ECO316: Applied Game Theory
Lecture 11

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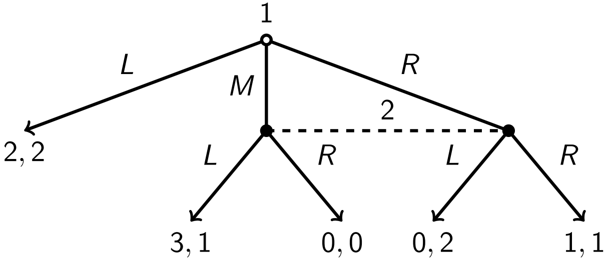
Extensive Games with Imperfect Information

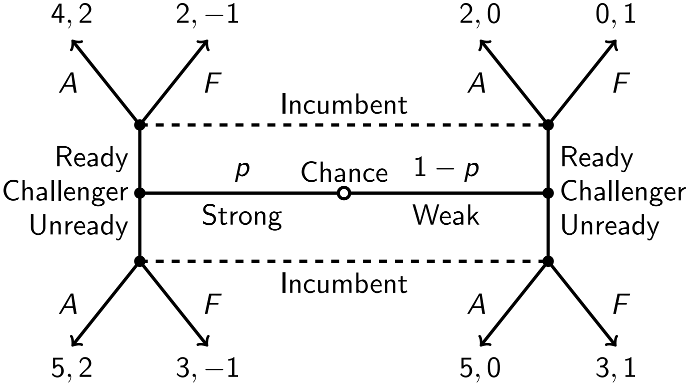
- Extensive games with perfect information: players are perfectly informed about past actions
- Now consider games in which players are not perfectly informed about past actions
- Example: Firm may know whether another firm has entered industry, but may not know how much the entrant has decided to spend on R & D

Extensive Games with Imperfect Information: Example



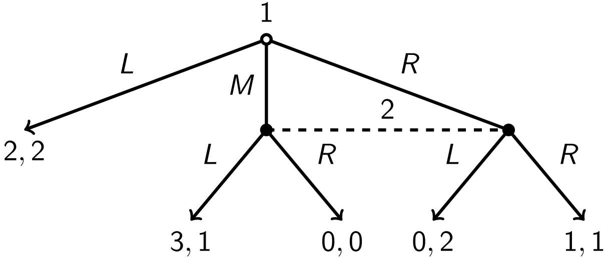
- Two players: player 1 and player 2
- Player 1 moves ⇒ player 2 moves if player 1 chose M or R
- Player 2 cannot distinguish between actions M and R

Extensive Games with Imperfect Information: Example



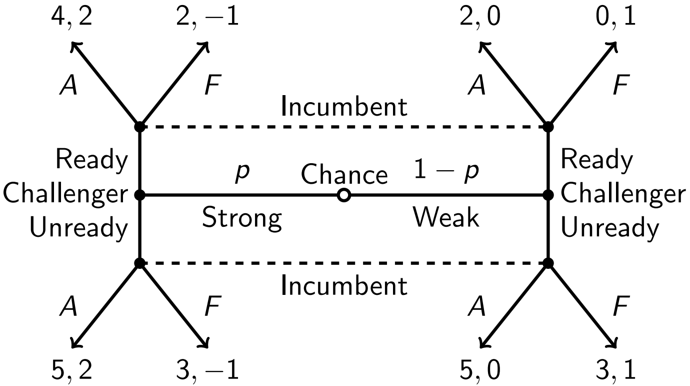
- Two players: Challenger and Incumbent + Chance
- Strong or Weak ⇒ Challenger moves ⇒ Incumbent moves
- Incumbent cannot distinguish between Strong and Weak

Information Sets



- **Information set**: histories that cannot be distinguished
- **Examples**: $\{\emptyset\}$, $\{M, R\}$

