ECO316: Applied Game Theory
Lecture 6

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Collective Choice

- Group of people has to choose one of several actions
- Group members' preferences differ
- How should the action of the group be selected?

Collective Choice

Two actions:

$B \succ A$

$A \succ B$

$B \succ A$

$B \succ A$ $A \succ B$

$A \succ B$ $B \succ A$

$A \succ B$ $A \succ B$

- Only information we have is whether each person prefers A or B, not the intensity of their preferences
- Knowing whether a person prefers A or B is equivalent to knowing her preference relation (because two alternatives)

Collective Choice

Two actions:

$B \succ A$

$A \succ B$

$B \succ A$

$B \succ A$ $A \succ B$

$A \succ B$ $B \succ A$

$A \succ B$ $A \succ B$

- If we know peoples' preferences and want to treat people equally and A and B symmetrically then it is natural to select the alternative that is favoured by the majority

Collective Choice

Two actions:

$B \succ A$

$A \succ B$

$B \succ A$

$B \succ A$ $A \succ B$

$A \succ B$

$B \succ A$

$A \succ B$ $A \succ B$

- But typically we do not know peoples' preferences
- Could ask each person to vote for a single alternative and then select the alternative with the most votes
- What are the *strategic* properties of this mechanism?

Majority Rule with Two Candidates

- **Players**: k citizens
- **Actions**: vote for A or B, for each citizen
- **Preferences**: each citizen has one of the following relations:
  - A **supporter**: A gets most votes $\succ$ tie $\succ$ B gets most votes
  - B **supporter**: B gets most votes $\succ$ tie $\succ$ A gets most votes

- What are the **Nash equilibria** of this strategic game?

Review: Weak Domination

Often reasonable to restrict our attention to Nash equilibria in which no player uses a **weakly dominated** action:

Player i's action a_i'' **weakly dominates** her action a_i' if a_i'' is at least as good as a_i' for i, for all of the other players' actions, **and** a_i'' is better than a_i' for i, for some of the other players' actions:

$$u_i(a_i'' a_{-i}) \geq u_i(a_i', a_{-i}), \text{ for every } a_{-i}$$
$$u_i(a_i'' a_{-i}) > u_i(a_i', a_{-i}), \text{ for some } a_{-i}$$

Note: $u_i(a_i, a_{-i})$ is player i's payoff from a_i and a_{-i}

Majority Rule with Two Candidates

- Suppose citizen i **supports** A
- How should she vote?
- Compare **possible outcomes** for citizen i for each action:

Other Votes:

		A wins by ≥ 2	A wins by 1	tie	B wins by 1	B wins by ≥ 2
Citizen i	Vote A	A	A	A	tie	B
	Vote B	A	tie	B	B	B

Election Outcomes

- Voting for A **weakly dominates** voting for B
- Similar argument for a B supporter

Majority Rule with Two Candidates

Nash equilibrium:

- Every citizen votes for her favourite candidate
- Only Nash equilibrium with no weakly dominated actions

Other Nash equilibria?

- Yes! Example: All citizens vote for A

Majority Rule with Three Candidates

- **Three candidates**: A, B and C
- Suppose citizen i **supports** A over B over C
- Let v_X denote the number of votes for candidate X
- Compare **possible outcomes** for citizen i for each action:

Other Votes:

	$v_A > v_B > v_C$	$v_A = v_B = v_C + 1$	$v_A = v_B + 1$, $v_B = v_C$	$v_A = v_B = v_C$	
Vote A	A	A	A	A	...
Vote C	A	ABC	AC	C	

Election Outcomes

- Voting for A **weakly dominates** voting for C

Majority Rule with Three Candidates

Dominated actions:

- Suppose citizen i **supports** A over B over C
- Voting for B is **not weakly dominated**
- Consider the following situation:

Candidates	Other Votes
A	15
B	30
C	31

- Vote A implies C wins
- Vote B implies B and C tie
- Voting B is **better** than voting A
- **Strategic voting is not necessarily "sincere"**

Majority Rule with Three Candidates

Nash equilibrium:

- Many Nash equilibria
- Example: Any action profile with a winning margin ≥ 3
- There are Nash equilibria with no weakly dominated actions and at least one citizen not voting for her favourite candidate

Alternative Voting Systems

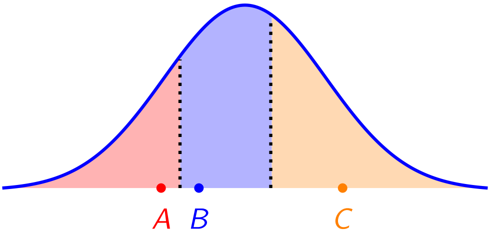
- Many voting schemes have been proposed
- The aim is to improve upon majority rule
- **Two alternatives**:
  - Approval voting
  - Single-transferable vote

