ECO316: Applied Game Theory
Lecture 2

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Based on materials by Martin J. Osborne

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Firm Competition

- Topic at the heart of classical economic theory
- **Standard model**:
  - Each firm takes price as given
  - Firms do not take other firms into account
  - Outcome is independent of the number of firms
- This lecture introduces **two alternative models**

Preview of the Two Alternative Models

- **Bertrand model**:
  - Each firm chooses a price
  - Customers buy from the firm with the lowest price
- **Cournot model**:
  - Each firm chooses a level of output
  - Price is determined by the output of all firms

Review: Strategic Games

Game theory studies situations in which rational decision-makers interact—a **strategic game** is a simple model consisting of

- a set of **players**
- a set of **actions**, for each player
- **preferences** over the set of action profiles, for each player

Action profile: a list of actions with one action for each player

Bertrand Model

A **Bertrand model** is a **strategic game** consisting of

- a set of **firms**
- a set of possible **prices**, for each firm
- **preferences** in the form of a **profit function**, for each firm

in which

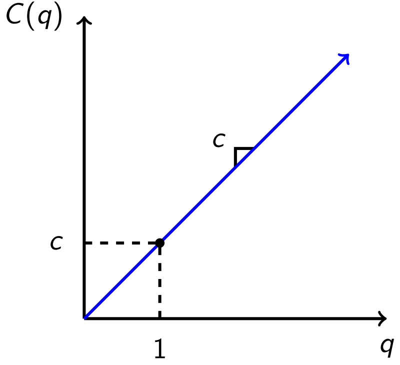
- each firm chooses a **price**
- **consumers** buy from the firm with the **lowest price**
- each firm produces enough output to satisfy **demand**

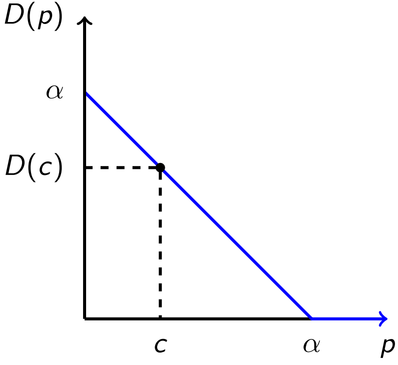
Example of a Bertrand Model

- **Players**: two firms
- **Actions**: any non-negative price p_i, for each firm i
- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Demand**: $D(p) = \alpha - p$ for $p \leq \alpha$ and $\alpha > c$

Example of a Bertrand Model: Cost and Demand

- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Demand**: $D(p) = \alpha - p$ for $p \leq \alpha$ and $\alpha > c$





Example of a Bertrand Model: Profit

- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Demand**: $D(p) = \alpha - p$ for $p \leq \alpha$ and $\alpha > c$

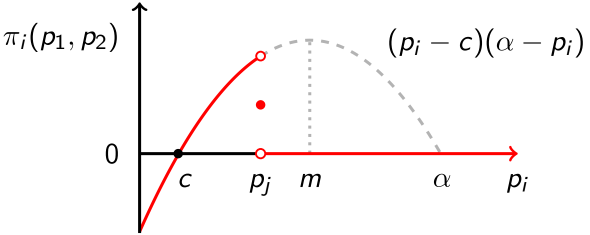
Profit: if $p_i \leq \alpha$ then firm $i \neq j$'s profit is

- $p_i < p_j$: $\quad \pi_i(p_1, p_2) = \underbrace{p_i D(p_i)}_{\text{revenue}} - \underbrace{cD(p_i)}_{\text{cost}} = (p_i - c)(\alpha - p_i)$
- $p_i = p_j$: $\quad \pi_i(p_1, p_2) = \underbrace{\frac{p_i D(p_i)}{2}}_{\text{revenue}} - \underbrace{\frac{cD(p_i)}{2}}_{\text{cost}} = \frac{1}{2}(p_i - c)(\alpha - p_i)$
- $p_i > p_j$: $\quad \pi_i(p_1, p_2) = 0$

