Optimal OFDM Channel Estimation with Carrier Frequency Offset and Phase Noise

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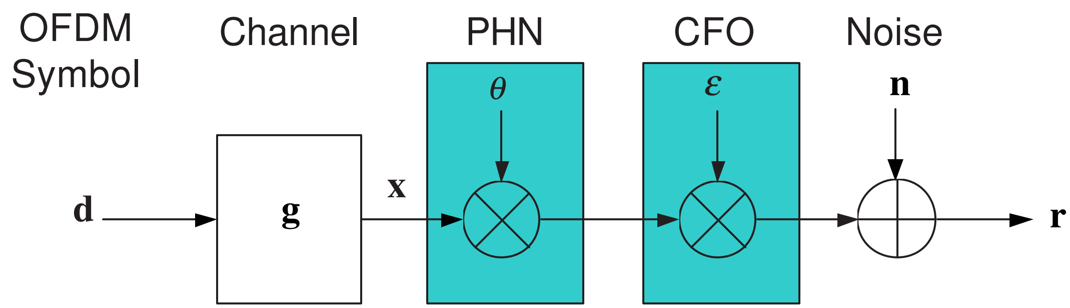
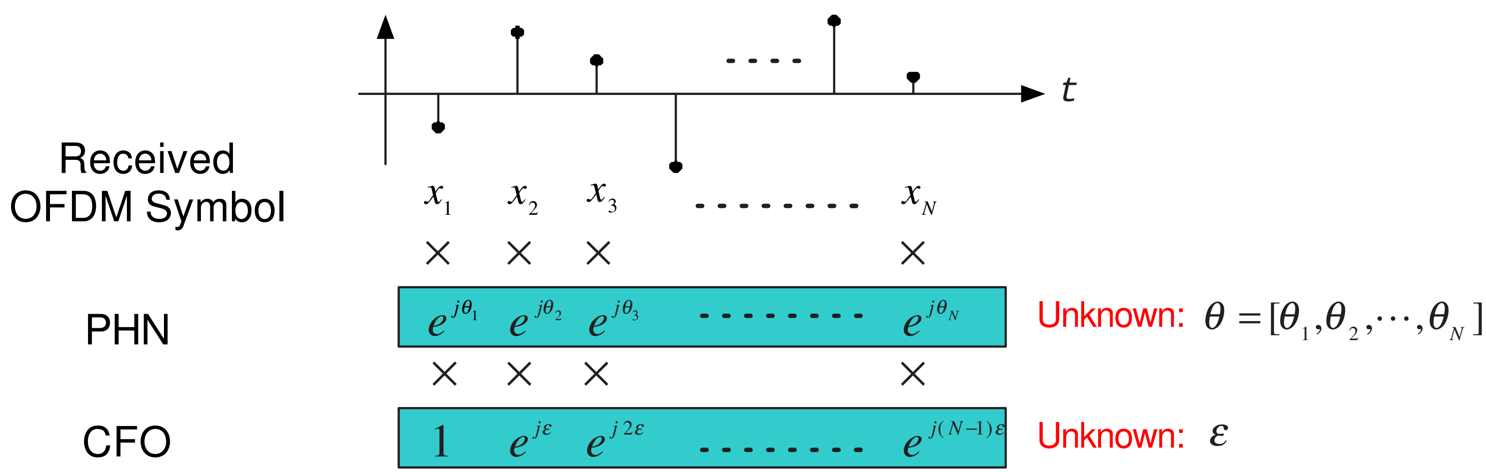
Outline

- **Problem description and signal model in matrix form.**
- Prior statistics of phase noise.
- Channel estimation ignoring frequency offset and phase noise.
- The optimal joint CFO/PHN/CIR estimator.
- Complexity reduction using conjugate gradient method.
- Simulation results.

