of θ_2 is a p-evident event, and both agents know θ when it occurs.

Now consider whether approximate common knowledge of θ is acquired. Suppose first that the agents do not communicate. We claim that approximate common knowledge of θ is acquired with probability tending to 1 as t tends to ∞. To see this, let p_t denote the probability that agent 1 has observed θ by time t. Regardless of the realized history, at each time t, it is common knowledge that agent 2 p_t-believes that agent 1 has observed θ. In the event that agent 1 has observed θ by time t, one can verify that θ is common p_t-belief; both agents know θ, both p_t-believe that both know θ, both p_t-believe that both p_t-believe that both know θ, and so on. The claim follows since $p_t \to 1$ as $t \to \infty$.

Suppose now that agent 1 sends a message to agent 2 at time t_0 indicating that she has observed θ. Agent 2 receives the message either one or two periods later, with equal probability. No other messages are sent. In this setup, common learning fails uniformly across all histories; that is, there exists some $\overline{p} \in (0,1)$ such that common $\overline{p}$-belief of θ is not attained at *any* finite history. Note that this failure occurs even though approximate common knowledge is eventually acquired in the absence of communication. The failure of common learning follows from an infection argument based on asymmetric information about the timing of the message.

Consider a fixed period t. The infection begins from finite histories at which agent 1 has not observed θ, that is, $t_0 > t$. At any such history, agent 1 assigns probability 1/2 to each value of θ. In particular, θ is not common p-belief for $p > 1/2$. Now consider histories at time t such that agent 2 has not received the message, that is, $t_1 > t$. There exists some $q \in (0,1)$ such that, at any such history h, agent 2 q-believes that agent 1 has not yet observed θ. In particular, θ is not common p-belief at h for $p > 1 - \min\{q, 1/2\}$. Now consider histories at time t such that $t_1 \leq t$. There exists some $q' \in (0,1)$ such that, at any such history, agent 2 q'-believes that $t_0 = t_1 - 1$, that is, that agent 1 observed θ just one period before agent 2 received the message. Agent 1, on the other hand, 1/2-believes that $t_1 = t_0 + 2$, that is, that agent 2 received the message two periods after it was sent. Regardless of whether the message is delivered in one period or two, one of the agents assigns significant probability to the other agent receiving information later than she actually did. Iterating these beliefs leads to higher order beliefs in histories in which agent 1 observes θ later and later, and ultimately to histories in which agent 1 has not observed θ by time t. Therefore, θ is not common p-belief for $p > 1 - \min\{q, q', 1/2\}$.

To illustrate the infection argument, Figure 2 depicts information sets at $t = 3$ without and with communication. Without communication, for a given θ, all histories lie in a single information set for agent 2. At this information set, 2's belief that 1 has not observed θ vanishes over time, making the event that 1 has observed θ approximate common

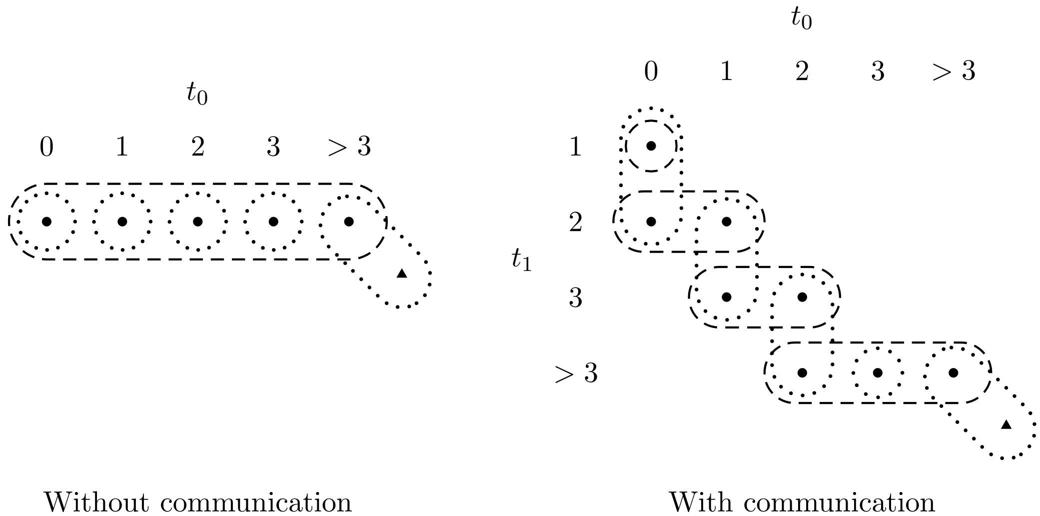

Figure 2: Information sets without and with communication at time $t = 3$ for a given value $\overline{\theta}$ of θ. Each circular dot represents a history for $\overline{\theta}$, and each triangular dot represents the history for $\theta \neq \overline{\theta}$ at which agent 1 has not observed θ. Dotted curves correspond to information sets for agent 1, dashed curves to information sets for agent 2.

knowledge. With communication, the overlapping structure of information sets for the two agents leads to the failure of common learning. For example, at the history $(t_0, t_1) = (1, 2)$, agent 1 believes the history $(1, 3)$, at which agent 2 believes the history $(2, 3)$, at which agent 1 believes the history $(2, > 3)$, at which agent 2 believes the history $(> 3, > 3)$, at which agent 1 believes the wrong value of θ, with a lower bound of $\min\{q, q', 1/2\}$ on each belief in this sequence. A similar argument applies at any time $t > 3$. At each $t > 3$, agent 2 has one information set with three histories (corresponding to the histories in which she has not yet received the message), and the other histories form an overlapping pattern of binary information sets like that depicted in the figure. Within information sets, beliefs are not vanishing over time; more precisely, $\min\{q, q', 1/2\}$ is a lower bound on the belief assigned to any history in any information set for either agent. Consequently, for any $p > 1 - \min\{q, q', 1/2\}$, θ is not common p-belief at any t, and the agents fail to efficiently coordinate.

