ECO316: Applied Game Theory

Lecture 5

Christopher R. Dobronyi

University of Toronto

July 17, 2018

Table of contents

Review: Mixed Strategies

A **mixed strategy** for a player in a strategic game is a probability distribution over the set of possible actions for that player

A **probability distribution** over a (finite) set of possible actions $\{a_1, \ldots, a_n\}$ is a list of numbers $(p_1, \ldots, p_n)$ such that

- $p_1 + \cdots + p_n = 1$
- $0 \leq p_i \leq 1$, for all $i = 1, \ldots, n$

Review: Mixed Strategy Nash Equilibrium

A profile of mixed strategies is a **mixed strategy Nash equilibrium** of a strategic game if

every player's strategy is optimal, given the other players' strategies,

or, equivalently, if

no player can change her strategy to increase her expected payoff, given the other players' strategies

Review: How to Find a Mixed Strategy Nash Equilibrium

Method: We can find every **mixed strategy Nash equilibrium** of a strategic game by calculating the **best response function** $b_i(\alpha_{-i})$ for each player i and identifying all of the profiles of mixed strategies α such that

$$\alpha_i = b_i(\alpha_{-i}), \text{ for all players } i$$

- Works well with two actions per player
- **Difficulty**: What if we have **many actions** per player?
- **Solution**: Eliminate actions that will never be chosen

Equivalent Definition of Nash Equilibrium

An action profile is a **Nash equilibrium** of a strategic game if

every player has beliefs about the other players' actions

and

every player's action is optimal, given her beliefs

and

her beliefs are correct

Beliefs

A **belief** of a player in a strategic game is a probability distribution over the set of lists of the other players' actions

Example:

Player 2

		$H(q)$	$L(1-q)$
Player 1	H	$2, 2$	$0, 3$
	L	$3, 0$	$1, 1$

- A belief of player 1 is a probability distribution over $\{H, L\}$
- A belief of player 1 is a pair $(q, 1-q)$ such that $0 \leq q \leq 1$
- q is the probability with which 1 believes 2 will choose H
- $1-q$ is the probability with which 1 believes 2 will choose L

Another Example of Beliefs

- **Players**: three people
- **Actions**: A or B, for each person
- **Preferences**: any preferences, for each person

A **belief** of a person in this strategic game is a probability distribution over the set of the pairs of the other two people, i.e.,

$$\{(A, A), (A, B), (B, A), (B, B)\}$$

Example: a belief for person 1 is $(1/4, 1/4, 1/2, 0)$ which assigns

- a probability of $1/4$ to (A, A)
- a probability of $1/4$ to (A, B)
- a probability of $1/2$ to (B, A)
- a probability of 0 to (B, B)

Best Responses to a Belief

The mixed strategy α_i of player i is a **best response** to player i's belief β_i if α_i maximizes player i's expected payoff, i.e.,

$$\sum_{a_{-i} \in A_{-i}} \beta_i(a_{-i}) U_i(\alpha_i, a_{-i}),$$

in which A_{-i} denotes the set of lists of the other players' actions and $\beta_i(a_{-i})$ is the probability that player i's belief assigns to a_{-i}

Note: $U_i(\alpha_i, a_{-i})$ is player i's expected payoff from α_i and a_{-i}

Never-Best Responses

An action for a player in a strategic game is a **never-best response** if it is not a best response to *any* belief

Since an action can be chosen with **positive probability** in a mixed strategy Nash equilibrium only if it is a **best response**, we obtain:

A never-best response cannot be chosen with positive probability in a mixed strategy Nash equilibrium of a strategic game

Strict Domination

How can we check whether an action is a **never-best response**? Let us start with an **easier question**: When is a player's *action* a best response to *some* list of *actions* of the other players?