- OFDM channel with PHN (Phase Noise) and CFO (Carrier Frequency Offset):



- Distortions caused by PHN and CFO:



I. Signal Model in Matrix Form

- Complex baseband received signal in one OFDM symbol interval:

$$\mathbf{r} = \mathbf{EPGF}^H\mathbf{d} + \mathbf{n}, \tag{1}$$

  - $\mathbf{r} \in \mathbb{C}^{N\times 1}$: received OFDM symbol with cyclic prefix removed;
  - $\mathbf{E} = \text{diag}([1, e^{j2\pi\epsilon/N}, \cdots, e^{j2\pi(N-1)\epsilon/N}]^T)$: CFO matrix;
  - $\mathbf{P} = \text{diag}([e^{j\theta_0}, \cdots, e^{j\theta_{N-1}}]^T)$: PHN matrix;
  - $\mathbf{G}$: channel circular convolution matrix, formed by CIR $\mathbf{g}$;
  - $\mathbf{F} \in \mathbb{C}^{N\times N}$: DFT matrix;
  - $\mathbf{d} \in \mathbb{C}^{N\times 1}$: vector of constant-modulus training symbols;
  - $\mathbf{n} \in \mathbb{C}^{N\times 1}$: complex white Gaussian noise with variance σ^2 per dimension.

- The objective is to, based on received $\mathbf{r}$, estimate three unknowns:
  - (1)ϵ, (2)$\boldsymbol{\theta} = [\theta_0, \cdots, \theta_{N-1}]^T$, (3)$\mathbf{g} = [g_0, \cdots, g_{L-1}]^T$.

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II. Prior Statistics of Phase Noise

- Two different models of PHN are available:
  - For free-running oscillator at the receiver, we assume a non-stationary Gaussian process, called *Wiener PHN*.
  - For oscillator controlled by a phase-locked loop (PLL), we assume a zero-mean coloured Gaussian process, called *Gaussian PHN*.

- The prior statistics of both types of PHN can be modeled as a multivariate Gaussian distribution:

$$p(\boldsymbol{\theta}) = \mathcal{N}(\mathbf{0}, \boldsymbol{\Phi}), \tag{2}$$

where the covariance matrix $\boldsymbol{\Phi}$ can be determined from the power spectral density (PSD) of the VCO output. The details are presented in the paper, but omitted here.

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III. Channel Estimation Ignoring CFO and PHN

- If we ignore CFO and PHN (i.e. assume $\mathbf{EP} = \mathbf{I}$), the received signal model becomes: $\mathbf{r} = \mathbf{GF}^H\mathbf{d} + \mathbf{n}$.

- Or, it can be written alternatively as

$$\mathbf{r} = \mathbf{F}^H\mathbf{DWg} + \mathbf{n}, \tag{3}$$

  - $\mathbf{D} = \text{diag}(\mathbf{d})$;
  - $\mathbf{F} = [\mathbf{W}|\mathbf{V}]$, $\mathbf{W} \in \mathbb{C}^{N\times L}$;
  - $\mathbf{g} \in \mathbb{C}^{L\times 1}$ is the CIR (Channel Impulse Response) of length $L < N$.

- Assuming constant-modulus training symbols, i.e. $\mathbf{D}^H\mathbf{D} = 2\rho^2\mathbf{I}$, maximizing $p(\mathbf{r}|\mathbf{g}) = \mathcal{CN}(\mathbf{F}^H\mathbf{DWg}, 2\sigma^2\mathbf{I})$ leads to the ML estimator

$$\hat{\mathbf{g}} = (2\rho^2)^{-1}\mathbf{W}^H\mathbf{D}^H\mathbf{Fr}. \tag{4}$$

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IV. The Complete Likelihood Function

- If we consider CFO and PHN, the optimal estimator requires joint estimation of three unknowns, $\epsilon, \boldsymbol{\theta}$ and $\mathbf{g}$, in

$$\mathbf{r} = \mathbf{EPF}^H\mathbf{DWg} + \mathbf{n}. \tag{5}$$

- We first write the "complete likelihood function"

$$p(\mathbf{r}, \epsilon, \boldsymbol{\theta}, \mathbf{g}) = p(\mathbf{r}|\epsilon, \boldsymbol{\theta}, \mathbf{g})p(\epsilon)p(\boldsymbol{\theta})p(\mathbf{g}), \tag{6}$$

which is proportional to the *a posteriori* distribution of the unknowns, $p(\epsilon, \boldsymbol{\theta}, \mathbf{g}|\mathbf{r})$.

- Since we assume no prior knowledge of ϵ and $\mathbf{g}$, $p(\epsilon)$ and $p(\mathbf{g})$ are constants and can be omitted. The prior of $\boldsymbol{\theta}$ is available, which is $p(\boldsymbol{\theta}) = \mathcal{N}(\mathbf{0}, \boldsymbol{\Phi})$ as discussed before.