Now consider the same example, except that the message sent by agent 1 reports the exact time at which she sends it. In this case, agent 2 can distinguish between the histories within each of her binary information sets in Figure 2. Since the two agents' information sets are no longer overlapping, there are histories from which it is not possible, through a chain of beliefs, to reach the history in which agent 1 has not observed θ. In this case, common learning occurs, and for any $p \in (1/2, 1)$, there exists a rationalizable strategy profile in which agents efficiently coordinate with high probability when t is sufficiently large.

3 Model

Two agents, 1 and 2, learn about a parameter θ in periods $t = 0, 1, \ldots$. The parameter θ is drawn before period 0 from the set $\Theta = \{\theta_1, \theta_2\}$ according to the common prior distribution $\Pr(\theta_1) = \Pr(\theta_2) = 1/2$, and remains fixed over time. In the baseline learning process each agent i in each period t receives a signal $z_t^i \in \mathcal{Z}_i = \{\theta_1, \theta_2, u\}$. Conditional on the parameter θ, agent 1 receives a signal equal to θ at a random time t_0 distributed independently of θ according to a distribution $G(\cdot)$ with full support on $\mathbb{N}$. She receives the signal u in all other periods. Note that after receiving the signal $z_t^1 = \theta_k$, agent 1 knows that the parameter is $\theta = \theta_k$. If $z_t^1 = \theta$ for some $t \leq T$, we say that agent 1 has observed θ by T. Also note that the signal u carries no information about the value of θ, and hence, absent communication, agent 1's belief about θ remains equal to her prior belief until she observes θ. Agent 2 receives the signal θ in each period and hence she observes θ at time 0. The distribution of agent 2's signals is not important for the results; it matters only that agent 2 eventually learns the value of the parameter and that the timing is independent of agent 1's signals.

In addition to direct signals about θ, the agents communicate according to a protocol characterized by three elements: the number of messages $N \in \mathbb{N}_+ \cup \{\infty\}$ that the agents exchange, the distribution $F(\cdot)$ of delay times, and the selection of each message. There is no uncertainty about any element of the protocol, which is common knowledge between the agents. We describe message selection below; first we focus on timing.

In addition to the signals z_t^i, in each period t, each agent i privately observes a communication signal m_t^i from a set $\mathcal{M}_i$ containing at least 2 elements, including s (for "silence"), which is interpreted as not receiving a message from the other agent. The signals m_t^i are determined by the following stochastic process. As soon as agent 1 first observes θ in period t_0, she sends a message in $\mathcal{M}_2 \setminus \{s\}$ to agent 2 that may depend on θ and t_0. This message is received by agent 2 at some date $t_1 > t_0$ with the delay $t_1 - t_0$ distributed according to $F(\cdot)$ with support on $\mathbb{N}_+$. We allow for $F(\cdot)$ to be defective so that messages may be "lost".[2] If $N = 1$ or $t_1 = \infty$, there is no further communication; each agent receives s in every period except t_1. Otherwise, at time t_1, agent 2 sends a message in $\mathcal{M}_1 \setminus \{s\}$ which is received by agent 1 at some time $t_2 > t_1$ with the delay $t_2 - t_1$ distributed according to $F(\cdot)$. The agents continue alternately sending messages in this way at each t_k with $k < N$ (or until $t_k = \infty$ for some k). Delay times are independent across messages and independent of t_0. In every period $t \neq t_n$ for any odd n, agent 2 receives s, and similarly agent 1 receives s in every period $t \neq t_n$ for any even $n \geq 2$.

Letting $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ and $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$, the set of states is given by $\Omega = \Theta \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$.[3] The information of agent i at time t is captured by the natural projection of $\Theta \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$ onto $\mathcal{Z}_i^{t+1} \times \mathcal{M}_i^{t+1}$, where S^t denotes the t-fold Cartesian product of

[2] A distribution $F(\cdot)$ over $\mathbb{N}_+$ is defective if $\lim_{n\to\infty} F(n) < 1$.

[3] Note that Ω contains many states with 0 probability finite histories. For example, receiving $m_t^i \neq s$ in two consecutive periods happens with 0 probability. When there is no risk of confusion, we ignore finite histories that occur with probability 0.

the set S with itself.[4] We write $h_t^i(\omega) \in \mathcal{Z}_i^{t+1} \times \mathcal{M}_i^{t+1}$ for the private history of agent i at time t in state ω, and $h_t(\omega) = \left(h_t^1(\omega), h_t^2(\omega)\right)$ for the t-history at ω. We abuse notation by writing θ for the event $\{\theta\} \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$.

Let H_t^i denote the set of private histories $h_t^i(\omega)$ for agent i at time t and $H^i = \cup_t H_t^i$. Message selection is determined according to a pair of *selection rules*

$$\mu_i : H^i \longrightarrow \mathcal{M}_j \setminus \{s\}$$

for $i, j \in \{1, 2\}$, $i \neq j$. Whenever a message is sent at some time t_k, the selection rules determine which message is sent as a function of the sender's private history. For example, if k is odd, the communication signal $m_{t_k}^2$ received by agent 2 at time t_k corresponds to the message sent by agent 1 at time t_{k-1}, and thus we have $m_{t_k}^2 = \mu_1\left(h_{t_{k-1}}^1\right)$. Like the other elements of the protocol, the message selection rules are commonly known to the agents.

We write $t_0(\omega)$ for the realized time at which agent 1 first observes the parameter. For $n \geq 1$, we write $t_n(\omega)$ for the realized time at which the nth confirmation message is received. The realizations $t_n(\omega)$ satisfy $t_0 = \min\{t : z_t^1 = \theta\}$ and for $n \geq 1$, recursively, $t_n = \min\{t > t_{n-1} : m_t^i \neq s \text{ for } i = 1 \text{ or } 2\}$.