Information Sets



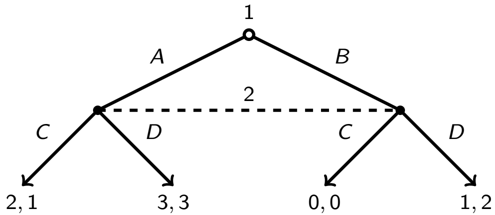
- **Information set**: histories that cannot be distinguished
- **Example**: {(Strong, Ready), (Weak, Ready)}

Extensive Games with Imperfect Information

An **extensive game with imperfect information** consists of

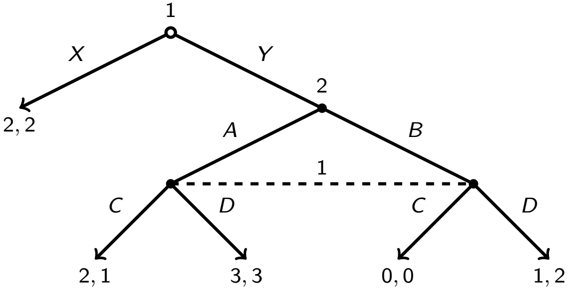
- a set of **players**
- a set of **terminal histories**
- a **player function** (can assign player or chance)
- a set of **probabilities** used by chance
- an **information set**, for each non-terminal history
- **preferences** over terminal histories, for each player

Examples



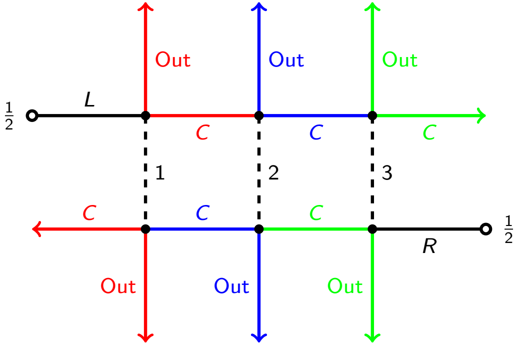
Models same situation as (simultaneous) strategic game

Examples



Player 1 moves then both players move simultaneously

Examples

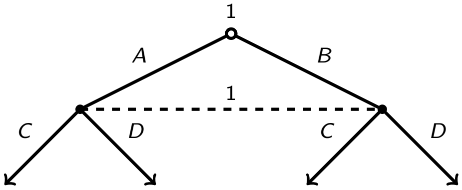


- Player 1 does not know whether she is moving first or last
- Player 2 does not know if she is moving after player 1 or 3
- Player 3 does not know whether she is moving first or last

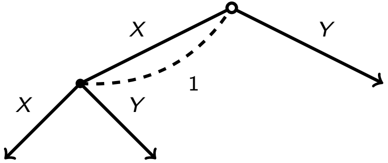
Perfect and Imperfect Recall

- Game has **perfect recall** if at every point in the game, every player "remembers" whatever she knew in the past
- Game has **imperfect recall** if it does not have perfect recall
- Will restrict our attention to games with perfect recall

Examples of Imperfect Recall



Player does not know action she made at start of game



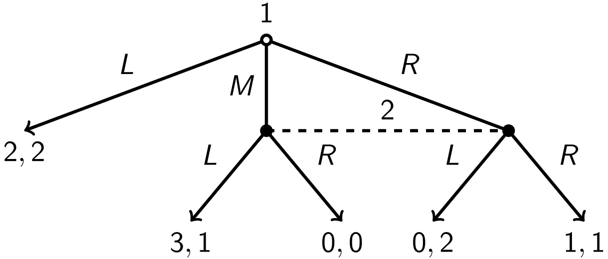
Player does not know whether she is at start of game or not

Pure Strategies and Strategic Form

A **pure strategy** for a player in an extensive game with imperfect information is a function that assigns an action to every information set at which the player moves