IV. Target of Optimization

- Taking the logarithm, the "complete negative log-likehood function" can be written as

$$\begin{aligned}\mathcal{L}(\epsilon,\boldsymbol{\theta},\mathbf{g}) = & -\log p(\mathbf{r}|\epsilon,\boldsymbol{\theta},\mathbf{g}) - \log p(\boldsymbol{\theta}) \\ = & \frac{1}{2\sigma^2}(\mathbf{r}-\mathbf{EPF}^H\mathbf{DWg})^H(\mathbf{r}-\mathbf{EPF}^H\mathbf{DWg}) + \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Phi}^{-1}\boldsymbol{\theta}.\end{aligned} \quad (7)$$

- The objective is to find the jointly optimal estimates

$$(\hat{\epsilon},\hat{\boldsymbol{\theta}},\hat{\mathbf{g}}) = \arg\min_{\epsilon,\boldsymbol{\theta},\mathbf{g}} \mathcal{L}(\epsilon,\boldsymbol{\theta},\mathbf{g}). \quad (8)$$

- The estimator proposed here is "optimal" in the sense of maximizing the "complete likelihood function". It can be derived in 3 optimization steps.

IV. The Optimal Estimator: CIR and PHN Estimation

1. *CIR Estimation*: Solve $\partial\mathcal{L}(\epsilon, \boldsymbol{\theta}, \mathbf{g})/\partial\mathbf{g}^* = \mathbf{0}$, we obtain

$$\hat{\mathbf{g}} = (2\rho^2)^{-1}\mathbf{W}^H\mathbf{D}^H\mathbf{F}\mathbf{P}^H\mathbf{E}^H\mathbf{r}. \tag{9}$$

– Substituting $\mathbf{g} = \hat{\mathbf{g}}$ back into $\mathcal{L}(\epsilon, \boldsymbol{\theta}, \mathbf{g})$ produces $\mathcal{L}(\epsilon, \boldsymbol{\theta})$.

2. *PHN Estimation*: Solve $\partial\mathcal{L}(\epsilon, \boldsymbol{\theta})/\partial\boldsymbol{\theta} = \mathbf{0}$, we obtain

$$\hat{\boldsymbol{\theta}} = [\mathrm{Re}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H) + 2\sigma^2\rho^2\boldsymbol{\Phi}^{-1}]^{-1}\mathrm{Im}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H)\mathbf{1}, \tag{10}$$

where $\mathbf{C} = \mathbf{R}^H\mathbf{F}^H\mathbf{D}\mathbf{V}$ and $\mathbf{R} = \mathrm{diag}(\mathbf{r})$.

– Substituting $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$ back into $\mathcal{L}(\epsilon, \boldsymbol{\theta})$ produces $\mathcal{L}(\epsilon)$

3. *CFO Estimation*: To minimize $\mathcal{L}(\epsilon)$, we require searching over the range $-0.5 < \epsilon < 0.5$:

$$\begin{aligned}\hat{\epsilon} = \quad & \arg\min_{\epsilon} \mathbf{1}^T\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H\mathbf{1} - \mathbf{1}^T\mathrm{Im}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H)^T \\ & \times[\mathrm{Re}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H) + 2\sigma^2\rho^2\mathbf{\Phi}^{-1}]^{-1}\mathrm{Im}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H)\mathbf{1}.\end{aligned} \tag{11}$$

- Combining the 3 optimization steps leads to the complete *Joint CFO/PHN/CIR Estimation* (JCPCE) algorithm:

$$\begin{aligned}
\text{Step 1:}\quad & \hat{\epsilon} = \arg\min_{\epsilon} \mathbf{1}^T\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H\mathbf{1} - \mathbf{1}^T\mathrm{Im}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H)^T \\
& \times[\mathrm{Re}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H) + 2\sigma^2\rho^2\mathbf{\Phi}^{-1}]^{-1}\mathrm{Im}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H)\mathbf{1}; \\
& \hat{\mathbf{E}} = \mathrm{diag}([1, e^{j2\pi\hat{\epsilon}/N}, \cdots, e^{j2\pi(N-1)\hat{\epsilon}/N}]^T); \\
\text{Step 2:}\quad & \hat{\boldsymbol{\theta}} = [\mathrm{Re}(\hat{\mathbf{E}}\mathbf{C}\mathbf{C}^H\hat{\mathbf{E}}^H) + 2\sigma^2\rho^2\mathbf{\Phi}^{-1}]^{-1}\mathrm{Im}(\hat{\mathbf{E}}\mathbf{C}\mathbf{C}^H\hat{\mathbf{E}}^H)\mathbf{1}; \\
& \hat{\mathbf{P}} = \mathrm{diag}([e^{j\hat{\theta}_0}, \cdots, e^{j\hat{\theta}_{N-1}}]^T); \\
\text{Step 3:}\quad & \hat{\mathbf{g}} = (2\rho^2)^{-1}\mathbf{W}^H\mathbf{D}^H\mathbf{F}\hat{\mathbf{P}}^H\hat{\mathbf{E}}^H\mathbf{r}.
\end{aligned}$$

