ECO316: Applied Game Theory
Lecture 4

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Strategic Game with no Nash Equilibrium

		Player 2	
		L	R
Player 1	T	$1, 0$	$0, 4$
	B	$0, 1$	$2, 0$

No Nash equilibrium: What happens if this game is played?

- Player 1 might choose T sometimes and B sometimes
- Player 2 might choose L sometimes and R sometimes
- How often should each player choose each action?

Pure and Mixed Strategies

A **mixed strategy** for a player in a strategic game is a probability distribution over the set of possible actions for that player

A **probability distribution** over a (finite) set of possible actions $\{a_1, \ldots, a_n\}$ is a list of numbers $(p_1, \ldots, p_n)$ such that

- $p_1 + \cdots + p_n = 1$
- $0 \leq p_i \leq 1$, for all $i = 1, \ldots, n$

A **pure strategy** for a player in a strategic game is a mixed strategy that assigns a probability of 1 to a single action

Example of Mixed Strategies

		Player 2	
		L	R
Player 1	T	$1, 0$	$0, 4$
	B	$0, 1$	$2, 0$

Possible mixed strategies:

- **Player 1**: $(p, 1-p)$ in which p is the probability that she chooses T and $1-p$ is the probability that she chooses B
- **Player 2**: $(q, 1-q)$ in which q is the probability that she chooses L and $1-q$ is the probability that she chooses R

Example of Mixed Strategies

		Player 2	
		L	R
Player 1	T	$1, 0$	$0, 4$
	B	$0, 1$	$2, 0$

Possible pure strategies:

- **Player 1**:
  - $(1, 0)$: assign a probability of 1 to T
  - $(0, 1)$: assign a probability of 1 to B
- **Player 2**:
  - $(1, 0)$: assign a probability of 1 to L
  - $(0, 1)$: assign a probability of 1 to R

Mixed Equilibrium: Optimal Actions

Player 2

		$L(q)$	$R(1-q)$
Player 1	T	$1,0$	$0,4$
	B	$0,1$	$2,0$

Optimal action for player 1 if player 2 uses the strategy $(q, 1-q)$?

- **Expected payoff of** T: $q \cdot 1 + (1-q) \cdot 0 = q$
- **Expected payoff of** B: $q \cdot 0 + (1-q) \cdot 2 = 2(1-q)$

$$
\begin{aligned}
T \succ B \quad &\Leftrightarrow \quad q > 2(1-q) \quad \Leftrightarrow \quad q > \tfrac{2}{3} \\
T \sim B \quad &\Leftrightarrow \quad q = 2(1-q) \quad \Leftrightarrow \quad q = \tfrac{2}{3} \\
B \succ T \quad &\Leftrightarrow \quad q < 2(1-q) \quad \Leftrightarrow \quad q < \tfrac{2}{3}
\end{aligned}
$$

Mixed Equilibrium: Optimal Actions

		Player 2	
		L	R
Player 1	$T(p)$	$1, 0$	$0, 4$
	$B(1-p)$	$0, 1$	$2, 0$

Optimal action for player 2 if player 1 uses the strategy $(p, 1-p)$?

- **Expected payoff of** L: $p \cdot 0 + (1-p) \cdot 1 = 1 - p$
- **Expected payoff of** R: $p \cdot 4 + (1-p) \cdot 0 = 4p$

$$L \succ R \quad \Leftrightarrow \quad 1 - p > 4p \quad \Leftrightarrow \quad p < \frac{1}{5}$$
$$L \sim R \quad \Leftrightarrow \quad 1 - p = 4p \quad \Leftrightarrow \quad p = \frac{1}{5}$$
$$R \succ L \quad \Leftrightarrow \quad 1 - p < 4p \quad \Leftrightarrow \quad p > \frac{1}{5}$$

Mixed Equilibrium: Optimal Actions

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$1, 0$	$0, 4$
	$B(1-p)$	$0, 1$	$2, 0$

Player 1:

$$T \succ B \quad \Leftrightarrow \quad q > \frac{2}{3}$$

$$T \sim B \quad \Leftrightarrow \quad q = \frac{2}{3}$$

$$B \succ T \quad \Leftrightarrow \quad q < \frac{2}{3}$$

Player 2:

$$L \succ R \quad \Leftrightarrow \quad p < \frac{1}{5}$$

$$L \sim R \quad \Leftrightarrow \quad p = \frac{1}{5}$$

$$R \succ L \quad \Leftrightarrow \quad p > \frac{2}{3}$$

Mixed Equilibrium: Equilibrium

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$1, 0$	$0, 4$
	$B(1-p)$	$0, 1$	$2, 0$