Approval Voting

Approval voting:

- Each citizen “approves” a **set of candidates**
- Candidate with the most approvals wins
- **Weakly dominated actions**:
  - Approving a set including least favourite candidate
  - Approving a set that does not include favourite candidate
- **Example**:
  - Three candidates: A, B and C
  - Suppose a citizen **supports** A over B over C
  - Only $\{A\}$ and $\{A, B\}$ are not weakly dominated

Example of Approval Voting



- If every citizen votes for favourite candidate then C wins
- But citizens on left are almost indifferent between A and B
- If left citizens approve $\{A, B\}$ then A and B tie

Single Tranferable Vote

Single tranferable vote:

- Each citizen **ranks** all candidates
- Process for **determining winner**:
  - If some candidate has $> 50\%$ of first-place votes the she wins
  - Else, remove candidate with fewest first-place votes and repeat

Example of Single Transferable Vote

- **Example**:

# of Citizens	Rankings
6	A, B, C, D
5	B, A, C, D
3	C, B, A, D
1	D, C, A, B

- **Procedure**:

	# First-Place:				
Round	A	B	C	D	
1	6	5	3	1	Eliminate D
2	6	5	4		Eliminate C
3	7	8			B Wins

- **Note**: Majority rule selects A if all vote for favourite candidate

Single Transferable Vote

Single transferable vote:

- Analysis of Nash equilibria is **difficult**
- No simple general results (problem set has example)

Alternative Voting Systems

Summary:

- Many alternatives to majority rule
- No voting system is perfect
- For some preferences, voting systems like approval voting or single tranferable vote can generate outcomes that reflect citizens' preferences better than majority rule

Motivation for Bayesian Games

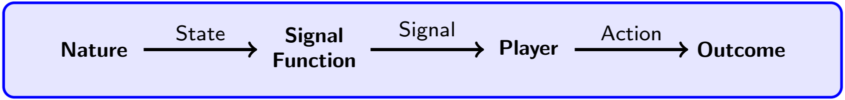
- So far, citizens differ in their **preferences**
- Suppose, instead, that citizens differ in their **information**
- **Example**:
  - Candidate A is preferred in state a
  - Candidate B is preferred in state b
  - Some voters "know" the state while others do not
  - How do citizens vote in equilibrium?
- We need **new tools** to answer this quesion
- Let us take a detour to introduce **Bayesian games**

Bayesian Games

A **Bayesian game** is a model consisting of

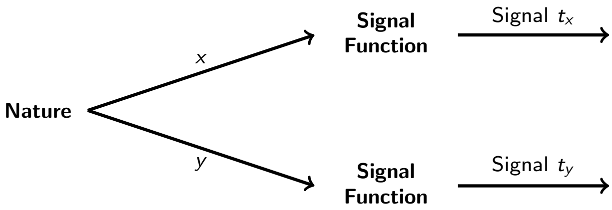
- a set of **players**
- a set of **states**
- a set of **actions**, for each player
- a set of **signals** and a **signal function**, for each player
- a **belief** over states (consistent with signal), for each player
- **preferences** over action profiles and states, for each player

Description of States, Signals and Signal Functions



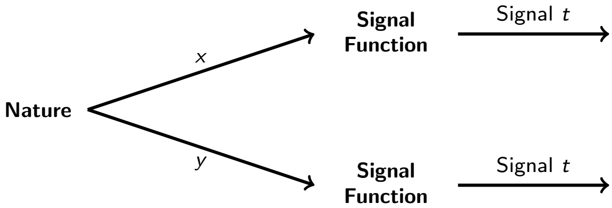
- A **state** is realized at the start of the game
- Players **do not observe** the state
- Rather, each player observes a **signal**
- **Signal function** determines a player's signal in each state

Example: Signals are Sometimes Informative



- States x and y
- Signal t_x is received if the state is x
- Signal t_y is received if the state is y
- Receiving signal t_x implies that the state is x
- Receiving signal t_y implies that the state is y

Example: Signals are not Always Informative



- States x and y
- Signal t is received in both states x and y
- Receiving signal t implies nothing about the state

Beliefs

- Each player forms a **belief** about the state given her signal
- A belief is a **probability distribution** over states
- **Example**: A player believes that the state is x with probability $1/2$ and the state is y with probability $1/2$

Beliefs must be Consistent with Signals: Bayes' Rule

General form:

$$P(x|t) = \frac{P(t|x) \cdot P(x)}{P(t)}$$

- $P(x)$: probability of **state** x
- $P(t)$: probability of **signal** t
- $P(x|t)$: probability of **state** x **given signal** t
- $P(t|x)$: probability of **signal** t **given state** x