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V. Complexity Reduction: Special Structure of Ψ

- Letting $\boldsymbol{\Psi} = \frac{1}{2\sigma^2\rho^2}\boldsymbol{\Phi}$, the dominant complexity of the previous algorithm is associated with the evaluation of $[\text{Re}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H) + \boldsymbol{\Psi}^{-1}]^{-1}$, which in general has complexity $\mathcal{O}(N^3)$.

- Fortunately, complexity reduction is available by noticing the following:
  - For *Wiener PHN*, $\boldsymbol{\Psi}^{-1}$ is a tridiagonal matrix;

$$\boldsymbol{\Psi}^{-1} = \frac{2\sigma^2\rho^2}{\alpha_\phi^2}\begin{bmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ \mathbf{0} & & & -1 & 1 \end{bmatrix}. \tag{12}$$

  - For *Gaussian PHN*, $\boldsymbol{\Psi}$ is a Toeplitz matrix, but can be closely approximated as a circulant matrix $\tilde{\boldsymbol{\Psi}}$, making the evaluation of $\tilde{\boldsymbol{\Psi}}^{-1}$ simple.

V. Complexity Reduction: Conjugate Gradient Method

- Letting $\mathbf{q} = \mathrm{Im}(\hat{\mathbf{E}}\mathbf{C}\mathbf{C}^H\hat{\mathbf{E}}^H)\mathbf{1}$, the evaluation of $[\mathrm{Re}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H) + \boldsymbol{\Psi}^{-1}]^{-1}\mathbf{q}$ can be accomplished by the conjugate gradient method as follows:

Initialization:	
	$\boldsymbol{\theta}_0 = \mathbf{0}$
	$\boldsymbol{\gamma}_0 = [\mathrm{Re}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H) + \tilde{\boldsymbol{\Psi}}^{-1}]\boldsymbol{\theta}_0 - \mathbf{q} = -\mathbf{q}$
	$\boldsymbol{\nu}_0 = -\boldsymbol{\gamma}_0 = \mathbf{q}$
For	$k = 0 : i-1$
	$\alpha_k = \boldsymbol{\gamma}_k^H\boldsymbol{\gamma}_k / (\boldsymbol{\nu}_k^H[\mathrm{Re}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H) + \tilde{\boldsymbol{\Psi}}^{-1}]\boldsymbol{\nu}_k)$
	$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k + \alpha_k\boldsymbol{\nu}_k$
	$\boldsymbol{\gamma}_{k+1} = \boldsymbol{\gamma}_k + \alpha_k[\mathrm{Re}(\mathbf{E}\mathbf{C}\mathbf{C}^H\mathbf{E}^H) + \tilde{\boldsymbol{\Psi}}^{-1}]\boldsymbol{\nu}_k$
	$\beta_{k+1} = \frac{\boldsymbol{\gamma}_{k+1}^H\boldsymbol{\gamma}_{k+1}}{\boldsymbol{\gamma}_k^H\boldsymbol{\gamma}_k}$
	$\boldsymbol{\nu}_{k+1} = -\boldsymbol{\gamma}_{k+1} + \beta_{k+1}\boldsymbol{\nu}_k$
End	

V. Complexity Reduction: Overall Complexity

- Using the Conjugate Gradient algorithm, we have control over the number of iteration i. Exact matrix inversion corresponds to $i = N$.

- By utilizing the special structure of $\mathbf{\Psi}$, we have reduced the complexity from $\mathcal{O}(N^3)$ to $\mathcal{O}(3N^2)$ for Wiener PHN, and $\mathcal{O}(N^2 \log N)$ for Gaussian PHN.