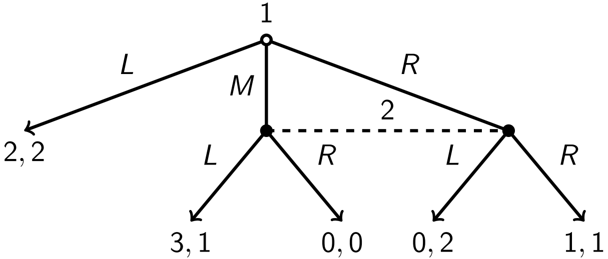
Given set of pure strategies for each player, can define **strategic form** of extensive game with imperfect information, as before

Pure Strategies: Example



- **1's pure strategies**: Choose L, M or R at info. set $\{\emptyset\}$
- **2's pure strategies**: Choose L or R at info. set $\{M, R\}$

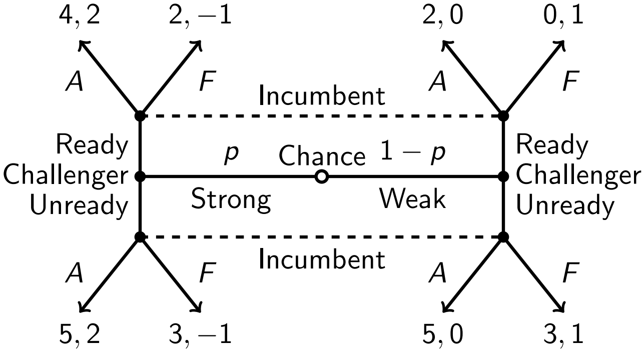
Strategic Form: Example



Player 2

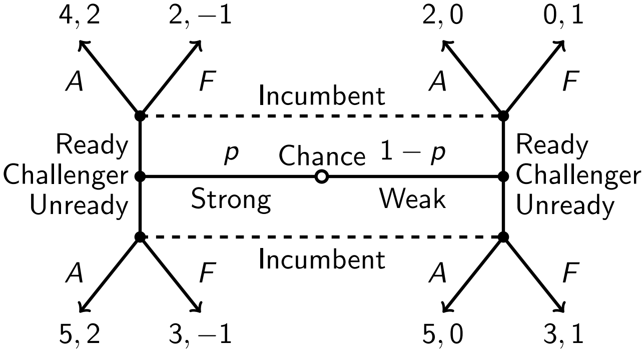
		L	R
Player 1	L	2, 2	2, 2
	M	3, 1	0, 0
	R	0, 2	1, 1

Pure Strategies: Example



- Example of a strategy for **Challenger**:
  - Ready at info. set {Strong}
  - Unready at info. set {Weak}
- Example of a strategy for **Incumbent**:
  - A at info. set $\{(\text{Strong}, \text{Ready}), (\text{Weak}, \text{Ready})\}$
  - F at info. set $\{(\text{Strong}, \text{Unready}), (\text{Weak}, \text{Unready})\}$

Strategic Form: Example



	AA	AF	$FA \cdots FF$
RR	$4p + 2(1-p), 2p$	$4p, 2p + (1-p)$	$\cdots$
RU	$4p + 5(1-p), 2p$	$4p + 3(1-p), 2p + (1-p)$	$\cdots$
UR	$5p + 2(1-p), 2p$	$5p, 2p + (1-p)$	$\cdots$
UU	$5, 2p$	$5p + 3(1-p), 2p + (1-p)$	$\cdots$

Nash Equilibrium

A **Nash equilibrium** of an extensive game with imperfect information is a Nash equilibrium of the strategic form of the game

To find all Nash equilibria:

- Construct the strategic form of the game
- Find all Nash equilibria of the resulting strategic form game