Example of a Bertrand Model: Profit

- **Profit**: if $p_i \le \alpha$ then firm $i \neq j$'s profit is

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(\alpha - p_i), & \text{if } p_i < p_j, \\ \frac{1}{2}(p_i - c)(\alpha - p_i), & \text{if } p_i = p_j, \\ 0, & \text{if } p_i > p_j \end{cases}$$



Review: Nash Equilibrium

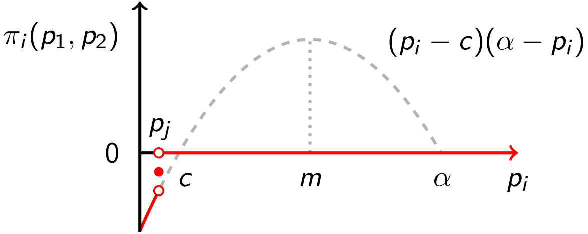
An action profile is a **Nash equilibrium** of a strategic game if

every player's action is optimal, given the other players' actions,

or, equivalently, if

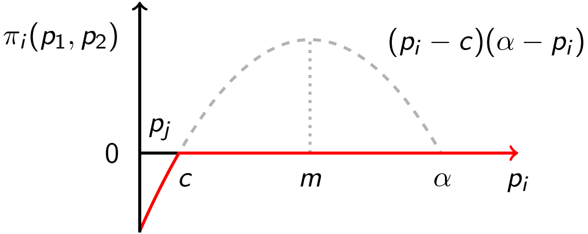
no player can change her action to make herself strictly better off, given the other players' actions

Example of a Bertrand Model: Optimal Actions



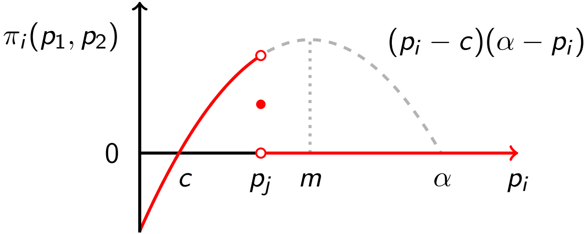
- If $p_j < c$ then any price $p_i > p_j$ is **optimal** for firm i

Example of a Bertrand Model: Optimal Actions



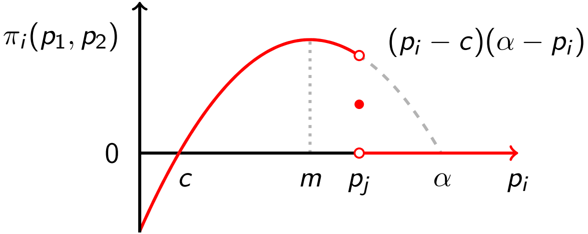
- If $p_j < c$ then any price $p_i > p_j$ is **optimal** for firm i
- If $p_j = c$ then any price $p_i \geq c$ is **optimal** for firm i

Example of a Bertrand Model: Optimal Actions



- If $p_j < c$ then any price $p_i > p_j$ is **optimal** for firm i
- If $p_j = c$ then any price $p_i \geq c$ is **optimal** for firm i
- If $c < p_j \leq m$ then there is **no optimal price** for firm i
  - if $p_i \geq p_j$ then any price slightly less than p_j is strictly better
  - if $p_i < p_j$ then any price slightly larger than p_i is strictly better

Example of a Bertrand Model: Optimal Actions



- If $p_j < c$ then any price $p_i > p_j$ is **optimal** for firm i
- If $p_j = c$ then any price $p_i \geq c$ is **optimal** for firm i
- If $c < p_j \leq m$ then there is **no optimal price** for firm i
  - if $p_i \geq p_j$ then any price slightly less than p_j is strictly better
  - if $p_i < p_j$ then any price slightly larger than p_i is strictly better
- If $p_j > m$ then a price of m is **optimal** for firm i