Player 1:

$p = 1$ **optimal** $\Leftrightarrow$ $q > \frac{2}{3}$

any p **optimal** $\Leftrightarrow$ $q = \frac{2}{3}$

$p = 0$ **optimal** $\Leftrightarrow$ $q < \frac{2}{3}$

Player 2:

$q = 1$ **optimal** $\Leftrightarrow$ $p < \frac{1}{5}$

any q **optimal** $\Leftrightarrow$ $p = \frac{1}{5}$

$q = 0$ **optimal** $\Leftrightarrow$ $p > \frac{2}{3}$

Mixed Equilibrium: Equilibrium

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$1,0$	$0,4$
	$B(1-p)$	$0,1$	$2,0$

Player 1:

$p = 1$ **optimal** $\Leftrightarrow$ $q > \frac{2}{3}$

any p **optimal** $\Leftrightarrow$ $q = \frac{2}{3}$

$p = 0$ **optimal** $\Leftrightarrow$ $q < \frac{2}{3}$

Player 2:

$q = 1$ **optimal** $\Leftrightarrow$ $p < \frac{1}{5}$

any q **optimal** $\Leftrightarrow$ $p = \frac{1}{5}$

$q = 0$ **optimal** $\Leftrightarrow$ $p > \frac{2}{3}$

$\left\{\left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right\}$ is a **mixed strategy Nash equilibrium**

Mixed Strategy Nash Equilibrium

A profile of mixed strategies is a **mixed strategy Nash equilibrium** of a strategic game if

every player's strategy is optimal, given the other players' strategies,

or, equivalently, if

no player can change her strategy to increase her expected payoff, given the other players' strategies

How to Find a Mixed Strategy Nash Equilibrium

Method: We can find every **mixed strategy Nash equilibrium** of a strategic game by calculating the **best response function** $b_i(\alpha_{-i})$ for each player i and identifying all of the profiles of mixed strategies α such that

$$\alpha_i = b_i(\alpha_{-i}), \text{ for all players } i$$

Examples of Best Response Functions: Player 1

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$1, 0$	$0, 4$
	$B(1-p)$	$0, 1$	$2, 0$

We already found the best response functions for this game:

$p = 1$ **optimal** $\Leftrightarrow$ $q > \frac{2}{3}$

any p **optimal** $\Leftrightarrow$ $q = \frac{2}{3}$

$p = 0$ **optimal** $\Leftrightarrow$ $q < \frac{2}{3}$

Examples of Best Response Functions: Player 1

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$1, 0$	$0, 4$
	$B(1-p)$	$0, 1$	$2, 0$

We already found the best response functions for this game:

$$b_1(q) = \begin{cases} \{1\}, & \text{if } q > \frac{2}{3}, \\ \{p : 0 \leq p \leq 1\}, & \text{if } q = \frac{2}{3}, \\ \{0\}, & \text{if } q < \frac{2}{3} \end{cases}$$

Examples of Best Response Functions: Player 2

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$1, 0$	$0, 4$
	$B(1-p)$	$0, 1$	$2, 0$

We already found the best response functions for this game:

$$q = 1 \text{ optimal} \iff p < \frac{1}{5}$$

$$\text{any } q \text{ optimal} \iff p = \frac{1}{5}$$

$$q = 0 \text{ optimal} \iff p > \frac{1}{5}$$

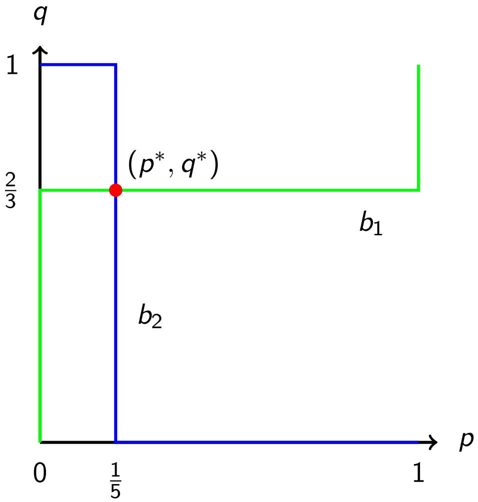
Examples of Best Response Functions: Player 2

