ECO316: Applied Game Theory

Lecture 7

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Auctions

Many types of goods are transacted by auction:

- Art
- Fish and flowers
- Treasury bills
- Oil tracts
- Wireless spectrum (for cell phones, TV, etc.)
- Ads on search engines (e.g. Google)
- Products on eBay
- Government contracts

Auctions

General framework:

- Good for sale
- Bidders have values for the good
- Bidders submit bids for the good
- Bids determine the winner and price
- Winner pays the price and receives the good

Questions:

- How should a bidder bid?
- What type of auction should we use?
- What if bidders have a common value for the good?

Types of Auctions

Ascending auction (English auction):

- Price rises
- Last bidder wins
- Examples: art and fish auctions

Descending auction (Dutch auction):

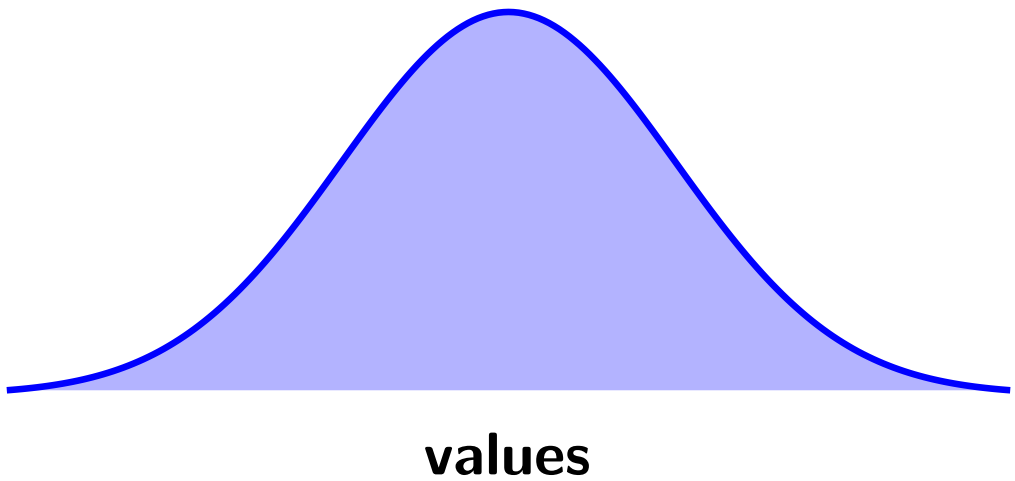
- Price falls
- First bidder wins
- Examples: flower auctions

Sealed-bid auction (simultaneous auction):

- Bidders submit bids
- Bidder with highest bid wins
- Examples: oil tracts, ads, eBay, contracts

Independent Private Values

- Bidders' values are drawn independent from distribution
- Bidders know their own values
- Bidders do not know the other players' values
- Bidders know the distribution of other players' values



Sealed-Bid Auction with Independent Private Values

Bayesian game:

- **Players**: n bidders
- **States**: all lists of values $(v_1, \ldots, v_n)$
- **Actions**: a set of possible bids, for each bidder
- **Signals**: bidder i's signal function is $t_i(v_1, \ldots, v_n) = v_i$
- **Beliefs**: values drawn independently from known distribution
- **Preferences**:
  - **win**: value minus payment determined by bids
  - **lose**: minus payment determined by bids

Strategies and Nash Equilibrium

Strategies: A bidder's strategy specifies a bid for each of her possible valuations

Nash equilibrium: A profile of strategies is a Nash equilibrium if the bid specified by each player's strategy for each of her possible valuations maximizes her expected payoff given the other players' strategies and her belief about the other players' valuations

Other Auctions with Independent Private Values

- Bayesian game formulates a sealed-bid auction
- Also formulates other auctions with independent private values

Ascending and descending auctions:

- How would you instruct someone bidding on your behalf?
- Give them a maximum value that you are willing to pay?
- Makes sense if your value does not depend on others' values

Ascending Auctions with Independent Private Values

Example:

- Four bidders
- Bidder i bids if price is smaller than a limit m_i
- Bidders drop out as the price rises
- Once the price reaches the second highest limit
  - there will only be one bidder left
  - this bidder will win and receive the good
  - this bidder will pay the second highest limit

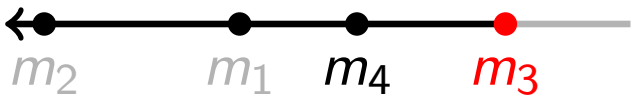
Ascending Auctions with Independent Private Values

We can, therefore, model an **ascending auction** with independent private values using our **sealed-bid** bayesian game in which

- each bidder's bid is her limit
- winner is the bidder with the highest limit
- price is the second highest limit

This model is a **second-price sealed-bid auction**

Descending Auctions with Independent Private Values



Example:

- Four bidders
- Bidder i bids if price is no larger than a limit m_i
- Once the price reaches the highest limit
  - the bidder with this limit will enter
  - this bidder will win and receive the good
  - this bidder will pay her limit

Ascending Auctions with Independent Private Values

We can, therefore, model a **descending auction** with independent private values using our **sealed-bid** bayesian game in which