Review: How to Find a Nash Equilibrium

Method: We can find every **Nash equilibrium** of a strategic game by **checking every action profile** to see if there is a player that can change her action to make herself strictly better off, while holding the other players' actions constant

Tip: Group similar profiles together if there are a lot of profiles

Example of a Bertrand Model: Nash Equilibrium

Proof that (c, c) is a Nash equilibrium:

We must show that no firm i can obtain a profit strictly larger than $\pi_i(c, c) = 0$ when the other firm chooses a price of c such that

$$\pi_1(p_1, c) \leq 0, \text{ for all } p_1,$$
$$\pi_2(c, p_2) \leq 0, \text{ for all } p_2$$

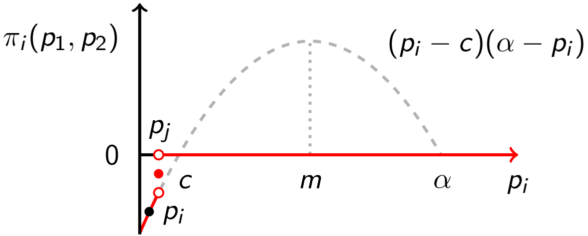
If firm 1 chooses a price of p_1 and firm 2 chooses a price of c then

$$p_1 < c \text{ implies } \pi_1(p_1, c) < 0,$$
$$p_1 \geq c \text{ implies } \pi_1(p_1, c) = 0$$

We can make a similar argument for firm 2

Example of a Bertrand Model: Unique Nash Equilibrium

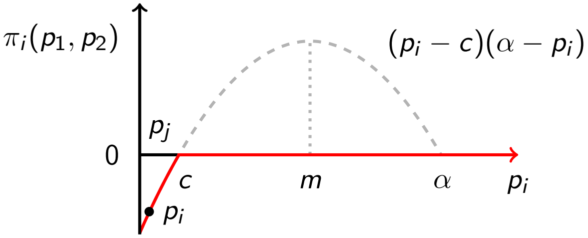
Proof that (c, c) is the **only** Nash equilibrium:



- $p_i < c$ & $p_i \leq p_j$: Not a Nash equilibrium because firm i can deviate to c to obtain a payoff of 0 instead of $\pi_i(p_1, p_2) < 0$

Example of a Bertrand Model: Unique Nash Equilibrium

Proof that (c, c) is the **only** Nash equilibrium:



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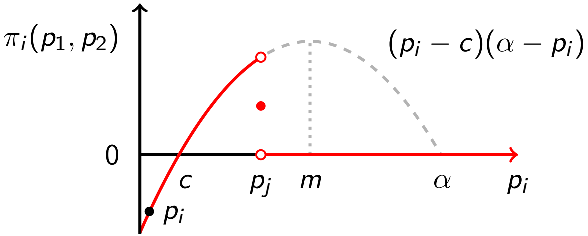

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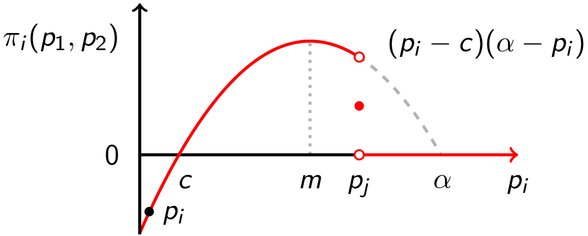

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Example of a Bertrand Model: Unique Nash Equilibrium

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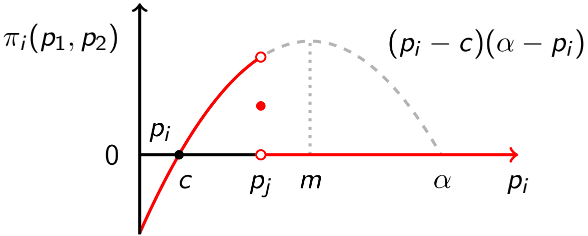
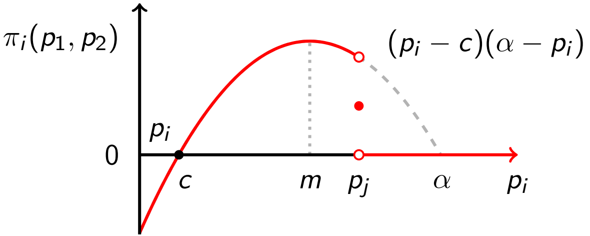