Let $f(\cdot)$ and $g(\cdot)$ denote the densities of $F(\cdot)$ and $G(\cdot)$ respectively. We assume that $0 < f(1) < 1$, and in addition that $f(\cdot)$ and $g(\cdot)$ satisfy the regularity condition that $\lim_{t\to\infty} \frac{g(t-1)}{f*g(t)}$ exists, where $f * g(\cdot)$ denotes the density of the convolution $F * G(\cdot)$ of $F(\cdot)$ and $G(\cdot)$. The expression $\frac{g(t-1)}{f*g(t)}$, when multiplied by $f(1)$, is equal to the probability that agent 1 observed θ at time $t-1$ conditional on the first message being received at time t. The regularity condition says that this belief converges as t grows large. Roughly speaking, the condition holds as long as the tails of $F(\cdot)$ and $G(\cdot)$ vary in a well-behaved way. For

[4]The natural projection of $\Theta \times \mathcal{Z}^\infty \times \mathcal{M}^\infty$ onto $\mathcal{Z}_i^{t+1} \times \mathcal{M}_i^{t+1}$ maps each state to the sequence of signals received by agent i in periods $0, \ldots, t$.

example, it is easy to show that the condition holds for any $F(\cdot)$ if $G(\cdot)$ is a geometric distribution, or for any $F(\cdot)$ with bounded support if $G(\cdot)$ is a power-law distribution.

4 Preliminaries

For convenience, we review the definitions of p-belief, common p-belief, and p-evident events due to Monderer and Samet [9]. Let Σ denote the Borel σ-algebra of Ω endowed with the product topology (with the discrete topology on each factor). For $E \in \Sigma$ and $p \in [0,1]$, let

$$B_p^{i,t}(E) = \left\{\omega : \Pr\left(E \mid h_t^i(\omega)\right) \geq p\right\}.$$

If $\omega \in B_p^{i,t}(E)$ then we say that agent i p-believes E at time t in state ω. We say that agent i *knows* E if she 1-believes E and E occurs. An event $E \in \Sigma$ is *p-evident* at time t if $E \subseteq \bigcap_{i=1,2} B_p^{i,t}(E)$, that is, if both agents p-believe E at time t in every state in E. An event E is *common p-belief* at time t in state ω if and only if there exists an event F such that F is p-evident at time t, $\omega \in F$, and $F \subseteq \bigcap_{i=1,2} B_p^{i,t}(E)$.[5] We denote by $C_p^t(E)$ the set of all states at which E is common p-belief at time t.

Definition 1. 1. *(CEMS) Agents* commonly learn Θ *if, for each $\theta \in \Theta$ and $q \in (0,1)$, there exists some T such that for all $t > T$,*

$$\Pr\left(C_q^t(\theta) \mid \theta\right) > q.$$

2. *Common learning of* Θ uniformly fails *if there is some $q < 1$ such that, for each $\theta \in \Theta$,*

$$\Pr\left(C_q^t(\theta) \mid \theta\right) = 0.$$

[5]By Monderer and Samet (1989), this definition is equivalent to the usual definition of common p-belief based on intersections of higher order p-beliefs.

for every t.[6]

Uniform failure of common learning is stronger than the negation of common learning insofar as approximate common knowledge may be acquired with positive probability that does not approach 1.

It is easy to see that, in the absence of communication, agents commonly learn Θ in our setting. Consider the event F_k^t that $\theta = \theta_k$ and agent 1 has observed θ by time t. At any state in F_k^t, each agent assigns probability at least $G(t)$ at time t to F_k^t (in fact, agent 1 knows F_k^t). Hence whenever $q < G(t)$, F_k^t is q-evident at t. Moreover, F_k^t implies that both agents know $\theta = \theta_k$, and thus θ_k is common q-belief at t on F_k^t. Conditional on θ_k, the event F_k^t occurs with probability $G(t)$. Therefore, for sufficiently large t, θ_k is common q-belief with probability at least q.

The following definition captures a distinction that plays an important role in determining whether communication can generate higher order uncertainty that persists over time.

Definition 2. *Communication is* fast *(relative to learning) if*

$$\limsup_{t\to\infty} \Pr(t_0 \leq t \mid t_1 > t) < 1.$$

Communication is slow *(relative to learning) if*

$$\lim_{t\to\infty} \Pr(t_0 \leq t \mid t_1 > t) = 1.$$

The definition states that communication is fast if, when agent 2 has not yet received the first message, she always assigns some non-vanishing probability to agent 1 not having

[6]The term "uniformly" refers to the requirement that q be uniform across all finite histories that occur with positive probability.

observed θ. The idea behind the terminology is that, if message delays tend to be short, agent 2 would eventually think that if agent 1 had observed θ, then she probably should have received the message. For example, consider geometric distributions. Suppose $g(t) = \lambda(1-\lambda)^t$ and $f(t) = \delta(1-\delta)^{t-1}$ with supports $\mathbb{N}$ and $\mathbb{N}_+$, respectively. Suppose that agent 2 has not received the first message after many periods. Agent 2 knows that either agent 1 did not observe θ for many periods, or the message from agent 1 was delayed for many periods. The faster communication is relative to learning, the greater the probability agent 2 assigns to the first explanation; indeed, $\lim_{t\to\infty} \Pr(t_0 \leq t \mid t_1 > t) = \frac{\lambda}{\delta} < 1$ when communication is relatively fast ($\delta > \lambda$) and $\lim_{t\to\infty} \Pr(t_0 \leq t \mid t_1 > t) = 1$ when communication is relatively slow ($\delta \leq \lambda$).[7]

Before identifying conditions under which communication destroys common learning, we note that slow communication trivially preserves common learning.

Proposition 1. *If communication is slow then the agents commonly learn* Θ*.*

All proofs are in the appendix. The proof is based on the observation that with slow communication, regardless of her private history, agent 2 eventually assigns high probability to the event that agent 1 has observed θ. Hence agent 1's uncertainty about agent 2's information becomes irrelevant as t grows large. This implies that the event that agent 1 has observed θ is eventually approximately evident.

Any delay distribution that assigns positive probability to messages being lost—that is, for which $F(\cdot)$ is defective—is slow according to Definition 2 and thus does not destroy common learning.[8] In particular, in contrast to the failure of approximate common knowledge in Rubinstein's email game, in our setting, communication by a protocol similar to that of the email game in which each message is either delivered in one period or never

[7]See the appendix for a proof of these limits.