- In practice, $i \ll N$. In this case, the complexity becomes $\mathcal{O}(i \times 3N)$ for Wiener PHN, and $\mathcal{O}(i \times N \log N)$ for Gaussian PHN.

- Simulations demonstrate that even for $i = 5$, no significant performance degradation results.

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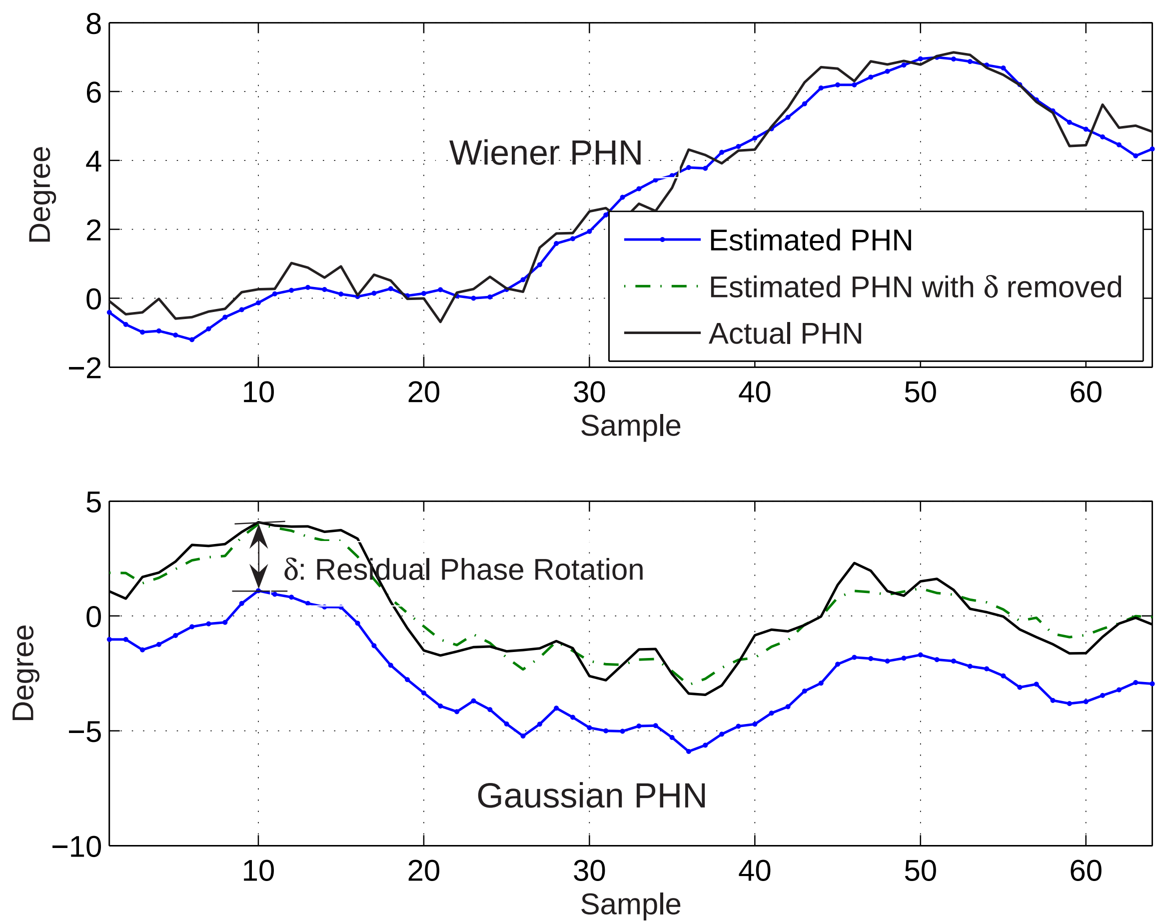
- *Unresolvable residual common phase rotation*: A residual phase rotation δ cannot be estimated for Gaussian PHN. $\delta \sim \mathcal{N}(0, \mathbf{1}^T \mathbf{\Phi} \mathbf{1}/N^2)$.