- $p_i = c$ & $p_j > c$: Not a Nash equilibrium because firm i can deviate to $p_i < p < p_j$ to obtain positive payoff instead of 0

Example of a Bertrand Model: Unique Nash Equilibrium

Proof that (c, c) is the **only** Nash equilibrium:



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Example of a Bertrand Model: Unique Nash Equilibrium

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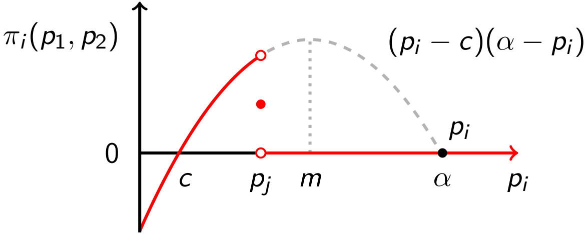
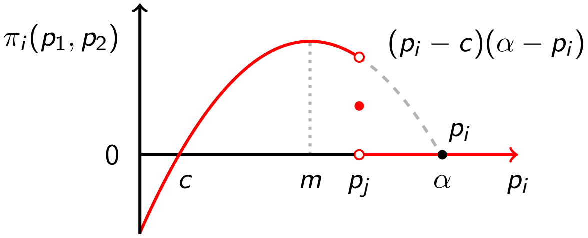


- $p_i \geq p_j > c$: Not a Nash equilibrium because firm i can deviate to p slightly below p_j if $p_j \leq m$ and to m if $p_j > m$

Example of a Bertrand Model: Unique Nash Equilibrium

Proof that (c, c) is the **only** Nash equilibrium:



- $p_i \geq p_j > c$: Not a Nash equilibrium because firm i can deviate to p slightly below p_j if $p_j \leq m$ and to m if $p_j > m$

Example of a Bertrand Model: Summary

There is a **unique** Nash equilibrium at (c, c):
It only takes **two firms** to get the **competitive outcome**!

Questions:

- What happens with more than two firms?
- What about other cost functions?
- What about other demand functions?
- What if prices must be integers (e.g. 1 cent, 2 cents, etc.)?
- Is there a way for firms to collude?
- What if firms interact repeatedly?

Cournot Model

A **Cournot model** is a **strategic game** consisting of

- a set of **firms**
- a set of possible **quantities**, for each firm
- **preferences** in the form of a **profit function**, for each firm

in which

- each firm chooses a **quantity**
- **price** is determined by an **inverse demand function**

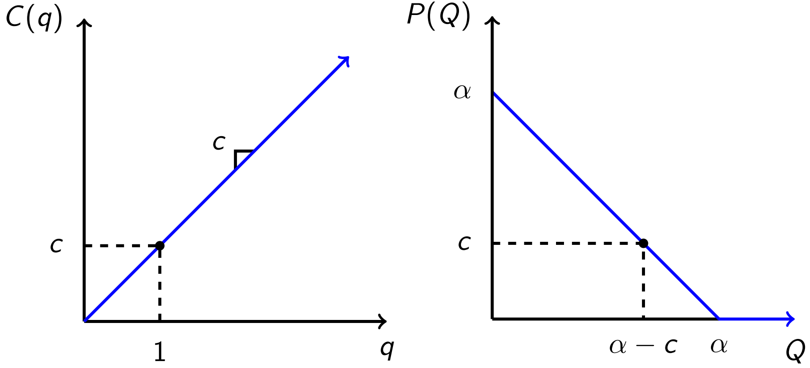
Example of a Cournot Model

- **Players**: two firms
- **Actions**: any non-negative quantity q_i, for each firm i
- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Inverse demand**: $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $\alpha > c$