Player i's action a_i'' **strictly dominates** her action a_i' if a_i'' is strictly better than a_i' for player i, for all of the other players' actions:

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i}), \text{ for all } a_{-i}$$

Note: $u_i(a_i, a_{-i})$ is player i's payoff from a_i and a_{-i}
Note: If a_i'' strictly dominates a_i' then a_i' is strictly dominated by a_i''

Example of Strict Domination

Others' Actions

		a'_{-i}	a''_{-i}	a'''_{-i}
i's Actions	T	1	4	0
	B	2	6	3

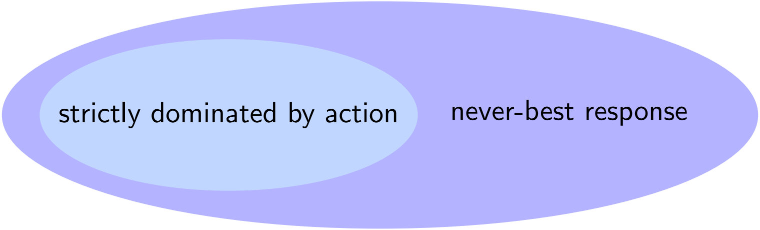
i's Payoffs

B **strictly dominates** T **because** B **is always better than** T: player i's payoff from choosing B is higher than her payoff from choosing T, for all of the other players' actions

Strict Domination and Best Responses

If an action for a player in a strategic game is strictly dominated by another action then it is a never-best response

Actions



Example of Strict Domination: Continued

		Others' Actions		
		$a'_{-i}(q')$	$a''_{-i}(q'')$	$a'''_{-i}(q''')$
i's Actions	T	1	4	0
	B	2	6	3
		i's Payoffs		

T **is not a best response to any belief** (q', q'', q'''):

$$2 \cdot q' + 6 \cdot q'' + 3 \cdot q''' > 1 \cdot q' + 4 \cdot q'' + 0 \cdot q'''$$

Another Example: Prisoner's Dilemma

Player 2

		A	B
Player 1	A	$2, 2$	$0, 3$
	B	$3, 0$	$1, 1$

A is a **never-best response** because it is strictly dominated by B:

- A is worse than B if the other player chooses A
- A is worse than B if the other player chooses B

Challenging Example: Cournot Model

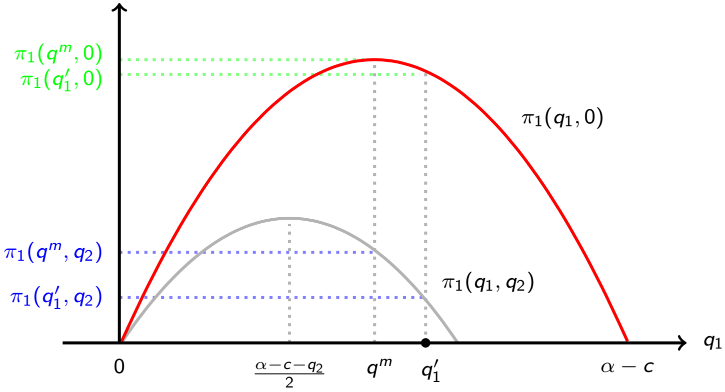
- **Players**: two firms
- **Actions**: any non-negative quantity q_i, for each firm i
- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Inverse demand**: $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $\alpha > c$

- **Claim**: Each quantity $q_i' > q^m$ is **strictly dominated** by q^m:

$$\pi_1(q^m, q_2) > \pi_1(q_1', q_2), \text{ for all } q_1' > q^m \text{ and } q_2 \geq 0$$

Challenging Example: Cournot Model

- $\pi_1(q^m, 0) > \pi_1(q_1', 0)$ because q^m is the maximizer
- $\pi_1(q^m, q_2) - \pi_1(q_1', q_2)$ is increasing in q_2



Never-Best Response that is not Strictly Dominated

Player 2

		$L(q)$	$R(1-q)$
	T	1	1
Player 1	M	4	0
	B	0	4

1's payoffs

T is a **never-best response**:

- $B \succ T$ whenever $q < 3/4$
- $M \succ T$ whenever $q > 1/4$

T is not **strictly dominated** by an action:

- B is worse than T if player 2 chooses L
- M is worse than T if player 2 chooses R

Never-Best Response that is not Strictly Dominated

		Player 2	
		$L(q)$	$R(1-q)$
Player 1	$T(0)$	1	1
	$M(1/2)$	4	0
	$B(1/2)$	0	4

1's payoffs

Suppose that player 1 uses the mixed strategy $(0, 1/2, 1/2)$:

- Expected payoff if player 2 chooses L: $1/2 \cdot 4 = 2$
- Expected payoff if player 2 chooses R: $1/2 \cdot 4 = 2$
- T is **strictly dominated** by the **mixed strategy** $(0, 1/2, 1/2)$

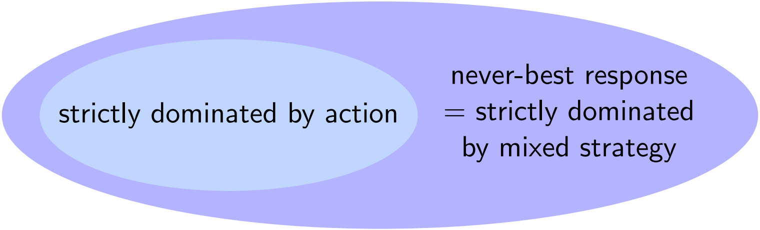
Strict Domination

Player i's mixed strategy α_i **strictly dominates** her action a_i if her expected payoff to (α_i, a_{-i}) is larger than her expected payoff to (a_i, a_{-i}), for every list a_{-i} of the other players' actions

Never-Best Response and Strict Domination

An action is a **never-best response** if and only if it is strictly dominated by a mixed strategy

Actions



Never-Best Responses and Strict Domination: Summary

Summary:

- Never-best responses cannot be used in a Nash equilibrium
- Strictly dominated by action $\Rightarrow$ never-best response
- Strictly dominated by mixed strategy $\Leftrightarrow$ never-best response

Difficulty:

- Often hard to check for strict domination by mixed strategy

Drawing Conclusions from Rationality

Nash equilibrium:

- Each player's strategy is optimal, given her beliefs
- Each player's beliefs are **correct**

Two-player intuition:

- 1 will not choose a never-best response
- 2 will not choose a never-best response
- 1 knows 2 will not choose a never-best response
- 2 knows 1 will not choose a never-best response
- 1 knows 2 knows 1 will not choose a never-best response
- 2 knows 1 knows 2 will not choose a never-best response
- 1 knows 2 knows 1 knows 2 will not choose a never-best ...

Drawing Conclusions from Rationality: Example

Player 2

Player 1		L	C	R
	T	0, 4	4, 0	2, 1
	M	1, 0	3, 1	3, 2
	B	0, 2	2, 3	1, 1

- 1's strategy is optimal given her beliefs ⇒ 1 will not choose B
- 2's beliefs are correct ⇒ 2 believes 1 will not choose B
- 2's strategy is optimal given her beliefs ⇒ 2 will not choose C
- 1's beliefs are correct ⇒ 1 believes 2 will not choose C
- 1's strategy is optimal given her beliefs ⇒ 1 will not choose T
- 2's beliefs are correct ⇒ 2 believes 1 will not choose T
- 2's strategy is optimal given her beliefs ⇒ 2 will not choose L

Iterated Elimination of Strictly Dominated Actions

Process of sequentially deleting strictly dominated actions:
iterated elimination of strictly dominated actions (IESDA)

- Order of elimination does not matter
- Every Nash equilibrium survives IESDA
- Not every action profile that survives is a Nash equilibrium
- If only one action profile survives then it is a Nash equilibrium