[8]To see this, note that $\Pr(t_0 \leq t \mid t_1 > t) = 1 - \frac{1-G(t)}{1-F*G(t)}$. If F is defective, $1 - F * G$ is bounded away from 0, whereas $1 - G(t)$ vanishes as $t \to \infty$.

delivered preserves common learning.

5 Main Results

In this section we identify general conditions on the communication protocol under which communication does or does not cause common learning to fail. We begin with two negative results, each based on a different infection argument.

The first negative result generalizes the example from Section 2. We say that communication is *undated* if $\mathcal{M}_i = \Theta \cup \{s\}$ and $\mu_i(h_t^i(\omega)) \equiv \theta(\omega)$ for each i. We say that communication is *dated* if $\mathcal{M}_i = (\Theta \times \mathbb{N}) \cup \{s\}$ and $\mu_i(h_t^i(\omega)) \equiv (\theta(\omega), t)$ for each i. Note that, since agents learn only about timing after the initial observation of θ, any selection rules satisfying $\mu_i(h_t^i) \neq \mu_i(h_{t'}^i)$ whenever $t \neq t'$ generate the same beliefs as dated communication. In particular, dated communication is equivalent to message selection rules given by $\mu_i(h_t^i) \equiv h_t^i$ that report the sender's entire private history.

Proposition 2. *If communication is fast and undated then common learning uniformly fails.*

Asymmetric information about the dispatch and the delivery times of messages leads to a general infection of beliefs along the lines of that described in Section 2. This infection is very powerful: for fast, undated communication, it destroys common learning across all histories. At any finite history in which agent 1 has observed θ, at least one agent believes that the other sent a message later than she actually did. Iterating this belief, both agents have higher order beliefs that no message has been sent, and hence that agent 1 has not observed θ, in which case agent 1 assigns probability $1/2$ to each value of θ. It follows that θ is not approximate common knowledge.

The following lemma, which is used in the proofs of Propositions 2 and 3, formalizes the infection argument based on a given ordering of histories at each time t. Different orderings correspond to different forms of infection. Let $H_t \subseteq H_t^1 \times H_t^2$ denote the set of all t-histories that occur with positive probability, and let $\underline{h}_t \in H_t$ be the t-history in which agent 1 has not observed θ, i.e. $t_0 > t$.[9]

Lemma 1 (Infection lemma)**.** *Suppose that there exist $\underline{p} > 0$ and, for each t, a strict partial order $\prec_t$ on H_t such that, for each $h_t \in H_t \setminus \{\underline{h}_t\}$,*

$$\Pr\left(\{h' : h' \prec_t h_t\} \middle| h_t^i\right) \geq \underline{p} \tag{1}$$

for some $i \in \{1, 2\}$. Then common learning uniformly fails.

In order for (1) to be satisfied, the history $\underline{h}_t$ must be minimal under $\prec_t$ for each t. At any other t-history, iterating condition (1) shows that some agent believes histories ranked lower under $\prec_t$, at which some agent believes still lower histories, and so on, until we reach $\underline{h}_t$. Since agent 1 is uncertain about θ at $\underline{h}_t$, this chain of beliefs implies that there is higher order uncertainty about θ at every history.

The condition of the Infection Lemma captures the key feature underlying standard infection arguments. For example, in the email game (Rubinstein [12]), states may be identified with the number of sent messages. If states are ordered in the natural way, with more messages corresponding to a higher place in the ordering, then approximate common knowledge fails to be acquired because the static analogue of (1) holds: at each state, one agent assigns non-vanishing probability to lower states.

The proof of Proposition 2 applies the Infection Lemma using an ordering of histories based on the timing of messages rather than on the number of messages. For $h \in H_t$, let ω be such that $h_t(\omega) = h$ and let $m(h)$ denote the number of messages received at h

[9] That is, $\underline{h}_t$ is the t-history in which, for each $\tau \leq t$, $m_\tau^i = s$ for each $i = 1, 2$ and $z_\tau^1 = u$.

(i.e. $m(h) = \max\{k : t_k(\omega) \leq t\}$), with the convention that $m(h) = -1$ if $t_0(\omega) > t$. For $s \leq m(h)$, let $t_s(h)$ denote the delivery time $t_s(\omega)$. We define the *lexicographic ordering* $\prec_t^L$ by

$$h \prec_t^L h' \quad \text{if} \quad \sum_{s=0}^{m(h)} 2^{-t_s(h)} < \sum_{s=0}^{m(h')} 2^{-t_s(h')}$$

for $h, h' \in H_t$. As its name suggests, this ordering corresponds to a lexicographic ordering by delivery times, first considering t_0, then t_1, and so on; that is, if $t_0(h) > t_0(h')$ then $h \prec_t^L h'$, and more generally, if for some l we have $t_k(h) = t_k(h')$ for all $k < l$ and $t_l(h) > t_l(h')$, then $h \prec_t^L h'$. When the condition of the Infection Lemma holds for the lexicographic ordering, we say that there is *infection across delivery times.*

To illustrate the infection across delivery times, consider the example from Section 2 in which communication consists of one undated message delivered at time $t_0 + 1$ or $t_0 + 2$ with equal probability. Any history $h \in H_t \setminus \{\underline{h}_t\}$ falls into one of the following three categories:

$$\text{(i) } t_1(h) = t_0(h) + 2 \leq t, \text{ (ii) } t_1(h) = t_0(h) + 1 \leq t \text{ , or (iii) } t_0(h) \leq t < t_1(h).$$

To show that infection across delivery times occurs, we must show that for any history in each category, at least one agent has a non-vanishing belief in histories that are lower under the lexicographic ordering. At any history h in category (i), the message was delayed by two periods. We show in the Appendix that fast communication implies that there exists some $q \in (0, 1)$ such that agent 2 q-believes the history h' in which the message was sent just one period before she received it, that is, $t_0(h') = t_0(h)+1$. The lexicographic ordering ranks h' below h. At any history h in category (ii), the message was delayed by just one period. Agent 1 1/2-believes the history h' in which the message was delayed by two periods, that is, $t_1(h') = t_0(h) + 2$, and therefore $t_1(h') = t_1(h) + 1$. Since $t_0(h') = t_0(h)$, the lexicographic ordering again ranks h' below h. Finally, at any history h in category (iii), agent 2 has not yet received the message. Fast communication implies that there exists some $q' \in (0,1)$ such that agent 2 q'-believes that agent 1 has not yet observed θ, that is, q'-believes the history $\underline{h}_t$ (which is minimal under the lexicographic ordering). Therefore, for each t, the lexicographic ordering satisfies (1) with $\underline{p} = \min\{q, 1/2, q'\}$.