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$1, 0$	$0, 4$
	$B(1-p)$	$0, 1$	$2, 0$

We already found the best response functions for this game:

$$b_2(p) = \begin{cases} \{1\}, & \text{if } \; p < \frac{1}{5}, \\ \{q : 0 \leq q \leq 1\}, & \text{if } \; p = \frac{1}{5}, \\ \{0\}, & \text{if } \; p > \frac{1}{5} \end{cases}$$

Example of Best Response Functions: Plot



Summary of Example

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$1, 0$	$0, 4$
	$B(1-p)$	$0, 1$	$2, 0$

Summary:

- No **pure strategy** Nash equilibria
- Unique **mixed strategy** Nash equilibrium at $\left\{\left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right\}$

Expected Payoffs

Payoffs are **not** purely ordinal if players maximize expected payoffs

Example: The following functions have the **same order**

$$u(a) = 0, \;\; u(b) = 1, \;\; u(c) = 4$$
$$v(a) = 0, \;\; v(b) = 3, \;\; v(c) = 4$$

but represent **different preferences**:

$$\tfrac{1}{2}u(a) + \tfrac{1}{2}u(c) = 2 > 1 = u(b)$$
$$\tfrac{1}{2}v(a) + \tfrac{1}{2}v(c) = 2 < 3 = v(b)$$

Expected Payoffs: Affine Transformations

The expected values of the payoff functions $u(x)$ and $v(x)$ represent the same preferences over "lotteries" if and only if

$$v(x) = \alpha u(x) + \beta, \text{ for some } \alpha > 0 \text{ and } \beta$$

Example: The following functions represent **different preferences**

$$u(a) = 0, \quad u(b) = 1, \quad u(c) = 4$$
$$v(a) = 0, \quad v(b) = 3, \quad v(c) = 4$$

because

$$v(a) = \alpha u(a) + \beta \;\Rightarrow\; 0 = \alpha \cdot 0 + \beta \;\Rightarrow\; \beta = 0$$
$$v(b) = \alpha u(b) + \beta \;\Rightarrow\; 3 = \alpha \cdot 1 + \beta \;\Rightarrow\; \alpha = 3$$
$$v(c) = \alpha u(c) + \beta \;\Rightarrow\; 4 = \alpha \cdot 4 + \beta \;\Rightarrow\; \alpha = 1$$

Expected Payoffs: Affine Transformations

The expected values of the payoff functions $u(x)$ and $v(x)$ represent the same preferences over "lotteries" if and only if

$$v(x) = \alpha u(x) + \beta, \text{ for some } \alpha > 0 \text{ and } \beta$$

Example: The following functions represent **same preferences**

$$u(a) = 0, \; u(b) = 1, \; u(c) = 4$$
$$v(a) = 1, \; v(b) = 3, \; v(c) = 9$$

because

$$v(a) = \alpha u(a) + \beta \;\Rightarrow\; 1 = \alpha \cdot 0 + \beta \;\Rightarrow\; \beta = 1$$
$$v(b) = \alpha u(b) + \beta \;\Rightarrow\; 3 = \alpha \cdot 1 + \beta \;\Rightarrow\; \alpha = 2$$
$$v(c) = \alpha u(c) + \beta \;\Rightarrow\; 9 = \alpha \cdot 4 + \beta \;\Rightarrow\; \alpha = 2$$

Back or Stravinsky: Optimal Actions

		Player 2	
		$B(q)$	$S(1-q)$
Player 1	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

Optimal action for player 1 if player 2 uses the strategy $(q, 1-q)$?

- **Expected payoff of** B: $q \cdot 2 + (1-q) \cdot 0 = 2q$
- **Expected payoff of** S: $q \cdot 0 + (1-q) \cdot 1 = 1 - q$

$$B \succ S \quad \Leftrightarrow \quad 2q > 1 - q \quad \Leftrightarrow \quad q > \frac{1}{3}$$
$$B \sim S \quad \Leftrightarrow \quad 2q = 1 - q \quad \Leftrightarrow \quad q = \frac{1}{3}$$
$$S \succ B \quad \Leftrightarrow \quad 2q < 1 - q \quad \Leftrightarrow \quad q < \frac{1}{3}$$

Back or Stravinsky: Optimal Actions

		Player 2	
		B	S
Player 1	$B(p)$	$2, 1$	$0, 0$
	$S(1-p)$	$0, 0$	$1, 2$

Optimal action for player 2 if player 1 uses the strategy $(p, 1-p)$?

