ECO316: Applied Game Theory

Lecture 1

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Based on materials by Martin J. Osborne

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Course Information

- **Lectures**: Tuesday and Thursday 12:00 to 14:00 in SS1073
- **Tutorials**: Tuesday and Thursday 14:00 to 15:00 in SS1073
- **Office Hours**: Tuesday 15:00 to 16:00 in GE346

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Recommended Textbook

- **Title**: An Introduction to Game Theory
- **Author**: Martin J. Osborne
- **Publisher**: Oxford University Press
- **ISBN**: 9780195128956

Game Theory

- Analytical tools for studying **strategic interaction**
- Used in economics, political science, computer science, etc.
- This course covers theory and applications

Applications in Economics

Applications in this course include

- firm competition
- electoral competition
- collective choice
- auctions
- collusion
- signaling

Preference Relations

- Let X denote a set of **alternatives**
- A decision-maker has **preferences** over the alternatives in X
- Her preferences are summarized by a **preference relation** $\succeq$
- A preference relation **compares pairs** of alternatives such that

$x \succeq y$ is read "x is at least as good as y"

Example of a Preference Relation

Let $X = \{\text{Apple}, \text{Banana}, \text{Orange}\}$. I like **Banana** at least as much as **Apple**, **Apple** at least as much as **Orange** and **Orange** at least as much as **Banana**. My preference relation $\succeq$ satisfies

$$\begin{aligned} \text{Banana} &\succeq \text{Apple} \\ \text{Apple} &\succeq \text{Orange} \\ \text{Orange} &\succeq \text{Banana} \end{aligned}$$

Strict and Indifference Relations

Let $\succ$ denote the **strict relation** induced by $\succeq$ such that

$$\boxed{x \succ y \text{ if } x \succeq y \text{ and not } y \succeq x}$$

in which

$$\boxed{x \succ y \text{ is read ``} x \text{ is preferred to } y\text{''}}$$

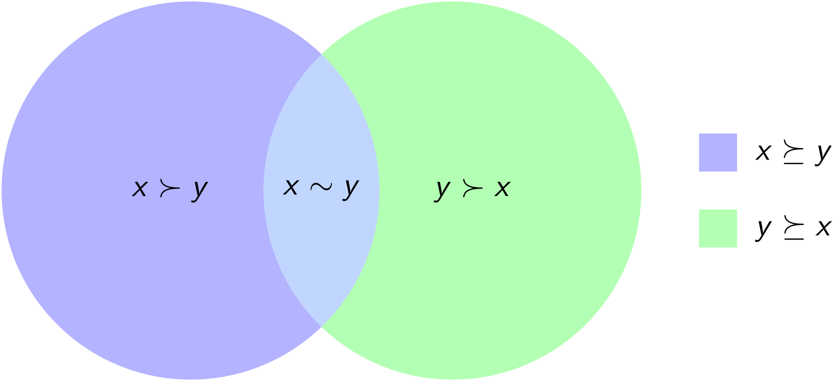
Let $\sim$ denote the **indifference relation** induced by $\succeq$ such that

$$\boxed{x \sim y \text{ if } x \succeq y \text{ and } y \succeq x}$$

in which

$$\boxed{x \sim y \text{ is read ``} x \text{ is indifferent to } y\text{''}}$$

Venn Diagram of Relations



Rational Preference Relations

A preference relation $\succeq$ is **rational** if it is **complete**, i.e.,

$$\boxed{x \succeq y \text{ or } y \succeq x \text{ (or both), for all } x \text{ and } y \text{ in } X}$$

and **transitive**, i.e.,

$$\boxed{x \succeq y \text{ and } y \succeq z \text{ implies } x \succeq z}$$

Examples of Rational and Irrational Preference Relations

$$X = \{\text{Apple}, \text{Orange}, \text{Banana}\}$$

Not complete:

$$\begin{aligned} \text{Banana} &\succeq \text{Apple} \\ \text{Apple} &\succeq \text{Orange} \\ \text{Orange} &\ ?\ \text{Banana} \end{aligned}$$

Not transitive:

$$\begin{aligned} \text{Banana} &\succeq \text{Apple} \\ \text{Apple} &\succeq \text{Orange} \\ \text{Orange} &\succ \text{Banana} \end{aligned}$$