Communication being undated is not necessary to destroy common learning. Depending on the other features of the communication protocol, common learning may fail under *any* message selection rule, even if agents communicate all of their information in each message (for example, when $\mu_i(h_t^i) \equiv h_t^i$ for each i).

The following definitions identify asymptotic properties of delay distributions that are important for common learning with arbitrary message selection rules.

Definition 3. *Following Shimura and Watanabe [13], we say that a distribution F is* O-subexponential *if*

$$\liminf_{x\to\infty} \frac{1-F(x)}{1-F*F(x)} > 0.$$

A distribution F is not heavy-tailed *if*

$$\lim_{x\to\infty} \frac{1-F(x)}{1-F*F(x)} = 0.$$

The class of O-subexponential distributions generalizes the class of subexponential distributions, which corresponds to a particular notion of heavy tails.[10] The subexponential class includes common heavy-tailed distributions such as log-normal, Pareto, and Lévy distributions. For exponential or geometric distributions, as x grows large, the tail of the distribution becomes vanishingly small relative to the tail of its convolution with itself; in our terminology, exponential and geometric distributions are not heavy-tailed. If, on the

[10]A distribution F is *subexponential* if $\lim_{x\to\infty} \frac{1-F(x)}{1-F*F(x)} = 1/2$.

other hand, F is O-subexponential, then the tails of F and $F * F$ vanish at the same rate.

In our setting, O-subexponentiality of the delay distribution corresponds directly to a simple property of agents' beliefs. If the delay distribution is O-subexponential, then an agent who expects a confirmation message that she has not received after many periods doubts that the last message she sent has been received. If, on the other hand, F is not heavy-tailed, then each agent eventually assigns high probability to her last message having been delivered even if she has not received a confirmation message.

Proposition 3. *If communication is fast, the delay distribution F is O-subexponential, and the number N of messages is infinite, then common learning uniformly fails (regardless of the message selection rule).*

Proposition 3 follows from an infection argument based on the number of delivered messages similar to that underlying Rubinstein's email game. If communication is fast, agent 2 doubts that agent 1 has observed θ until the first message is delivered. O-subexponentiality implies that each agent doubts that the last message she sent has been received until she receives a confirmation. If the number of messages in the protocol is infinite, these doubts generate persistent higher order beliefs that agent 1 has not observed θ.

To make this intuition precise, define for each t the *message-based ordering* $\prec_t^M$ by

$$h_t \prec_t^M h_t' \quad \text{if} \quad m(h_t) < m(h_t').$$

When the condition of the Infection Lemma holds for the message-based ordering, we say that there is *infection across messages*. The proof of Proposition 3 shows that infection across messages occurs under the conditions of the proposition.

Propositions 2 and 3 identify communication protocols that destroy common learning even though common learning arises in the absence of communication. The results naturally extend to a related setting in which only agent 1 observes θ directly. Consider a model identical to the one above except that $z_t^2 = u$ for every t and $\mu_1 \left(h^1_{t_0(\omega)}(\omega) \right) \neq \mu_1 \left(h^1_{t_0(\omega')}(\omega') \right)$ whenever $\theta(\omega) \neq \theta(\omega')$. The latter condition implies that agent 2 learns the value of θ when she receives the first message from agent 1. The negative results of Propositions 2 and 3 extend to this setting in a stronger form: they hold even without the assumption that communication is fast. Whether or not agent 2 observes θ directly, the infection arguments underlying the negative results are essentially the same. The only difference is that, in this case, failure of common learning follows from higher order uncertainty about whether agent 2 has received the first message, whereas in the original setting, failure requires higher order uncertainty about whether agent 1 has observed θ.

The following proposition identifies conditions (in addition to that of Proposition 1) under which dated communication preserves common learning.

Proposition 4. *Suppose that F is not defective. If messages are dated and*

1. *the number N of messages is finite, or*

2. *F is not heavy-tailed,*

then the agents commonly learn Θ.

The idea behind the proof of Proposition 4 is simple. Consider the case of finitely many messages (the proof for the second case is similar). Once the last message has been sent, the sender does not expect to receive a confirmation. Once enough time has elapsed, she becomes confident that the message has been delivered. Since the messages are dated, once this message is delivered, the recipient knows how much time has elapsed since the message was sent, and therefore knows the sender's belief. It follows that approximate common knowledge is eventually acquired. Note that dating of messages is essential for the second part of the argument. If the recipient does not learn when the message was sent, she may doubt the confidence of the sender, causing common learning to unravel (as in Proposition 2).

Proposition 4 extends to the alternative setting in which agent 2 does not observe θ directly. Approximate common knowledge arises some time after the first message has been delivered, in which case agent 2 knows θ; whether or not she observes θ before receiving the message is irrelevant.

Part 1 of Proposition 4 also extends to settings in which delays are not i.i.d. across messages. Whenever communication is dated and almost surely ends in finite time, agents commonly learn Θ. Since communication eventually breaks down, there must be a first message that the recipient assigns positive probability to not receiving. Even if this message (or some subsequent message) is never delivered, the recipient eventually believes that all previous messages were delivered and that agent 1 observed θ. Because of this, higher order uncertainty about θ cannot persist and common learning occurs.

6 Discussion and conclusion

There exist distributions such that communication is neither fast nor slow, that is, such that $\limsup_{t\to\infty} \Pr(t_0 \leq t \mid t_1 > t) = 1$ and $\liminf_{t\to\infty} \Pr(t_0 \leq t \mid t_1 > t) < 1$. For such distributions, common learning may fail but not uniformly: for a given $p \in (0,1)$, common p-belief may be acquired in some periods but not in others even as t grows large.