Not Every Profile that Survives is a Nash Equilibrium

		Player 2		
		L	C	R
	T	$0, 4$	$4, 0$	$2, 1$
Player 1	M	$1, 0$	$3, 1$	$3, 2$
	B	$2, 2$	$2, 3$	$1, 1$

- Nash equilibria: (M, R)
- All action profiles survive IESDA

How to Find a Mixed Strategy Nash Equilibrium

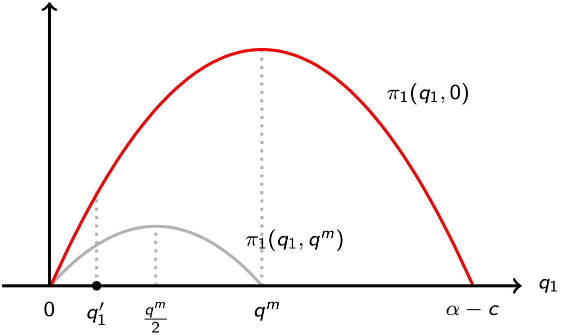
Method: We can find every **mixed strategy Nash equilibrium** of a strategic game by performing **iterated elimination of strictly dominated actions** then using our other method to find every mixed strategy Nash equilibrium of the remaining game

Challenging Example: Cournot Model Continued

- **Players**: two firms
- **Actions**: any non-negative quantity q_i, for each firm i
- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Inverse demand**: $P(Q) = \alpha - Q$ for $Q \leq \alpha$ and $\alpha > c$

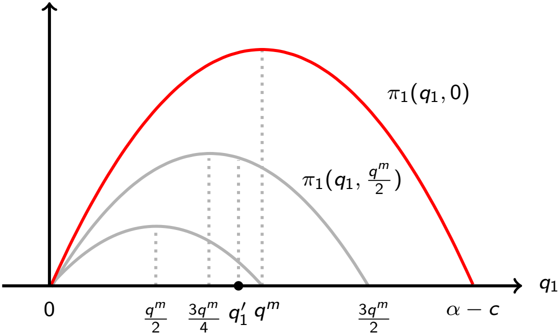
- **Claim**: Only $\left(\frac{\alpha-c}{3}, \frac{\alpha-c}{3}\right)$ survives IESDA

Challenging Example: Cournot Model Continued



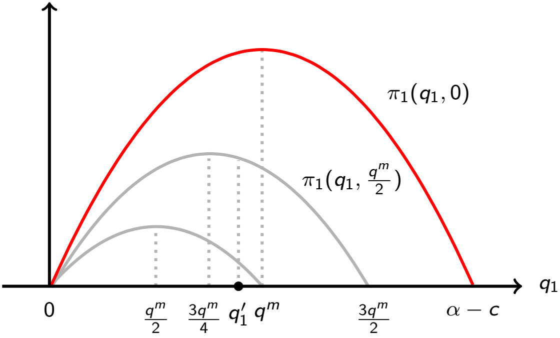
- Eliminate all $q_i > q^m$ by previous argument
- Largest quantity player 2 will choose is q^m
- Maximizer of $\pi_1(q_1, q_2)$ is decreasing in q_2
- Every $q_i' < \frac{q^m}{2}$ is strictly dominated by $\frac{q^m}{2}$

Challenging Example: Cournot Model Continued



- Eliminate all $q_i' < \frac{q^m}{2}$ by previous argument
- Smallest quantity player 2 will choose is $\frac{q^m}{2}$
- $\pi_1(\frac{3q^m}{4}, \frac{q^m}{2}) > \pi_1(q_1', \frac{q^m}{2})$ because $\frac{3q^m}{4}$ is the maximizer
- $\pi_1(\frac{3q^m}{4}, q_2) - \pi_1(\frac{3q^m}{4}, q_2)$ is increasing in q_2
- Every $q_i' > \frac{3q^m}{4}$ is strictly dominated by $\frac{3q^m}{4}$