Rational:

$$\begin{aligned} \text{Banana} &\sim \text{Apple} \\ \text{Apple} &\sim \text{Orange} \\ \text{Orange} &\sim \text{Banana} \end{aligned}$$

Rational Decision-Makers

For now, a decision-maker is **rational** if her preference relation is rational and she behaves **optimally** with respect to her preferences:

e.g. faced with the problem of choosing between alternatives x and y, the decision-maker will choose x only if x is at least as good as y

To be rational, a decision-maker does not need to be **well-informed**, **selfish** or **sensible** in an objective sense

Utility Functions

- It is often convenient to describe preferences using **utility**
- A **utility function** assigns a number $u(x)$ to each alternative
- $u(x)$ represents the decision-maker's **relative value** for x
- A utility function **represents** a rational preference relation $\succeq$ if

$$\boxed{x \succeq y \ \text{ if and only if } \ u(x) \geq u(y)}$$

Examples of Utility Representation

The rational **preference relation** $\succeq$ over fruit defined by

$$\boxed{\begin{aligned} \text{Banana} &\sim \text{Apple} \\ \text{Apple} &\sim \text{Orange} \\ \text{Orange} &\sim \text{Banana} \end{aligned}}$$

is **represented** by a **utility function** $u(x)$ such that

$$u(\text{Apple}) = u(\text{Orange}) = u(\text{Banana}) = 0$$

Examples of Utility Representation

The rational **preference relation** $\succeq$ over fruit defined by

$$\boxed{\begin{aligned} \text{Banana} &\sim \text{Apple} \\ \text{Apple} &\sim \text{Orange} \\ \text{Orange} &\sim \text{Banana} \end{aligned}}$$

is **represented** by a **utility function** $u(x)$ such that

$$u(\text{Apple}) = u(\text{Orange}) = u(\text{Banana}) = 5$$

Examples of Utility Representation

The rational **preference relation** $\succeq$ over fruit defined by

$$\begin{array}{c} \text{Banana} \succ \text{Apple} \\ \text{Banana} \succ \text{Orange} \\ \text{Orange} \succ \text{Apple} \end{array}$$

is **represented** by a **utility function** $u(x)$ such that

$$\begin{aligned} u(\text{Banana}) &= 2 \\ u(\text{Orange}) &= 1 \\ u(\text{Apple}) &= 0 \end{aligned}$$

Examples of Utility Representation

The rational **preference relation** $\succeq$ over fruit defined by

$$\begin{gathered}\text{Banana} \succ \text{Apple}\\ \text{Banana} \succ \text{Orange}\\ \text{Orange} \succ \text{Apple}\end{gathered}$$

is **represented** by a **utility function** $u(x)$ such that

$$\begin{aligned} u(\text{Banana}) &= 9\\ u(\text{Orange}) &= 5\\ u(\text{Apple}) &= 0\end{aligned}$$

Utility Functions are Ordinal

For now, only the **ranking** of alternatives matters—if a utility function $u(x)$ represents a preference relation $\succeq$ then $\succeq$ is also represented by every increasing function of $u(x)$:

A function $f(a)$ is **increasing** if $a > b$ implies $f(a) > f(b)$

Example: If $u(x)$ represents $\succeq$ then $\log(u(x))$ represents $\succeq$

Strategic Games

Game theory studies situations in which rational decision-makers interact—a **strategic game** is a simple model consisting of

- a set of **players**
- a set of **actions**, for each player
- **preferences** over the set of action profiles, for each player

Action profile: a list of actions with one action for each player

Example of a Strategic Game

- **Players**: two firms
- **Actions**: high (h) or low (ℓ) price, for each firm
- **Preferences**: firm i has a preference relation $\succeq_i$ such that

$$(\ell, h) \succ_1 (h, h) \succ_1 (\ell, \ell) \succ_1 (h, \ell)$$

$$(h, \ell) \succ_2 (h, h) \succ_2 (\ell, \ell) \succ_2 (\ell, h)$$

We can describe each firm's preferences using **utility**:

Firm 1's Utility:

		Firm 2	
		h	ℓ
Firm 1	h	2	0
	ℓ	3	1

Firm 2's Utility:

		Firm 2	
		h	ℓ
Firm 1	h	2	3
	ℓ	0	1

Example of a Strategic Game

- **Players**: two firms
- **Actions**: high (h) or low (ℓ) price, for each firm
- **Preferences**: firm i has a preference relation $\succeq_i$ such that

$$(\ell, h) \succ_1 (h, h) \succ_1 (\ell, \ell) \succ_1 (h, \ell)$$

$$(h, \ell) \succ_2 (h, h) \succ_2 (\ell, \ell) \succ_2 (\ell, h)$$

We can describe each firm's preferences using **utility**:

Combined:

		Firm 2	
		h	ℓ
Firm 1	h	$2, 2$	$0, 3$
	ℓ	$3, 0$	$1, 1$

Example of a Strategic Game

Since utility is ordinal for now, only the **ranking** of profiles matters:

Combined:

		Firm 2	
		h	ℓ
Firm 1	h	$2, 2$	$0, 3$
	ℓ	$3, 0$	$1, 1$

Same Game:

		Firm 2	
		h	ℓ
Firm 1	h	$3, 3$	$1, 4$
	ℓ	$4, 1$	$2, 2$

Same Game:

		Firm 2	
		h	ℓ
Firm 1	h	$2, 3$	$0, 4$
	ℓ	$3, 1$	$1, 2$

Different Game:

		Firm 2	
		h	ℓ
Firm 1	h	$2, 2$	$3, 0$
	ℓ	$3, 0$	$1, 1$

Example of a Strategic Game

Prisoner's Dilemma:

		Firm 2	
		h	ℓ
Firm 1	h	$2, 2$	$0, 3$
	ℓ	$3, 0$	$1, 1$

- The game in our example is named **Prisoner's Dilemma**
- Prisoner's Dilemma has been used to model many situations

Nash Equilibrium

An action profile is a **Nash equilibrium** of a strategic game if

every player's action is optimal, given the other players' actions,

or, equivalently, if

no player can change her action to make herself strictly better off, given the other players' actions

How to Find a Nash Equilibrium

Method: We can find every **Nash equilibrium** of a strategic game by **checking every action profile** to see if there is a player that can change her action to make herself strictly better off, while holding the other players' actions constant

Example of Method

Prisoner's Dilemma:

		Firm 2	
		h	ℓ
Firm 1	h	$2, 2$	$0, 3$
	ℓ	$3, 0$	$1, 1$

- (h, h) is not a Nash equilibrium because either firm can make itself better off by deviating to ℓ when the other firm plays h
- (h, ℓ) is not a Nash equilibrium because firm 1 can make itself better off by deviating to ℓ when firm 2 plays ℓ
- (ℓ, h) is not a Nash equilibrium because firm 2 can make itself better off by deviating to ℓ when firm 1 plays h
- (ℓ, ℓ) is a Nash equilibrium because neither firm can make itself better off by deviating to h when the other firm plays ℓ

Pareto Efficiency

An action profile is **Pareto efficient** if

there is no action profile in which every player is at least as well off and at least one player is strictly better off,

or, equivalently, if

no player can be made strictly better off without making at least one player strictly worse off

Example of Pareto Efficiency

Prisoner's Dilemma:

		Firm 2	
		h	ℓ
Firm 1	h	$2, 2$	$0, 3$
	ℓ	$3, 0$	$1, 1$

- (h, h) is Pareto efficient because at least one firm is strictly worse off in every other action profile
- (h, ℓ) is Pareto efficient because at least one firm is strictly worse off in every other action profile
- (ℓ, h) is Pareto efficient because at least one firm is strictly worse off in every other action profile
- (ℓ, ℓ) is not Pareto efficient because both firms prefer (h, h)

Example of Pareto Efficiency

Prisoner's Dilemma:

		Firm 2	
		h	ℓ
Firm 1	h	$2, 2$	$0, 3$
	ℓ	$3, 0$	$1, 1$

(h, h), (h, ℓ) and (ℓ, h) are **Pareto efficient**

The only **Nash equilibrium**, i.e., (ℓ, ℓ), is **not** Pareto efficient

Coordination Game

Coordination Game:

		Player 2	
		a	b
Player 1	a	$2, 2$	$0, 0$
	b	$0, 0$	$1, 1$

(a, a) and (b, b) are **Nash equilibria**

(a, a) is **Pareto efficient**

Bach or Stravinsky

Bach or Stravinsky:

		Player 2	
		b	s
Player 1	b	$2, 1$	$0, 0$
	s	$0, 0$	$1, 2$

(b, b) and (s, s) are **Nash equilibria**

(b, b) and (s, s) are **Pareto efficient**

Matching Pennies

Matching Pennies:

		Player 2	
		a	*b*
Player 1	*a*	1, 0	0, 1
	b	0, 1	1, 0

No **Nash equilibrium**

All action profiles are **Pareto efficient**

Investing in a Joint Project

- **Players**: n investors
- **Actions**: *Invest* or *Don't Invest*, for each investor
- **Preferences**: larger payoffs are better; an investor's payoff is

$$\begin{cases} 100, & \text{if she chooses } \textit{Invest} \text{ \& at least } k \text{ investors } \textit{Invest}, \\ -10, & \text{if she chooses } \textit{Invest} \text{ \& less than } k \text{ investors } \textit{Invest}, \\ 0, & \text{if she chooses } \textit{Don't Invest} \end{cases}$$

- **Difficulty**: There are 2^n action profiles (e.g., $2^{15} = 32,768$)
- Cannot check every individual profile for a Nash equilibrium

How to Find a Nash Equilibrium

Method: We can find every **Nash equilibrium** of a strategic game by **checking every action profile** to see if there is a player that can change her action to make herself strictly better off, while holding the other players' actions constant

Tip: Group similar profiles together if there are a lot of profiles

Example of Grouping

- **Preferences**: larger payoffs are better; an investor's payoff is

$$\begin{cases} 100, & \text{if she chooses } \textit{Invest} \text{ \& at least } k \text{ investors } \textit{Invest}, \\ -10, & \text{if she chooses } \textit{Invest} \text{ \& less than } k \text{ investors } \textit{Invest}, \\ 0, & \text{if she chooses } \textit{Don't Invest} \end{cases}$$

- *n* **investors Invest**: Nash equilibrium because deviating to *Don't Invest* yields a payoff of 0 instead of 100
- *n* **investors Don't Invest**: Nash equilibrium because deviating to *Invest* yields a payoff of -10 instead of 0
- 1 **to** $k-1$ **investors Invest**: no Nash equilibrium because deviating from *Invest* to *Don't Invest* yields 0 instead of -10
- *k* **to** $n-1$ **investors Invest**: no Nash equilibrium because deviating from *Don't Invest* to *Invest* yields 100 instead of 0

Traveler's Dilemma

- An airline has lost the suitcases of two travelers
- Each traveler specifies a value for her suitcase in $\{2, \ldots, 100\}$
- If they specify the same value then both get paid that value
- If they specify different values then
  - the traveler with the smaller value is paid that value plus $2
  - the other traveler is paid the smaller value minus $2

- The **Traveler's Dilemma** is described in words but we can **formulate** this situation as a strategic game

Example of How to Formulate a Strategic Game

- **Players**: two travelers
- **Actions**: a number v_i in $\{2, \ldots, 100\}$, for each traveler i
- **Preferences**: traveler $i \neq j$'s payoff is

$$\begin{cases} v_i + 2, & \text{if } v_i < v_j, \\ v_i, & \text{if } v_i = v_j, \\ v_j - 2, & \text{if } v_i > v_j \end{cases}$$

Nash Equilibrium of the Traveler's Dilemma

- **Preferences**: traveler $i \neq j$'s payoff is

$$\begin{cases} v_i + 2, & \text{if } v_i < v_j, \\ v_i, & \text{if } v_i = v_j, \\ v_j - 2, & \text{if } v_i > v_j \end{cases}$$

- $v_i < v_j$: Not a Nash equilibrium because traveler j can deviate from v_j to v_i to obtain a payoff of v_i instead of $v_i - 1$
- $v_i = v_j$ and $v_i \geq 3$: Not a Nash equilibrium because traveler i can deviate to $v_i - 1$ to obtain a payoff of $v_i + 1$ instead of v_i
- $v_i = v_j = 2$: Nash equilibrium—if a traveler increases her value then she will obtain a payoff of 0 instead of 2