In order to emphasize the role of communication and timing, our results focus on a setting with very simple direct learning of the parameter: each agent perfectly observes the parameter at some time. In a more general setting with gradual learning, we conjecture that our results hold if the first message is sent as soon as one agent p-believes the parameter (for some fixed $p \in (1/2, 1)$).

The contrast between Propositions 2 and 4 indicates that common knowledge can be much easier to acquire if messages are dated. Halpern and Moses [4] make a similar distinction in a different framework, and show that (full) common knowledge cannot be acquired if agents are uncertain about whether they measure time using perfectly synchronized clocks. In our setting, agents possess synchronized clocks, but undated communication can create enough uncertainty about timing to prevent common learning. One possible interpretation of undated communication is that messages report dates but clocks may not be synchronized; thus an agent cannot, relative to her own knowledge of the current date, perfectly infer the date at which a message was sent based only on the date reported by the sender. However, modeling this uncertainty explicitly would complicate the analysis by introducing learning about the relative clock times. Under this interpretation, our positive results for dated messages implicitly rely on synchronization of agents' clocks.

Our results indicate that timing plays a crucial role in determining whether common learning occurs in the presence of communication. Two features of timing are particularly important: the distributions of delays in learning and communication, and the extent to which time is reported in messages. If messages are undated, communication destroys common learning unless communication is sufficiently slow. With dated messages, communication can destroy common learning only under much more stringent conditions on the timing of communication.

A Appendix: Proofs

Claim 1. *Suppose* $g(t) = \lambda(1-\lambda)^t$ *and* $f(t) = \delta(1-\delta)^{t-1}$ *with supports* $\mathbb{N}$ *and* $\mathbb{N}_+$, *respectively. Then*

$$\lim_{t\to\infty} \Pr(t_0 \leq t \mid t_1 > t) = \begin{cases} \frac{\lambda}{\delta} & \text{if } \delta > \lambda, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. First suppose that $\delta \neq \lambda$. Note that

$$\begin{aligned}
F * G(t) &= \sum_{\tau=1}^{t} \sum_{s=0}^{\tau-1} g(s) f(\tau - s) \\
&= \sum_{\tau=1}^{t} \lambda\delta(1-\delta)^{\tau-1} \sum_{s=0}^{\tau-1} \left(\frac{1-\lambda}{1-\delta}\right)^{s} \\
&= \sum_{\tau=1}^{t} \lambda\delta(1-\delta)^{\tau-1} \frac{1-\left(\frac{1-\lambda}{1-\delta}\right)^{\tau}}{1-\frac{1-\lambda}{1-\delta}} \\
&= \sum_{\tau=1}^{t} \lambda\delta \frac{(1-\delta)^{\tau}-(1-\lambda)^{\tau}}{\lambda-\delta} \\
&= \frac{\lambda\delta}{\lambda-\delta}\left((1-\delta)\frac{1-(1-\delta)^{t}}{\delta} - (1-\lambda)\frac{1-(1-\lambda)^{t}}{\lambda}\right) \\
&= \frac{\lambda(1-\delta)\left(1-(1-\delta)^{t}\right) - \delta(1-\lambda)\left(1-(1-\lambda)^{t}\right)}{\lambda-\delta} \\
&= 1 - \frac{\lambda(1-\delta)^{t+1} - \delta(1-\lambda)^{t+1}}{\lambda-\delta}.
\end{aligned}$$

By Bayes' rule, we have

$$\begin{aligned}
\Pr(t_0 \leq t \mid t_1 > t) &= 1 - \frac{1-G(t)}{1-F*G(t)} \\
&= 1 - \frac{(1-\lambda)^{t+1}}{\frac{\lambda(1-\delta)^{t+1}-\delta(1-\lambda)^{t+1}}{\lambda-\delta}} \\
&= 1 - \frac{\lambda-\delta}{\lambda\left(\frac{1-\delta}{1-\lambda}\right)^{t+1} - \delta},
\end{aligned}$$

from which the result follows since $\left(\frac{1-\delta}{1-\lambda}\right)^{t+1}$ tends to 0 as $t \to \infty$ if $\delta > \lambda$ and tends to ∞ if $\delta < \lambda$.

For $\delta = \lambda$, we have

$$\begin{aligned} F * G(t) &= \sum_{\tau=1}^{t} \sum_{s=0}^{\tau-1} \lambda^2 (1-\lambda)^{\tau-1} \\ &= \lambda^2 \sum_{\tau=1}^{t} \tau (1-\lambda)^{\tau-1} \\ &= 1 - (1+\lambda t)(1-\lambda)^t. \end{aligned}$$

Hence we have

$$\Pr(t_0 \le t \mid t_1 > t) = 1 - \frac{1-\lambda}{1+\lambda t},$$

which tends to 1 as $t \to \infty$. □

Proof of Proposition 1. Let $D_t = \{\omega : t_0(\omega) \le t\}$ be the event that agent 1 has observed θ by time t. Since $\Pr(D_t) = G(t)$, for each $q \in (0,1)$ there exists T' such that $\Pr(D_t) > q$ for all $t > T'$. Since communication is slow, for each $q \in (0,1)$ there exists T'' such that $\Pr(D_t \mid t_1 > t) > q$ for all $t > T''$.

We claim that the event D_t is q-evident at every $t > T''$. At every state in D_t, agent 1 knows D_t at time t. Agent 2 knows D_t at time t whenever $t_1 \le t$, and q-believes D_t at time t whenever $t_1 > t$, proving the claim.

Note that both agents know θ at time t on D_t. Since D_t is q-evident, θ is common q-belief at time t on D_t for $t > T''$.

Letting $T = \max\{T', T''\}$, we have

$$\Pr(C_q^t(\theta) \mid \theta) > q,$$

for all $t > T$, as needed. □

Proof of Lemma 1 (Infection Lemma). Recall that strict partial orders are transitive and irreflexive.

Choose any $q \in \left(\frac{1}{2}, 1\right)$ such that $q > 1 - \underline{p}$. Suppose for contradiction that common q-belief of θ occurs with positive probability at time t. Then there exists a subset $S \subseteq H_t$ such that, at time t, S is q-evident and both agents q-believe θ on S.