Nash Equilibrium: Alternative Definition

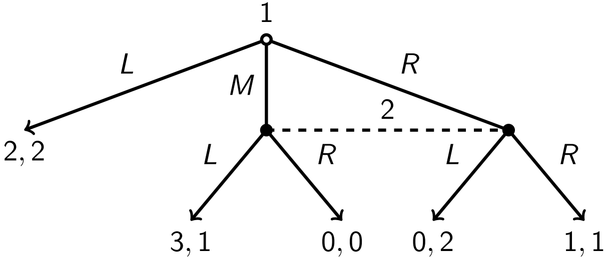
A **Nash equilibrium** of an extensive game with imperfect information is a strategy profile s^* such that, for every player i,

$$O(s_i^*, s_{-i}^*) \succeq_i O(s_i, s_{-i}^*), \text{ for every strategy } s_i \text{ of player } i,$$

in which $O(s)$ denotes outcome (terminal history) generated by s

That is, no player can be made better off by changing her strategy given the other players' strategies

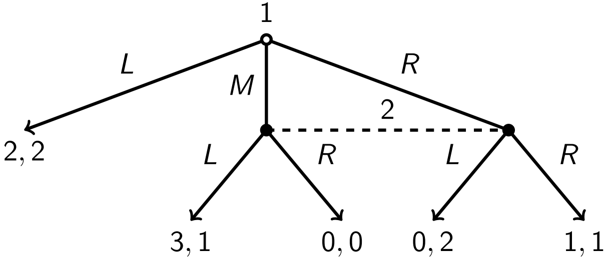
Nash Equilibrium: Example



(M, L) **is a Nash equilibrium**:

- Player 1's payoff: 3
- Player 1's payoff from deviating to L: 2
- Player 1's payoff from deviating to R: 0
- Player 2's payoff: 1
- Player 2's payoff from deviating to R: 0

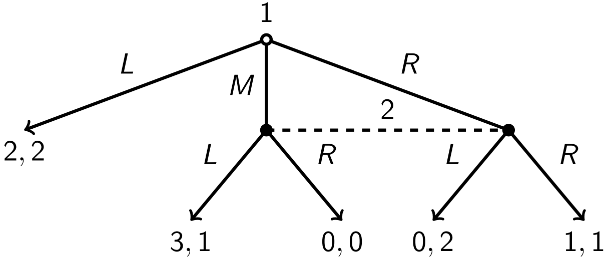
Nash Equilibrium: Example



(L, R) **is a Nash equilibrium**:

- Player 1's payoff: 2
- Player 1's payoff from deviating to M: 0
- Player 1's payoff from deviating to R: 1
- Player 2's payoff: 2
- Player 2's payoff from deviating to L: 2

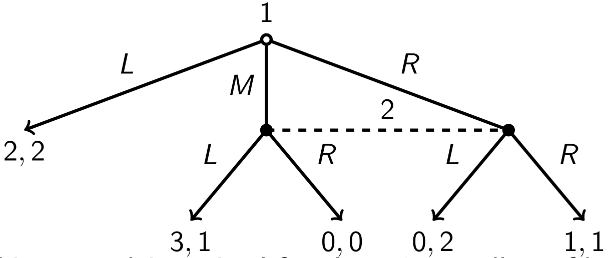
Nash Equilibrium: Example



(L, R) **is a Nash equilibrium**:

- But L is optimal for player 2 if player 1 were to choose M or R
- No proper subgame $\Rightarrow$ technically, this profile is SPE
- Clearly, we need new refinement of Nash equilibrium

Nash Equilibrium: Example



- In this game, L is optimal for player 2 regardless of her belief about whether player 1 chose action M or L
- But, for other payoffs, optimal action depends on belief
- If player 1 chooses M and/or R with positive probability, player 2's belief can be derived from player 1's strategy
- But, if player 1 chooses L with certainty, player 2's belief cannot be derived from player 1's strategy
- Need to specify player 2's belief as part of equilibrium

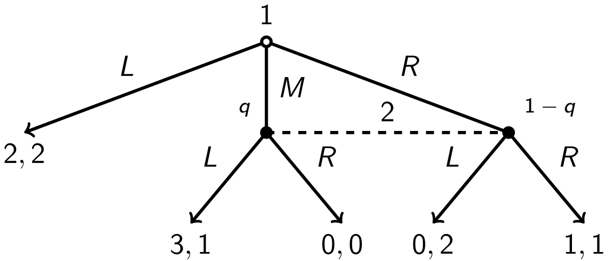
Beliefs and Assessments

A **belief** in an extensive game with imperfect information is a probability distribution over the histories in an information set