Challenging Example: Cournot Model Continued



The process continues until only $\left(\frac{\alpha-c}{3}, \frac{\alpha-c}{3}\right)$ remains

Example of Weak Domination

		Player 2	
		A	B
Player 1	A	$2, 2$	$0, 3$
	B	$3, 0$	$1, 1$

A is **strictly dominated** by B:

- A is **worse** than B if the other player chooses A
- A is **worse** than B if the other player chooses B

Example of Weak Domination

		Player 2	
		A	B
Player 1	A	$2, 2$	$0, 2$
	B	$2, 0$	$1, 1$

A is **weakly dominated** by B:

- A is **no better** than B if the other player chooses A
- A is **worse** than B if the other player chooses B

Weak Domination

Often reasonable to restrict our attention to Nash equilibria in which no player uses a **weakly dominated** action:

Player i's action a_i'' **weakly dominates** her action a_i' if a_i'' is at least as good as a_i' for i, for all of the other players' actions, **and** a_i'' is better than a_i' for i, for some of the other players' actions:

$$u_i(a_i'' a_{-i}) \geq u_i(a_i', a_{-i}), \text{ for every } a_{-i}$$
$$u_i(a_i'' a_{-i}) > u_i(a_i', a_{-i}), \text{ for some } a_{-i}$$

Another Example of Weak Domination

		Others' Actions		
		a'_{-i}	a''_{-i}	a'''_{-i}
i's Actions	T	1	6	0
	B	2	6	3

i's Payoffs

T **is weakly dominated by** B:

- T is **worse** than B if the other players choose a'_{-i}
- T is **no better** than B if the other players choose a''_{-i}
- T is **worse** than B if the other players choose a'''_{-i}

Third Example of Weak Domination: Bertrand Model

- **Players**: two firms
- **Actions**: any non-negative price p_i, for each firm i
- **Cost**: cost of producing q for firm i is $C_i(q) = cq$ for $c > 0$
- **Demand**: $D(p) = \alpha - p$ for $p \leq \alpha$ and $\alpha > c$

$p_i < c$ is **weakly dominated** by c:

		Firm 2	
		$p_2 < p_1$	$p_2 \geq p_1$
Firm 1	$p_1 < c$	0	< 0
	$p_1 < c$	0	0

1's payoffs

Weak Domination and Nash Equilibrium

		Player 2	
		L	R
Player 1	T	$1, 1$	$0, 0$
	B	$0, 0$	$0, 0$

Weakly dominated actions can be used in a Nash equilibrium:

- (B, R) is a Nash equilibrium
- B is weakly dominated by T
- R is weakly dominated by L

Iterated Elimination of Weakly Dominated Actions

Process of sequentially deleting weakly dominated actions:
iterated elimination of weakly dominated actions (IEWDA)

- Order of elimination matters
- Not every Nash equilibrium survives IEWDA
- Not every action profile that survives is a Nash equilibrium
- If only one action profile survives then it is a Nash equilibrium

Weak Domination and Nash Equilibrium: Continued

		Player 2	
		L	R
Player 1	T	$1, 1$	$0, 0$
	B	$0, 0$	$0, 0$

Iterated elimination of weakly dominated actions:

- Delete B because B is weakly dominated by T
- Delete R because R is weakly dominated by L
- (B, R) is eliminated even though it is a Nash equilibrium
- (T, L) is a Nash equilibrium because it is the only survivor

Comparison of Strict and Weak Domination

Iterated Elimination of Strictly Dominated Actions:

- Order of elimination does not matter
- Every Nash equilibrium survives IESDA
- Not every action profile that survives is a Nash equilibrium
- If only one action profile survives then it is a Nash equilibrium

Iterated Elimination of Weakly Dominated Actions:

- Order of elimination matters
- Not every Nash equilibrium survives IEWDA
- Not every action profile that survives is a Nash equilibrium
- If only one action profile survives then it is a Nash equilibrium