- **Expected payoff of** B: $p \cdot 1 + (1-p) \cdot 0 = p$
- **Expected payoff of** S: $p \cdot 0 + (1-p) \cdot 2 = 2(1-p)$

$$B \succ S \quad \Leftrightarrow \quad p > 2(1-p) \quad \Leftrightarrow \quad p > \tfrac{2}{3}$$
$$B \sim S \quad \Leftrightarrow \quad p = 2(1-p) \quad \Leftrightarrow \quad p = \tfrac{2}{3}$$
$$S \succ B \quad \Leftrightarrow \quad p < 2(1-p) \quad \Leftrightarrow \quad p < \tfrac{2}{3}$$

Back or Stravinsky: Optimal Actions

			Player 2	
			$B(q)$	$S(1-q)$
Player 1	$B(p)$		$2,1$	$0,0$
	$S(1-p)$		$0,0$	$1,2$

Player 1:

$B \succ S \quad \Leftrightarrow \quad q > \frac{1}{3}$

$B \sim S \quad \Leftrightarrow \quad q = \frac{1}{3}$

$S \succ B \quad \Leftrightarrow \quad q < \frac{1}{3}$

Player 2:

$B \succ S \quad \Leftrightarrow \quad p > \frac{2}{3}$

$B \sim S \quad \Leftrightarrow \quad p = \frac{2}{3}$

$S \succ B \quad \Leftrightarrow \quad p < \frac{2}{3}$

Back or Stravinsky: Equilibrium

		Player 2	
		$B(q)$	$S(1-q)$
Player 1	$B(p)$	$2, 1$	$0, 0$
	$S(1-p)$	$0, 0$	$1, 2$

Player 1:

$p = 1$ **optimal** $\Leftrightarrow$ $q > \frac{1}{3}$

any p **optimal** $\Leftrightarrow$ $q = \frac{1}{3}$

$p = 0$ **optimal** $\Leftrightarrow$ $q < \frac{1}{3}$

Player 2:

$q = 1$ **optimal** $\Leftrightarrow$ $p > \frac{2}{3}$

any q **optimal** $\Leftrightarrow$ $p = \frac{2}{3}$

$q = 0$ **optimal** $\Leftrightarrow$ $p < \frac{2}{3}$

Back or Stravinsky: Equilibrium

		Player 2	
		$B(q)$	$S(1-q)$
Player 1	$B(p)$	$2,1$	$0,0$
	$S(1-p)$	$0,0$	$1,2$

Player 1:

$p=1$ **optimal** $\Leftrightarrow$ $q > \frac{1}{3}$

any p **optimal** $\Leftrightarrow$ $q = \frac{1}{3}$

$p=0$ **optimal** $\Leftrightarrow$ $q < \frac{1}{3}$

Player 2:

$q=1$ **optimal** $\Leftrightarrow$ $p > \frac{2}{3}$

any q **optimal** $\Leftrightarrow$ $p = \frac{2}{3}$

$q=0$ **optimal** $\Leftrightarrow$ $p < \frac{2}{3}$

Back or Stravinsky: Equilibrium

		Player 2	
		$B(q)$	$S(1-q)$
Player 1	$B(p)$	$2, 1$	$0, 0$
	$S(1-p)$	$0, 0$	$1, 2$

Player 1:

$p = 1$ **optimal** $\Leftrightarrow$ $q > \frac{1}{3}$

any p **optimal** $\Leftrightarrow$ $q = \frac{1}{3}$

$p = 0$ **optimal** $\Leftrightarrow$ $q < \frac{1}{3}$

Player 2:

$q = 1$ **optimal** $\Leftrightarrow$ $p > \frac{2}{3}$

any q **optimal** $\Leftrightarrow$ $p = \frac{2}{3}$

$q = 0$ **optimal** $\Leftrightarrow$ $p < \frac{2}{3}$

Back or Stravinsky: Equilibrium

		Player 2	
		$B(q)$	$S(1-q)$
Player 1	$B(p)$	$2,1$	$0,0$
	$S(1-p)$	$0,0$	$1,2$

Player 1:

$p = 1$ **optimal** $\Leftrightarrow$ $q > \frac{1}{3}$

any p **optimal** $\Leftrightarrow$ $q = \frac{1}{3}$

$p = 0$ **optimal** $\Leftrightarrow$ $q < \frac{1}{3}$

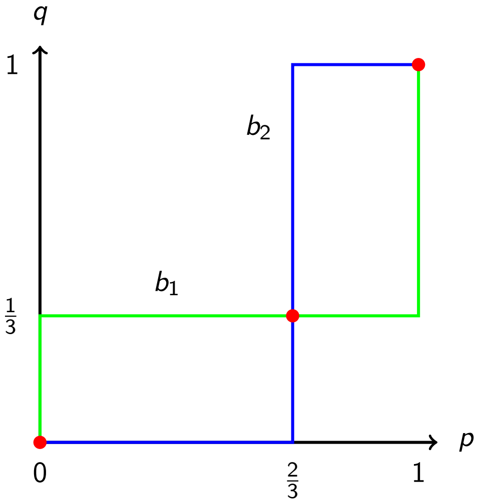
Player 2:

$q = 1$ **optimal** $\Leftrightarrow$ $p > \frac{2}{3}$

any q **optimal** $\Leftrightarrow$ $p = \frac{2}{3}$

$q = 0$ **optimal** $\Leftrightarrow$ $p < \frac{2}{3}$

Bach of Stravinsky: Plot of Best Response Functions



Bach or Stravinsky: Summary

		Player 2	
		$B(q)$	$S(1-q)$
Player 1	$B(p)$	$2, 1$	$0, 0$
	$S(1-p)$	$0, 0$	$1, 2$

Summary:

- Two **pure strategy** Nash equilibria:

$$\{(0,1),(0,1)\} \text{ and } \{(1,0),(1,0)\}$$

- Three **mixed strategy** Nash equilibria:

$$\{(0,1),(0,1)\}, \quad \{(1,0),(1,0)\} \text{ and } \left\{\left(\tfrac{2}{3},\tfrac{1}{3}\right),\left(\tfrac{1}{3},\tfrac{2}{3}\right)\right\}$$

Existence of a Mixed Strategy Nash Equilibrium

- There exists at **least one mixed strategy Nash equilibrium** in both of the examples that we have considered so far
- Same is true for any game with **finitely many actions**
- Main result of John Nash's Ph.D. thesis:

> Every strategic game in which every player has finitely many actions has a mixed strategy Nash equilibrium

Another Example: Optimal Actions

			Player 2	
			$L(q)$	$R(1-q)$
Player 1		T	$0,1$	$0,2$
		B	$2,2$	$0,1$

Optimal action for player 1 if player 2 uses the strategy $(q, 1-q)$?

- **Expected payoff of** T: $q \cdot 0 + (1-q) \cdot 0 = 0$
- **Expected payoff of** B: $q \cdot 2 + (1-q) \cdot 0 = 2q$

$$\begin{array}{ccccc} T \succ B & \Leftrightarrow & 0 > 2q & \Leftrightarrow & q < 0 \\ T \sim B & \Leftrightarrow & 0 = 2q & \Leftrightarrow & q = 0 \\ B \succ T & \Leftrightarrow & 0 < 2q & \Leftrightarrow & q > 0 \end{array}$$

Another Example: Optimal Actions

Player 2

Player 1		L	R
	$T(p)$	0, 1	0, 2
	$B(1-p)$	2, 2	0, 1

Optimal action for player 2 if player 1 uses the strategy $(p, 1-p)$?

- **Expected payoff of** L: $p \cdot 1 + (1-p) \cdot 2 = 2 - p$
- **Expected payoff of** R: $p \cdot 2 + (1-p) \cdot 1 = 1 + p$

$$L \succ R \quad \Leftrightarrow \quad 2 - p > 1 + p \quad \Leftrightarrow \quad p < \frac{1}{2}$$

$$L \sim R \quad \Leftrightarrow \quad 2 - p = 1 + p \quad \Leftrightarrow \quad p = \frac{1}{2}$$

$$R \succ L \quad \Leftrightarrow \quad 2 - p < 1 + p \quad \Leftrightarrow \quad p > \frac{1}{2}$$

Another Example: Optimal Actions

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$0,1$	$0,2$
	$B(1-p)$	$2,2$	$0,1$