A **belief system** in an extensive game with imperfect information is a function that assigns a belief to every information set

An **assessment** in an extensive game with imperfect information is a pair (s, μ) in which s is a strategy profile and μ is a belief system

Beliefs and Assessments: Example



Example of belief system: at information set $\{M, R\}$,

- Player 2 believes player 1 chose M with probability q
- Player 2 believes player 1 chose R with probability $1 - q$

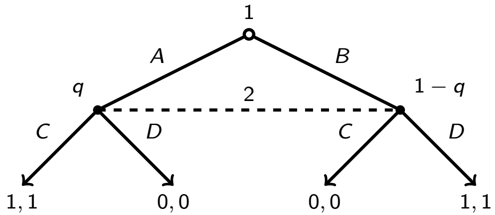
Example of assessment: $((M, R), (q, 1 - q))$

Sequential Rationality and Weak Consistency

An assessment is **sequentially rational** if, for every information set of every player, the player's strategy is a best response to the other players' strategies given her belief at the information set

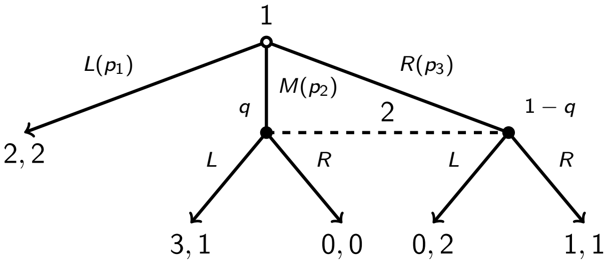
An assessment is **weakly consistent** if, for every information set reached with positive probability given the strategy profile, the probability assigned to each history by the belief system is derived using Bayes' rule

Sequential Rationality: Example



- Expected payoff from C: $q \cdot 1 + (1-q) \cdot 0 = q$
- Expected payoff from D: $q \cdot 0 + (1-q) \cdot 1 = 1 - q$
- Sequential rationality requires for player 2 to
  - choose C only if $q \geq 1 - q$, or equivalently, $q \geq 1/2$
  - choose D only if $q \leq 1 - q$, or equivalently, $q \leq 1/2$

Weak Consistency: Example

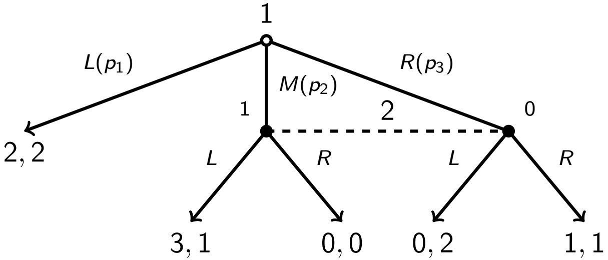


- Assessment in which player 1 chooses L and player 2 holds any belief at her information set is weakly consistent: player 2's information set is not reached given player 1's strategy
- If player 2's information set is reached with positive probability then weak consistency requires $q = \frac{p_2}{p_2+p_3}$ and $1 - q = \frac{p_3}{p_2+p_3}$

Weak Sequential Equilibrium

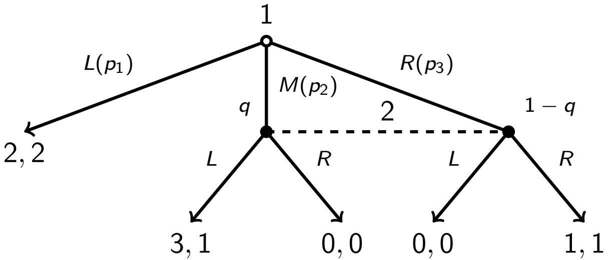
An assessment is a **weak sequential equilibrium** of an extensive game with imperfect information if it is sequentially rational and weakly consistent