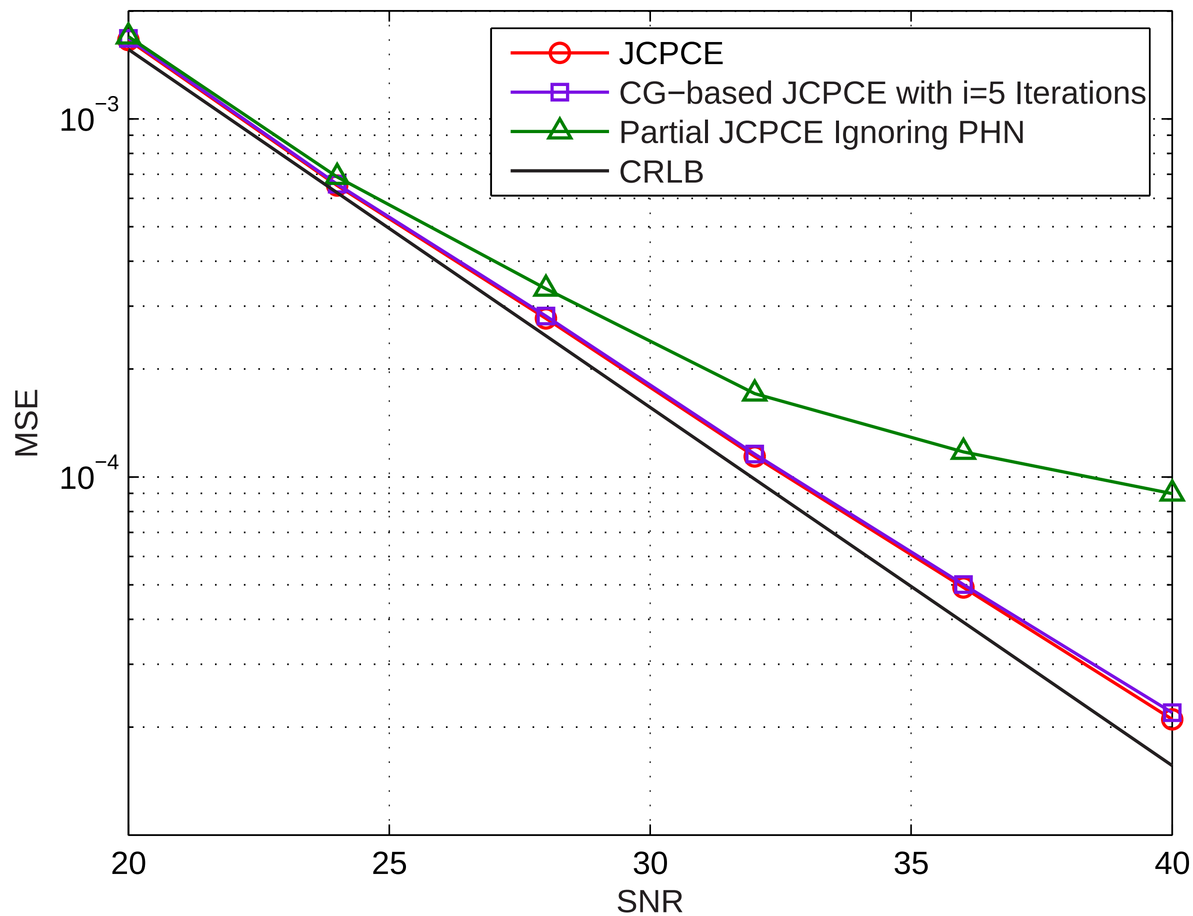
III. Simulation Settings

- The effect of residual phase rotation is that the channel estimate $\hat{\mathbf{g}}$ is off by a small unknown phase δ, which does not introduce ICI.

