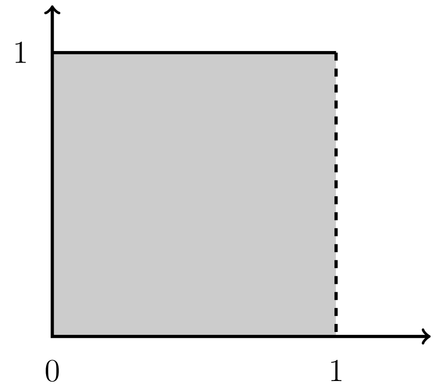
Finding a Symmetric Nash Equilibrium in a First-Price Sealed-Bid Auction

ECO316: Summer 2018 (Instructor: Christopher R. Dobronyi)

Consider a first-price sealed-bid auction with two bidders: Each bidder has a value for an object that is being sold by an auctioneer. Each bidder knows her own value but does not know the value of the other bidder. Each bidder correctly believes that the other bidder's value is drawn independently from the following uniform distribution between 0 and 1:



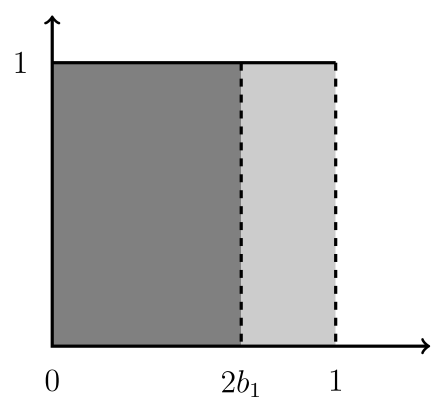
Each bidder chooses a non-negative bid. The bidder with the highest bid wins and receives the object. (You can ignore ties.) If a bidder wins then her payoff is her value minus her bid. If a bidder does not win then her payoff is zero.

1. Suppose that bidder 2's strategy is to bid $v_2/2$, for each value $0 \leq v_2 \leq 1$. Find the expected payoff of bidder 1 when she bids b_1, for each value $0 \leq v_1 \leq 1$.

   Solution. The probability that bidder 1 wins when she bids b_1 is the probability that b_1 is larger than bidder 2's bid, given b_1 and bidder 2's strategy:

$$P(1 \text{ wins} \,|\, b_1) = P(b_2 < b_1 \,|\, b_1) = P(v_2/2 < b_1 \,|\, b_1) = P\left(v_2 < 2b_1 \,|\, b_1\right).$$

   This probability is given by the area under the distribution of values to the left of $2b_1$. The dark shaded region in the following figure illustrates this area when $0 \leq 2b_1 \leq 1$ (or, equivalently, when $0 \leq b_1 \leq 1/2$):



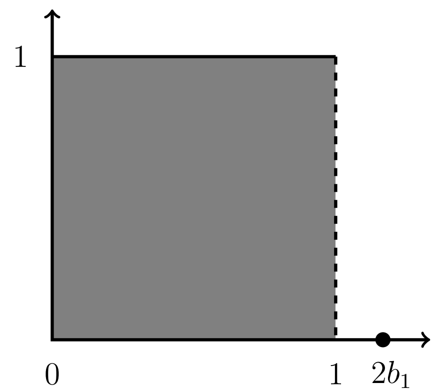
We can calculate this area because it is a rectangle with known dimensions and we know how to calculate the area of a rectangle. In doing so, we obtain:

$$P(1 \text{ wins} \,|\, b_1) = 2b_1, \quad \text{for all } 0 \leq b_1 \leq {}^1\!/\!_2.$$

Notice that I have written that this is true for all b_1 such that $0 \leq b_1 \leq {}^1\!/\!_2$. Since $0 \leq v_2 \leq 1$ and bidder 2's strategy is to bid exactly half her value, bidder 2 will never submit a bid larger than ${}^1\!/\!_2$. As a result,

$$P(1 \text{ wins} \,|\, b_1) = 1, \quad \text{for all } b_1 > {}^1\!/\!_2.$$

The dark shaded region in the following figure illustrates the area under the distribution of values to the left of $2b_1$ when $2b_1 > 1$ (or, equivalently, when $b_1 > {}^1\!/\!_2$):



If bidder 1 wins then her payoff is $v_1 - b_1$. If bidder 1 does not win then her payoff is 0. Therefore, bidder 1's expected payoff when she submits a bid of b_1 is

$$P(1 \text{ wins} \,|\, b_1) \cdot (v_1 - b_1) + P(1 \text{ loses} \,|\, b_1) \cdot 0 = P(1 \text{ wins} \,|\, b_1) \cdot (v_1 - b_1),$$

or, equivalently, using our knowledge of the probability that bidder 1 wins, we obtain:

$$\begin{cases} v_1 - b_1, & \text{if } b_1 > {}^1\!/\!_2, \\ 2b_1(v_1 - b_1), & \text{if } 0 \leq b_1 \leq {}^1\!/\!_2. \end{cases}$$

2. Does the auction have a Nash equilibrium in which each bidder i's strategy is to bid ${}^{v_i}\!/\!_2$, for each value $0 \leq v_i \leq 1$?

Solution. Suppose that bidder 2's strategy is to bid ${}^{v_2}\!/\!_2$, for each value $0 \leq v_2 \leq 1$. It is not optimal for bidder 1 to bid a value greater than ${}^1\!/\!_2$. (To see this, notice that bidding ${}^1\!/\!_2$ yields a payoff of $v_1 - {}^1\!/\!_2$ and bidding $b_1 > {}^1\!/\!_2$ yields a payoff of $v_1 - b_1 < v_1 - {}^1\!/\!_2$.) By part 1, we know that bidder 1's expected payoff is $2b_1(v_1 - b_1)$ when she bids b_1

such that $0 \leq b_1 \leq {}^1\!/_2$. We can find the maximizer of this function by taking its first derivative and setting it equal to zero. This process yields:

$$2v_1 - 4b_1 = 0.$$

This equality has one solution at $b_1 = {}^{v_1}\!/_2$. (It is worth noting that the value of this solution is always less than or equal to ${}^1\!/_2$.) This solution implies that if bidder 1 believes that bidder 2's value is distributed uniformly between 0 and 1, then it is optimal for bidder 1 to bid ${}^{v_1}\!/_2$ when player 2's strategy is to bid ${}^{v_2}\!/_2$. The same considerations apply to bidder 2. As a result, there exists a Nash equilibrium in which each bidder i's strategy is to bid ${}^{v_i}\!/_2$, for each value $0 \leq v_i \leq 1$.