Weak Sequential Equilibrium: Example



- L is optimal for player 2 at her information set, for any belief
- Sequential rationality $\Rightarrow$ player 2 must choose L in any WSE
- Player 2 chooses $L \Rightarrow$ player 1 must choose M in any WSE
- Weak consistency $\Rightarrow$ player 2 believes $q = 1$ and $1 - q = 0$
- **Unique WSE**: $((M, L), (1, 0))$

Weak Sequential Equilibrium: Another Example



- Player 2: $q \geq 1/2 \Rightarrow L$ optimal; $q \leq 1/2 \Rightarrow R$ optimal
- 2 chooses $L \Rightarrow$ 1 chooses $M \Rightarrow q = 1 \Rightarrow L$ optimal $\Rightarrow$ WSE
- 2 chooses $R \Rightarrow$ 1 chooses $L \Rightarrow$ any q weakly consistent; need $q \leq 1/2$ for R optimal $\Rightarrow \{(L, R), (q, 1-q) : q \leq 1/2\}$ is WSE

Signalling Games

- Some players informed of relevant variables, others are not
- Extensive game in which informed players move first
- Actions of informed players may "signal" their information
- **Example**: (Spence signalling model)
  - Employees know own ability
  - Employers observe college degrees but not ability
  - Higher ability $\Rightarrow$ lower cost of obtaining college degree
  - College degree may signal high ability to employer (not because college teaches skills but because only high ability employees find obtaining a college degree worthwhile, given the cost)

Two-Player Signalling Games

General two-player signalling game:

- Two players: Sender and Receiver
- Sender observes variable chosen by Chance, receiver does not
- Sender moves ⇒ Receiver observes choice ⇒ Receiver moves

Considerations:

- Sender may want to convey her information (e.g. high ability employee might want to convey that she is high ability)
- Sender may not want to convey her information (e.g. low ability employee might not want to be identified as low ability)

Entry Game

- Challenger contests an Incumbent's turf
- Challenger is *Strong* (well-prepared to fight) with prob. p
- Challenger is *Weak* (ill-prepared to fight) with prob. $1 - p$
- Challenger knows whether it is prepared, Incumbent does not
- Challenger chooses to *Ready* itself to fight or remain *Unready*
- Cost of choosing *Ready* is higher when *Weak*
- Incumbent observes Challenger's readiness
- Incumbent chooses to *Acquiesce* or *Fight*

Entry Game: Incumbent's Payoff

Incumbent's payoff depends on Challenger's strength:

$$\begin{cases} 1, & \text{if } \textit{Acquiesce}, \\ 2, & \text{if } \textit{Fight} \text{ and } \textit{Weak}, \\ 0, & \text{if } \textit{Fight} \text{ and } \textit{Strong} \end{cases}$$

Entry Game: Challenger's Payoff

Challenger's payoff when Strong:

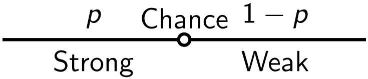
$$\begin{cases} 5, & \text{if } \textit{Acquiesce} \text{ and } \textit{Unready}, \\ 4, & \text{if } \textit{Acquiesce} \text{ and } \textit{Ready}, \\ 3, & \text{if } \textit{Fight} \text{ and } \textit{Unready}, \\ 2, & \text{if } \textit{Fight} \text{ and } \textit{Ready} \end{cases}$$

Challenger's payoff when Weak:

$$\begin{cases} 5, & \text{if } \textit{Acquiesce} \text{ and } \textit{Unready}, \\ 2, & \text{if } \textit{Acquiesce} \text{ and } \textit{Ready}, \\ 3, & \text{if } \textit{Fight} \text{ and } \textit{Unready}, \\ 0, & \text{if } \textit{Fight} \text{ and } \textit{Ready} \end{cases}$$