We will show that S contains $\underline{h}_t$. Following $\underline{h}_t$, agent 1 assigns probability 1/2 to the event θ' for $\theta' \neq \theta$. Since $q > 1/2$, these beliefs violate the hypothesis that both agents q-believe θ on S at time t, giving the desired contradiction.

Let $\hat{h}_t$ be a minimal element of S with respect to $\prec_t$; that is, let $\hat{h}_t \in S$ be such that there does not exist $h \in S$ satisfying $h \prec_t \hat{h}_t$. A minimal element exists since $\prec_t$ is transitive and S is finite.

We show that $\hat{h}_t = \underline{h}_t$. Suppose for contradiction that $\underline{h}_t \neq \hat{h}_t$. By assumption, we have

$$\Pr\left(\{h'_t : h'_t \prec_t \hat{h}_t\} \middle| \hat{h}^i_t\right) \geq \underline{p}$$

for some i. Since S is q-evident at time t, we also have

$$\Pr\left(S \mid \hat{h}^i_t\right) \geq q.$$

By the choice of q, $\underline{p} + q > 1$ and hence $\{h'_t : h'_t \prec_t \hat{h}_t\} \cap S \neq \emptyset$. Thus there exists $h'_t \in S$ such that $h'_t \prec_t \hat{h}_t$. Since $\prec_t$ is irreflexive, $h'_t \neq \hat{h}_t$, contradicting that $\hat{h}_t$ is a minimal element of S. Therefore, $\hat{h}_t = \underline{h}_t$ and $\underline{h}_t \in S$. $\square$

Proof of Proposition 2. By the Infection Lemma, it suffices to show that there exists $\underline{p} \in (0, 1)$ such that, for each t, the lexicographic ordering $\prec^L_t$ satisfies (1).

We claim that if communication is fast then there exists $q \in (0, 1)$ such that

$$\Pr\left(t_0 = t - 1 \mid t_1 = t\right) > q \tag{2}$$

for all $t \in \mathbb{N}_+$, and

$$\Pr(t_0 > t \mid t_1 > t) \geq q \tag{3}$$

for every $t \in \mathbb{N}$. Inequality (3) immediately follows from the definition of fast communication.

To prove (2), we need to show that fast communication implies that there exists $q \in (0, 1)$ such that

$$\frac{f(1)g(t-1)}{f * g(t)} \geq q$$

for every $t \geq 1$. Since $f(1)$ is positive, the claim follows if $\liminf_{t\to\infty} \frac{g(t-1)}{f*g(t)} > 0$. By the regularity assumption, it suffices to show that $\lim_{t\to\infty} \frac{g(t-1)}{f*g(t)} \neq 0$. Suppose for contradiction that $\lim_{t\to\infty} \frac{g(t-1)}{f*g(t)} = 0$. Since communication is fast, we have

$$\liminf_{t\to\infty} \frac{1 - G(t)}{1 - F * G(t)} > 0.$$

Hence there exists some $\delta > 0$ such that $\frac{1-G(t)}{1-F*G(t)} \geq \delta$ for every $t \geq 1$. Since $\lim_{t\to\infty} \frac{g(t-1)}{f*g(t)} = 0$, there exists some T such that

$$g(t-1) < \delta f * g(t)$$

for all $t \geq T$. By summing over $t \geq T$, the last inequality implies that

$$1 - G(t-1) < \delta(1 - F * G(t)),$$

contradicting the definition of δ (since $1 - G(t) \leq 1 - G(t-1)$).

We now prove that (1) holds for $\underline{p} = \min\{q/2, q(1 - f(1))\}$. We distinguish three cases.

First consider any $h \neq \underline{h}_t$ for which there is some $k \geq 0$ such that

$$t \geq t_{k+1}(h) > t_k(h) + 1. \tag{4}$$

Let k^* be the smallest k satisfying this condition and let i be the agent who receives message $k^* + 1$. We claim that following h, agent i $\underline{p}$-believes $\{h' : h' \prec_t^L h\}$ at time t. Since, in this case, both agents know the realizations of $t_1, \ldots, t_{k^*-1}$ (if $k^* > 1$), it suffices to show that agent i $\underline{p}$-believes the joint event that $t_0 \geq t_0(h)$ and $t_{k^*} > t_{k^*}(h)$. Suppose that $i = 2$ (the proof for $i = 1$ is similar only simpler). If $k^* = 0$, then agent 2 q-believes that $t_0 = t_1(h) - 1$, as needed. If $k^* > 0$, then agent 2 assigns independent probabilities of q to t_0 being equal to $t_1(h) - 1$ and

$$1 - f(1)\frac{f(t_{k^*+1}(h) - t_{k^*-1}(h) - 1)}{\sum_{s=1}^{t_{k^*+1}(h)-t_{k^*-1}(h)-1} f(s)f(t_{k^*+1}(h) - t_{k^*-1}(h) - s)} \tag{5}$$

to the event that $t_{k^*} > t_{k^*-1}(h) + 1$. Since, by assumption, $t_{k^*+1}(h) - t_{k^*-1}(h) - 1 > 1$, the denominator of (5) contains two terms equal to the numerator, and hence the entire expression is at least 1/2. Therefore, agent 2 $q/2$-believes $\{h' : h' \prec_t^L h\}$, as needed.