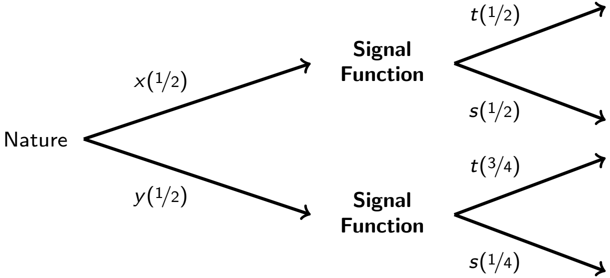
Beliefs must be Consistent with Signals: Bayes' Rule

Finite states $(x_1, \ldots, x_n)$:

$$P(x|t) = \frac{P(t|x) \cdot P(x)}{P(x_1) \cdot P(t|x_1) + \cdots + P(x_n) \cdot P(t|x_n)}$$

- $P(x)$: probability of **state** x
- $P(t)$: probability of **signal** t
- $P(x|t)$: probability of **state** x **given signal** t
- $P(t|x)$: probability of **signal** t **given state** x

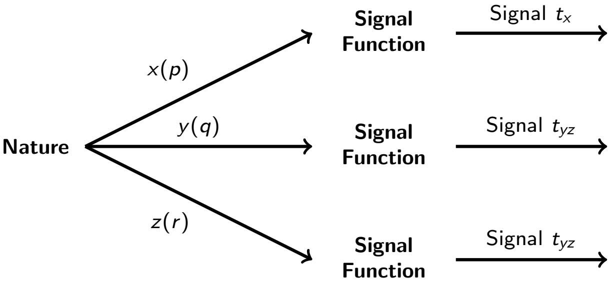
Bayes' Rule: Example 1



Probability the state is x after observing signal t:

$$P(x|t) = \frac{P(t|x) \cdot P(x)}{P(x) \cdot P(t|x) + P(y) \cdot P(t|y)} = \frac{1/2 \cdot 1/2}{1/2 \cdot 1/2 + 1/2 \cdot 3/4} = \frac{2}{5}$$

Bayes' Rule: Example 2



Consistent:

- Believes state is x after receiving t_x
- Believes state is y with probability $\frac{q}{q+r}$ after receiving t_{yz}
- Believes state is z with probability $\frac{r}{q+r}$ after receiving t_{yz}

Not consistent:

- Believes state is y after receiving t_x

Bayesian Game: Voting with Imperfect Information

- **Players**: two citizens
- **States**: a or b
- **Actions**: vote for A or B or abstain, for each player
- **Signals**: citizen i has signal function t_i such that
  - $t_1(a) = t_a$ and $t_1(b) = t_b$ (citizen 1 knows the state)
  - $t_2(a) = t_2(b) = t_{ab}$ (citizen 2 does not know the state)
- **Beliefs**:
  - 1 believes state is a after receiving t_a
  - 1 believes state is b after receiving t_b
  - 2 believes state is a with probability 0.9 after receiving t_{ab}
  - 2 believes state is b with probability 0.1 after receiving t_{ab}
- **Preferences**:
  - state a: payoff of 1 if A elected and 0 if B elected
  - state b: payoff of 0 if A elected and 1 if B elected

Voting with Imperfect Information: Expected Payoffs

Expected payoff by action profile and state:

State a

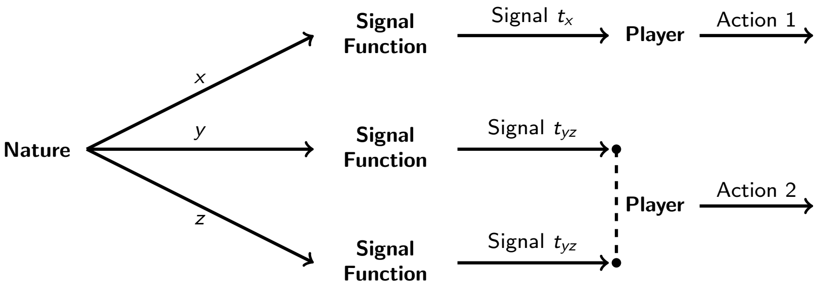
	A	B	Abs.
A	1	$1/2$	1
B	$1/2$	0	0
Abs.	1	0	$1/2$