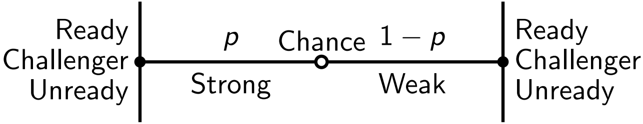
Entry Game: Tree Diagram

Challenger Strong with prob. p **and Weak with prob.** $1-p$:



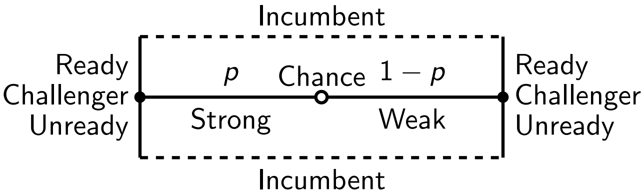
Entry Game: Tree Diagram

Challenger observes strength, chooses Ready or Unready:



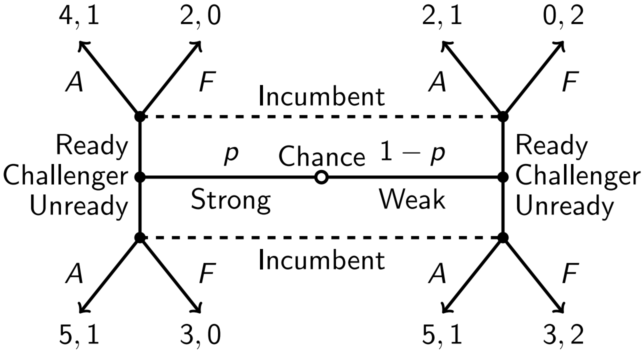
Entry Game: Tree Diagram

Incumbent observes readiness but not strength:

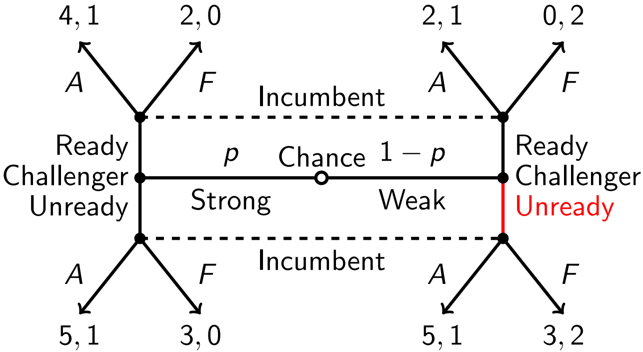


Entry Game: Tree Diagram

Incumbent chooses Acquiesce or Fight:

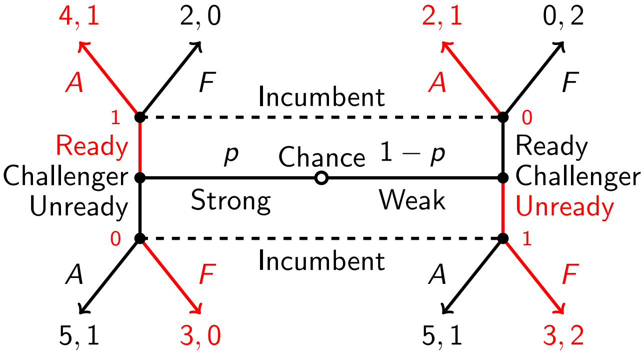


Entry Game: Weak Sequential Equilibrium



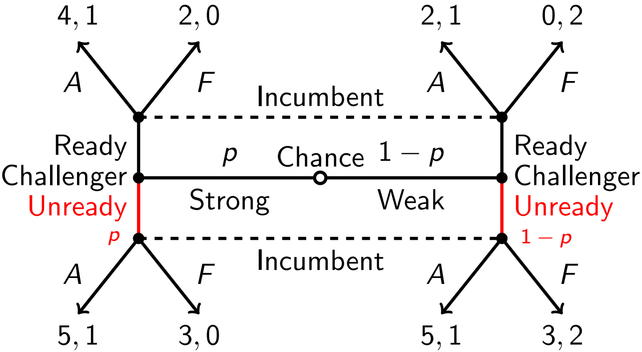
- *Weak* Challenger always prefers *Unready* to *Ready*
- *Weak* Challenger must choose *Unready* in any WSE