- **Note**: Q denotes **total output** such that $Q = q_1 + q_2$

Example of a Cournot Model: Cost and Demand

- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Inverse demand**: $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $\alpha > c$



Example of a Cournot Model: Profit

- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Inverse demand**: $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $\alpha > c$

Profit: if $Q \leq \alpha$ then firm $i \neq j$'s profit is

$$\pi_i(q_1, q_2) = \underbrace{P(Q)q_i}_{\text{revenue}} - \underbrace{cq_i}_{\text{cost}} = (\alpha - q_1 - q_2)q_i - cq_i$$

Example of a Cournot Model: Profit

- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Inverse demand**: $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $\alpha > c$

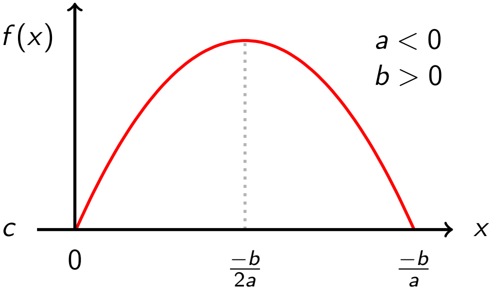
Profit: if $Q \leq \alpha$ then firm $i \neq j$'s profit is

$$\pi_i(q_1, q_2) = \underbrace{P(Q)q_i}_{\text{revenue}} - \underbrace{cq_i}_{\text{cost}} = -q_i^2 + (\alpha - c - q_j)q_i$$

This function is **quadratic** in q_i

Review: Quadratic Functions

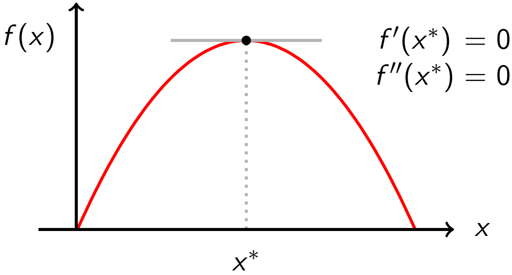
Quadratic function: $f(x) = ax^2 + bx + c$



- $a > 0$: graph has a **U-shape** with one **minimum**
- $a < 0$: graph has an **inverted U-shape** with one **maximum**
- Minimum/maximum at $\frac{-b}{2a}$

Review: Twice-Differentiable Functions

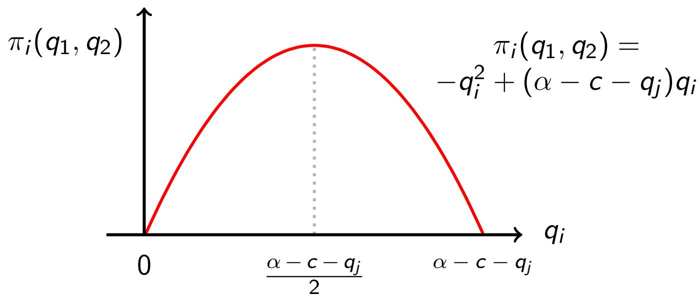
Twice-differentiable function: Derivatives $f'(x)$ and $f''(x)$ exist



- **Critical points**: values x^* such that $f'(x^*) = 0$
- $f'(x^*) = 0$ and $f''(x^*) > 0$: local **minimum** at x^*
- $f'(x^*) = 0$ and $f''(x^*) < 0$: local **maximum** at x^*

Example of a Cournot Model: Optimal Actions

Profit function:



Given quantity q_j, firm i's **optimal** quantity is $\frac{\alpha - c - q_j}{2}$

That is, firm i's **best response function** is $b_i(q_j) = \frac{\alpha - c - q_j}{2}$

How to Find a Nash Equilibrium

A **best response function** for player i in a strategic game is a function $b_i(a_{-i})$ that inputs the other players' actions a_{-i} and outputs the set of optimal actions for player i