Player 1:

$$T \succ B \quad \Leftrightarrow \quad q < 0$$

$$T \sim B \quad \Leftrightarrow \quad q = 0$$

$$B \succ T \quad \Leftrightarrow \quad q > 0$$

Player 2:

$$L \succ R \quad \Leftrightarrow \quad p < \tfrac{1}{2}$$

$$L \sim R \quad \Leftrightarrow \quad p = \tfrac{1}{2}$$

$$R \succ L \quad \Leftrightarrow \quad p > \tfrac{1}{2}$$

Another Example: Equilibrium

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$0, 1$	$0, 2$
	$B(1-p)$	$2, 2$	$0, 1$

Player 1:

$p = 1$ **optimal** $\Leftrightarrow$ $q < 0$

any p **optimal** $\Leftrightarrow$ $q = 0$

$p = 0$ **optimal** $\Leftrightarrow$ $q > 0$

Player 2:

$q = 1$ **optimal** $\Leftrightarrow$ $p < \frac{1}{2}$

any q **optimal** $\Leftrightarrow$ $p = \frac{1}{2}$

$q = 0$ **optimal** $\Leftrightarrow$ $p > \frac{1}{2}$

Another Example: Equilibrium

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$0, 1$	$0, 2$
	$B(1-p)$	$2, 2$	$0, 1$

Player 1:

$p = 1$ **optimal** $\Leftrightarrow$ $q < 0$

any p **optimal** $\Leftrightarrow$ $q = 0$

$p = 0$ **optimal** $\Leftrightarrow$ $q > 0$

Player 2:

$q = 1$ **optimal** $\Leftrightarrow$ $p < \frac{1}{2}$

any q **optimal** $\Leftrightarrow$ $p = \frac{1}{2}$

$q = 0$ **optimal** $\Leftrightarrow$ $p > \frac{1}{2}$

Another Example: Equilibrium

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$0,1$	$0,2$
	$B(1-p)$	$2,2$	$0,1$

Player 1:

$p = 1$ **optimal** $\Leftrightarrow$ $q < 0$

any p **optimal** $\Leftrightarrow$ $q = 0$

$p = 0$ **optimal** $\Leftrightarrow$ $q > 0$

Player 2:

$q = 1$ **optimal** $\Leftrightarrow$ $p < \frac{1}{2}$

any q **optimal** $\Leftrightarrow$ $p = \frac{1}{2}$

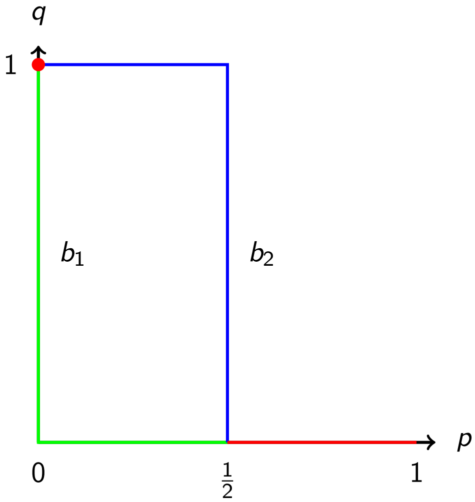
$q = 0$ **optimal** $\Leftrightarrow$ $p > \frac{1}{2}$

Another Example: Equilibrium

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	$0, 1$	$0, 2$
	$B(1-p)$	$2, 2$	$0, 1$

Player 1:			**Player 2**:		
$p = 1$ **optimal**	$\Leftrightarrow$	$q < 0$	$q = 1$ **optimal**	$\Leftrightarrow$	$p < \frac{1}{2}$
any p **optimal**	$\Leftrightarrow$	$q = 0$	**any** q **optimal**	$\Leftrightarrow$	$p = \frac{1}{2}$
$p = 0$ **optimal**	$\Leftrightarrow$	$q > 0$	$q = 0$ **optimal**	$\Leftrightarrow$	$p > \frac{1}{2}$

Another Example: Plot of Best Response Functions



Another Example: Summary

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(p)$	0, 1	0, 2
	$B(1-p)$	2, 2	0, 1

Summary:

- Two **pure strategy** Nash equilibria:

$$\{(0,1),(1,0)\} \text{ and } \{(1,0),(0,1)\}$$

- Infinite **mixed strategy** Nash equilibria:

$$\{(0,1),(1,0)\} \text{ and } \left\{(p,1-p),(0,1) : \tfrac{1}{2} \le p \le 1\right\}$$