State b

	A	B	Abs.
A	0	$1/2$	0
B	$1/2$	1	1
Abs.	0	1	$1/2$

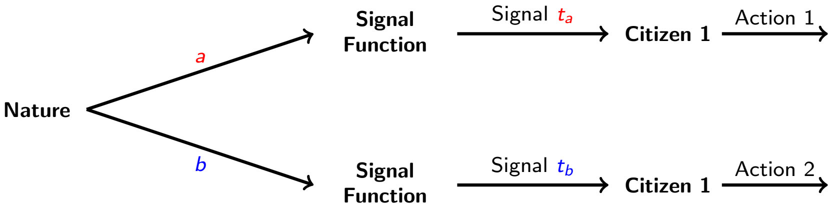
Example: Actions in a Bayesian Game

- **Strategic game**: each player chooses an action
- **Bayesian game**: each player chooses a **strategy**
  - A strategy is an action for each signal



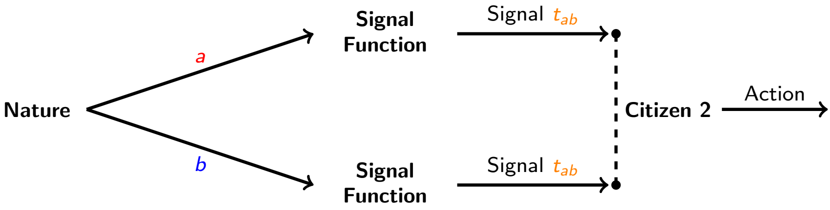
Voting with Imperfect Information: Citizen 1's Actions

- **Signals**: citizen i has signal function t_i such that
  - $t_1(a) = t_a$ and $t_1(b) = t_b$ (citizen 1 knows the state)
  - $t_2(a) = t_2(b) = t_{ab}$ (citizen 2 does not know the state)



Voting with Imperfect Information: Citizen 2's Actions

- **Signals**: citizen i has signal function t_i such that
  - $t_1(a) = t_a$ and $t_1(b) = t_b$ (citizen 1 knows the state)
  - $t_2(a) = t_2(b) = t_{ab}$ (citizen 2 does not know the state)



Nash Equilibrium of a Bayesian Game

An **Nash equilibrium of a Bayesian game** is a Nash equilibrium of the strategic game defined as follows:

- **Players**: the set of pairs (i, t_i), for each player i of the Bayesian game and each signal t_i for that player
- **Actions**: the set of actions for player (i, t_i) is the set of actions for player i in the associated Bayesian game
- **Preferences**: Expected payoffs given beliefs, for each (i, t_i)

Voting with Imperfect Information: Nash Equilibrium

	A	B	Abs.
A	1	1/2	1
B	1/2	0	0
Abs.	1	0	1/2

State a

	A	B	Abs.
A	0	1/2	0
B	1/2	1	1
Abs.	0	1	1/2

State b

Citizen 1: (citizen 1 knows the state)

- **State** a:
  - voting for A strictly dominates voting for B
  - voting for A weakly dominates abstaining
- **State** b:
  - voting for B strictly dominates voting for A
  - voting for B weakly dominates abstaining
- If no citizen uses a weakly dominated action then citizen 1
  - votes for A after receiving signal t_a
  - votes for B after receiving signal t_b

Voting with Imperfect Information: Nash Equilibrium

	A	B	Abs.
A	1	1/2	1
B	1/2	0	0
Abs.	1	0	1/2

State a

	A	B	Abs.
A	0	1/2	0
B	1/2	1	1
Abs.	0	1	1/2

State b

Citizen 2: (citizen 2 does not know the state)

- **Expected payoffs** given citizen 1's actions:
  - **Vote for** A: $0.9 \cdot 1 + 0.1 \cdot 0.5 = 0.95$
  - **Vote for** B: $0.9 \cdot 0.5 + 0.1 \cdot 1 = 0.46$
  - **Abstain**: $0.9 \cdot 1 + 0.1 \cdot 1 = 1$
- Optimal to **abstain** even though she is quite sure the state is a

Voting with Imperfect Information: Summary

Bayesian game with a **unique Nash equilibrium** in which **no player uses a weakly dominated action**:

- Citizen 1 votes for A when she receives signal t_a
- Citizen 1 votes for B when she receives signal t_b
- Citizen 2 abstains

- Citizen 2 suffers from the "swing voter's curse":
  - If she votes for A then her vote makes no difference if state is a (citizen 1 votes for in any case) and induces a worse outcome if state is b (in which the best candidate is B)
  - If she votes for B then her vote makes no difference if state is b (citizen 1 votes for in any case) and induces a worse outcome if state is a (in which the best candidate is A)

Voting with Imperfect Information: Conclusion

Conclusion:

- If citizen 2 were the only voter, she would vote for A
- Else, she must consider when her vote makes a difference
- In our example, she does not vote in the Nash equilibrium
- Imperfect information affects behaviour in many situations
- Several examples of this in the tutorial and problem set