Second, consider any t-history $h \neq \underline{h}_t$ with $m(h) > 0$ for which there is no k satisfying (4). Let i denote the agent who receives message $m(h) - 1$. Again, both agents know the realizations of $t_1, \ldots, t_{k^*-1}$, and thus it suffices to show that agent i $\underline{p}$-believes the joint event that $t_0 = t_0(h)$ and $t_{m(h)} > t_{m(h)}(h)$. Suppose that $i = 2$ (once again, the argument is simpler if $i = 1$). Let $\Delta_1 = t_{m(h)} - t_{m(h)-1}$ and $\Delta_2 = t_{m(h)+1} - t_{m(h)}$. Since, by assumption, $t_{m(h)}(h) = t_{m(h)-1}(h) + 1$, agent 2 assigns probability

$$1 - f(1)\frac{\Pr\left(\Delta_1 > t - t_{m(h)-1} - 1\right)}{\Pr\left(\Delta_1 + \Delta_2 > t - t_{m(h)-1}\right)}$$

to the event that $t_{m(h)} > t_{m(h)}(h)$. This last expression is at least $1 - f(1)$ since

$$\Pr(\Delta_1 \geq d - 1) \leq \Pr(\Delta_1 + \Delta_2 \geq d)$$

for any d because Δ_2 has support on $\mathbb{N}_+$. Since agent 2 assigns independent probability q to $t_0 = t_1(h) - 1$, she $q(1 - f(1))$-believes that $t_0 = t_0(h)$ and $t_{m(h)} > t_{m(h)}(h)$, as needed.

Finally, at any history h such that $m(h) = 0$, agent 2 q-believes $\underline{h}_t$. ☐

Proof of Proposition 3. By the Infection Lemma, it suffices to show that there exists $\underline{p} \in (0, 1)$ such that, for each t, the message-based ordering $\prec_t^M$ satisfies (1).

First, because communication is fast, there exists $q < 1$ such that at any history h_t with $m(h_t) = 0$, agent 2 q-believes $\underline{h}_t$. Second, consider a history h_t with $m(h_t) > 0$, and let i be the agent who receives message $m(h_t) + 1$. Since agent i has not received message $m(h_t) + 1$ by t, we have

$$\Pr(t_{m(h_t)} > t \mid h_t^i) = \frac{1 - F(t - t_{m(h_t)-1})}{1 - F * F(t - t_{m(h_t)-1})}.$$

Since F is O-subexponential, the right-hand side of this equation is bounded below by some $q' > 0$ uniformly across all values of $t - t_{m(h_t)-1}$. In particular, agent i q'-believes $\{h_t' : h_t' \prec_t^M h_t\}$. Letting $\underline{p} = \min\{q, q'\}$ establishes the required inequality. ☐

Proof of Proposition 4. For each case, we construct a system of events $D_{t,t'}$ such that (i) $\lim_{t' \to \infty} \lim_{t \to \infty} \Pr(D_{t,t'}) = 1$; (ii) for each $q \in (0, 1)$ and each t', $D_{t,t'}$ is q-evident at t whenever t is sufficiently large; and (iii) both agents know θ at t on $D_{t,t'}$. The existence of such a system implies common learning of Θ. To see this, fix $q \in (0, 1)$. Note first that (i) implies that there exist t' and T such that $\Pr(D_{t,t'}) > q$ whenever $t > T$. By (ii), there exists T' such that $D_{t,t'}$ is q-evident whenever $t > T'$. Letting $T'' = \max\{T, T'\}$, (iii) implies that $\Pr(C_q^t(\theta) \mid \theta) > q$ whenever $t > T''$, as needed.

For case 1 (finite N), let

$$D_{t,t'} = \{\omega : t_{N-1}(\omega) \leq t' \text{ and } t_N(\omega) \leq t\}.$$

Properties (i) and (iii) are immediate. All that remains is to prove that property (ii) holds. Since messages are dated, the agent who receives the Nth message knows $D_{t,t'}$ at t on $D_{t,t'}$. The other agent knows $t_{N-1}(\omega)$ and assigns probability at least $F(t-t')$ to t_N being at most t. Since F is not defective, this probability exceeds q when t is sufficiently large (given t').

For case 2, let

$$D_{t,t'} = \{\omega : t_1(\omega) \leq t' \text{ and } t_2(\omega) \leq t\}.$$

Properties (i) and (iii) are again immediate. For property (ii), note that, since messages are dated, agent 1 knows $D_{t,t'}$ at t on $D_{t,t'}$. If $t_3(\omega) \leq t$ then agent 2 knows $D_{t,t'}$ at t. Otherwise, she assigns probability at least $\frac{F(t-t')}{1-F*F(t-t')}$ to t_2 being at most t. Since F is not heavy-tailed, this probability exceeds q when t is sufficiently large (given t'). □

References

[1] K. Binmore and L. Samuelson. Coordinated action in the electronic mail game. *Games Econ. Behav.*, 35(1):6–30, 2001.

[2] M.W. Cripps, J.C. Ely, G.J. Mailath, and L. Samuelson. Common Learning. *Econometrica*, 76(4):909–933, 2008.

[3] J.D. Geanakoplos and H.M. Polemarchakis. We can't disagree forever. *J. Econ. Theory*, 28(1):192–200, 1982.

[4] J.Y. Halpern and Y. Moses. Knowledge and common knowledge in a distributed environment. *J. Assoc. Comput. Mach.*, 37(3):549–587, 1990.

[5] A. Heifetz. Comment on consensus without common knowledge. *J. Econ. Theory*, 70 (1):273–277, 1996.

[6] F. Koessler. Common knowledge and consensus with noisy communication. *Math. Soc. Sci.*, 42(2):139–160, 2001.

[7] D.K. Lewis. *Convention: a philosophical study.* Harvard University Press, 1969.

[8] G.J. Mailath and S. Morris. Repeated games with imperfect private monitoring: Notes on a coordination perspective. *CARESS Working Papers*, 1998.

[9] D. Monderer and D. Samet. Approximating common knowledge with common beliefs. *Games Econ. Behav.*, 1(2):170–190, 1989.

[10] S. Morris. Faulty Communication: Some Variations on the Electronic Mail Game. *Adv. Theoret. Econ.*, 1(1):Article 5, 2002.

[11] R. Parikh and P. Krasucki. Communication, consensus, and knowledge. *J. Econ. Theory*, 52(1):178–189, 1990.

[12] A. Rubinstein. The Electronic Mail Game: Strategic Behavior Under" Almost Common Knowledge". *Amer. Econ. Rev.*, 79(3):385–391, 1989.

[13] T. Shimura and T. Watanabe. Infinite divisibility and generalized subexponentiality. *Bernoulli*, 11(3):445–469, 2005.

[14] J. Steiner and C. Stewart. Communication Can Destroy Common Learning. Unpublished mimeo, 2009.