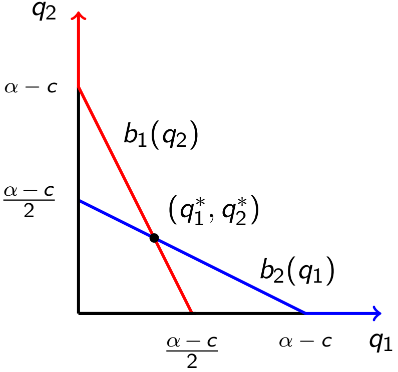
Method: We can find every **Nash equilibrium** of a strategic game by calculating the **best response function** $b_i(a_{-i})$ for each player i and identifying all of the action profiles a such that

$$a_i = b_i(a_{-i}), \text{ for all players } i$$

Example of a Cournot Model: Best Response Functions

Best response functions:

$$b_1(q_2) = \frac{\alpha - c - q_2}{2} \quad \text{and} \quad b_2(q_1) = \frac{\alpha - c - q_1}{2}$$



Example of a Cournot Model: Nash Equilibrium

Best response functions:

$$b_1(q_2) = \frac{\alpha - c - q_2}{2} \quad \text{and} \quad b_2(q_1) = \frac{\alpha - c - q_1}{2}$$

If (q_1^*, q_2^*) is a Nash equilibrium then it must satisfy

$$q_1^* = \frac{\alpha - c - q_2^*}{2} \quad \text{and} \quad q_2^* = \frac{\alpha - c - q_1^*}{2}$$

We can solve this system of equations using substitutiion:

$$q_i^* = \frac{\alpha - c - \left(\frac{\alpha - c - q_i^*}{2}\right)}{2} \quad \text{implies} \quad q_i^* = \frac{\alpha - c}{3}$$

Example of a Cournot Model: Properties

There is a **unique** Nash equilibrium at $\left(\frac{\alpha-c}{3}, \frac{\alpha-c}{3}\right)$

Positive profit: Since the total output in equilibrium is

$$Q^* = \frac{\alpha - c}{3} + \frac{\alpha - c}{3} = \frac{2(\alpha - c)}{3},$$

the price in equilibrium is

$$P(Q^*) = \alpha - Q^* = \alpha - \left(\frac{2(\alpha - c)}{3}\right) = \frac{\alpha + 2c}{3}$$

which is greater than c because $\alpha > c$

Example of a Cournot Model: Properties

There is a **unique** Nash equilibrium at $\left(\frac{\alpha-c}{3}, \frac{\alpha-c}{3}\right)$

Prices: The quantity q^m that a monopolist would choose solves

$$max_q \underbrace{(\alpha - q)q}_{\text{revenue}} - \underbrace{cq}_{\text{cost}} = -q^2 + (\alpha - c)q$$

This is a quadratic function: we can find q^m by using the properties of a quadratic function or by taking its **derivative**, i.e.,

$$\frac{d}{dq}\Big[(\alpha - q)q - cq\Big] = 0 \quad \text{implies} \quad q^m = \frac{\alpha - c}{2}$$

Since $q^m < Q^*$, we obtain: $c < P(Q^*) < P(q^m)$

Example of a Cournot Model: Summary

There is a **unique** Nash equilibrium at $\left(\frac{\alpha-c}{3}, \frac{\alpha-c}{3}\right)$

Questions:

- What happens with more than two firms?
- What about other cost functions?
- What about other demand functions?
- What if quantities must be integers (e.g. 1 unit, 2 units, etc.)?
- Is there a way for firms to collude?
- What if firms interact repeatedly?

Comparison of Models

- **Bertrand model**:
  - Strategic variable is price
  - Competitive outcome with only two firms
  - No firm makes positive profit in the Nash equilibrium
- **Cournot model**:
  - Strategic variable is quantity
  - Price is higher than in the competitive outcome
  - Price is lower than in the monopolistic outcome
  - Firms make positive profit in the Nash equilibrium
  - **Homework:** Show that the price in equilibrium goes to c and that the profit goes to 0 as number of firms increases