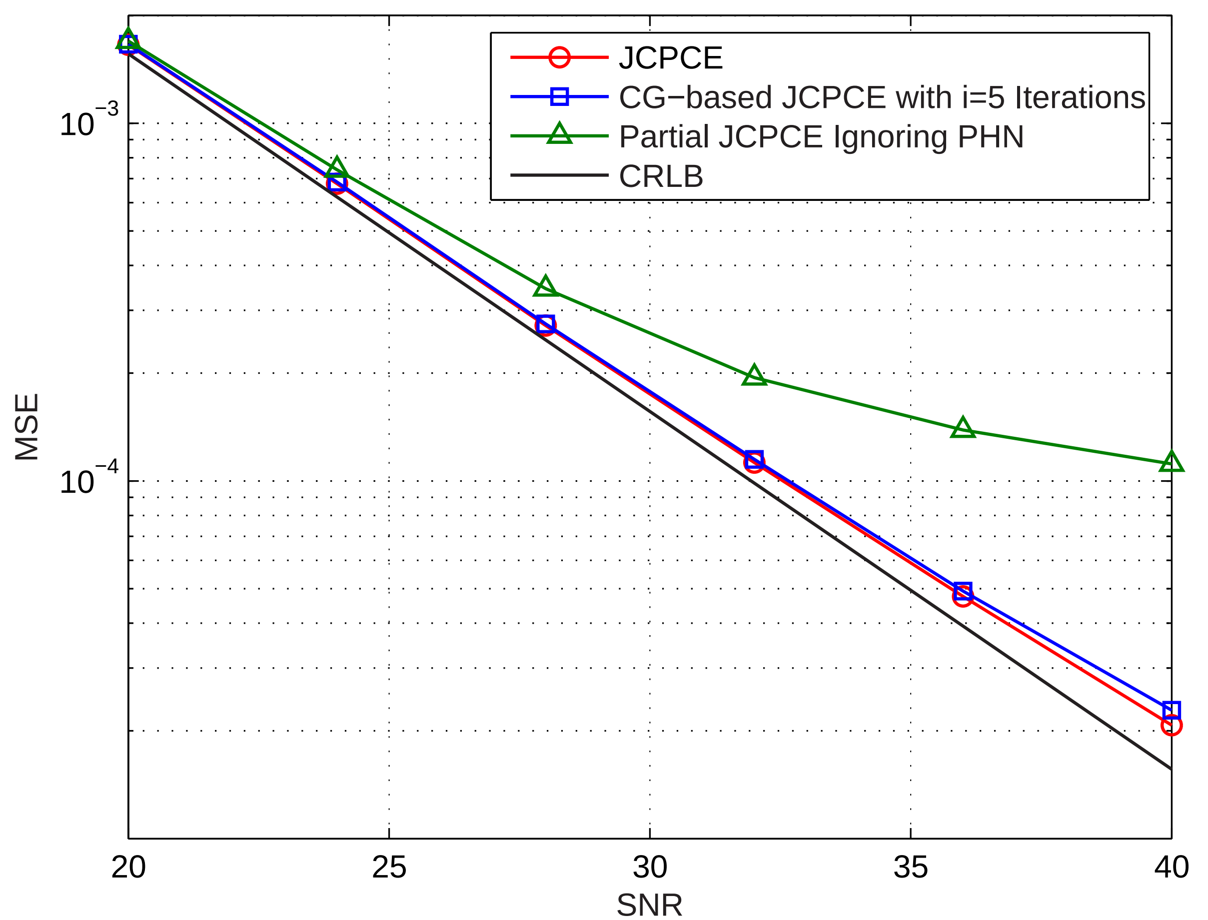
- We assume that δ can be perfectly corrected to facilitate easy assessment of channel estimation mean-squared error (MSE).

- The following system parameters are used in simulations:
  - A Rayleigh multipath fading channel with a delay of $L = 10$ taps;
  - An OFDM training symbol size of $N = 64$ subcarriers with each subcarrier modulated in QPSK;
  - Baseband sampling rate $f_s = 20$ MHz;
  - The Wiener PHN is generated as a random-walk process with incremental PHN of $\alpha_\phi = 0.6$ deg.
  - The Gaussian PHN has a standard deviation of $\theta_{rms} = 3$ deg. It is generated as i.i.d. Gaussian samples passed through a single pole Butterworth filter of 3dB bandwidth $\Omega_o = 100$ KHz.

- JCPCE performance in Wiener PHN:



- JCPCE performance in Gaussian PHN:



Summary

- Challenge: How do we optimally estimate the CIR of an OFDM channel in the presence of unknown PHN and CFO?

- Solution: We derived a *Joint CFO/PHN/CIR Estimation* (JCPCE) algorithm that optimizes the "complete likelihood function" incorporating the prior distribution of PHN. The CFO and PHN are accurately estimated together with the CIR.

- In addition, we reduced the complexity of the proposed estimator to an acceptable level using the Conjugate Gradient method.

- The complexity we cannot reduce within the optimal algorithm is the search operation in the estimation of CFO.

- This problem is solved with a sub-optimal algorithm to be presented at ICC 2006, titled "Near-Optimal Training-Based Estimation of Frequency Offset and Channel Response in OFDM with Phase Noise".

Thank You!