Entry Game: Weak Sequential Equilibrium



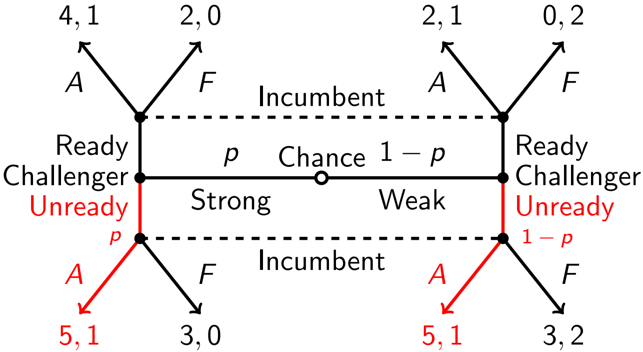
- Suppose *Strong* Challenger chooses *Ready*
- Incumbent believes *Ready* ⇒ *Strong* and *Unready* ⇒ *Weak*
- Incumbent chooses *A* after *Ready* and *F* after *Unready*
- *Ready* optimal for *Strong* Challenger given Incumbent strategy
- "Separating" WSE

Entry Game: Weak Sequential Equilibrium



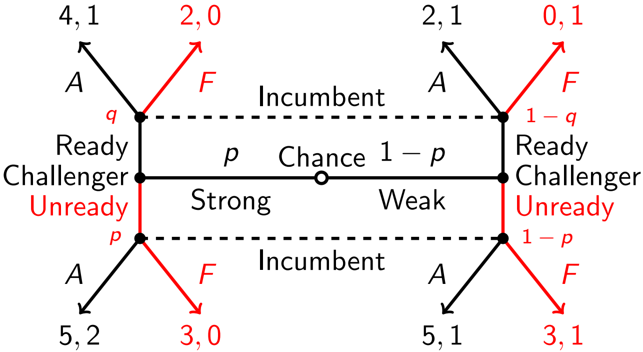
- Suppose *Strong* Challenger chooses *Unready*
- Incumbent believes *Unready* $\Rightarrow$ *Strong* with prob. p
- Incumbent's expected payoffs from A is 1
- Incumbent's expected payoffs from F is $2(1-p)$
- A optimal if $p \geq 1/2$; F optimal if $p \leq 1/2$

Entry Game: Weak Sequential Equilibrium



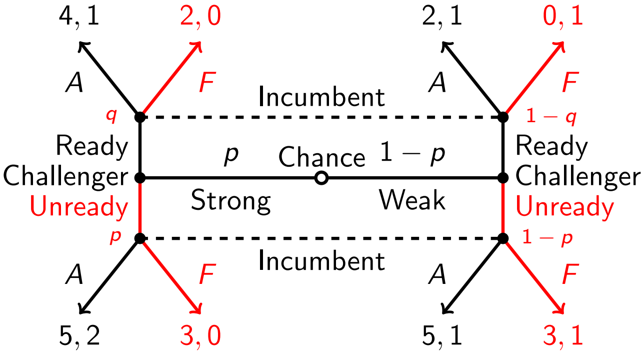
- Suppose $p \geq 1/2$ and that Incumbent chooses A
- *Unready* optimal, regardless of Incumbent strategy after *Ready*
- "Pooling" WSE when $p \geq 1/2$

Entry Game: Weak Sequential Equilibrium



- Suppose $p \leq 1/2$ and that Incumbent chooses F
- *Unready* optimal if Incumbent chooses F after *Ready*
- F optimal if $q \leq 1/2$
- Another "pooling" WSE when $p \geq 1/2$

Entry Game: Weak Sequential Equilibrium



Homework: Formally write each WSE