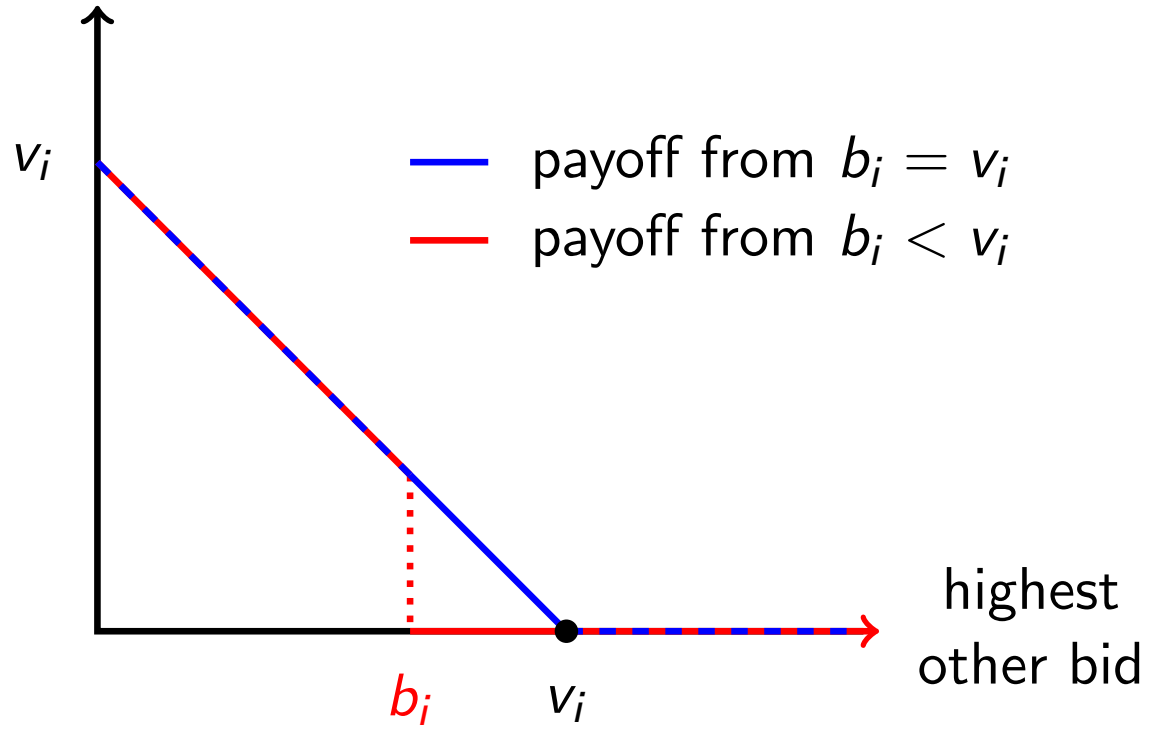
- each bidder's bid is her limit
- winner is the bidder with the highest limit
- price is the highest limit

This model is a **first-price sealed-bid auction**

Second-Price Sealed-Bid Auction

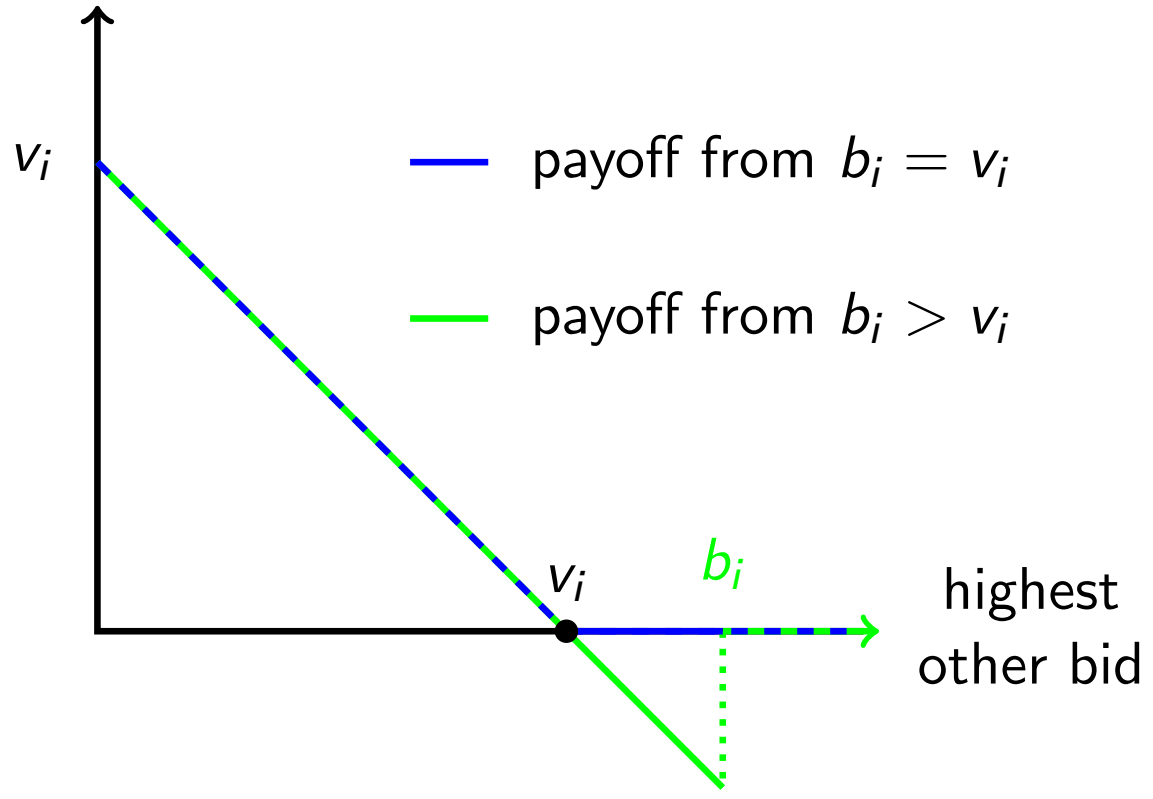
- **Players**: n bidders
- **States**: all lists of values $(v_1, \ldots, v_n)$
- **Actions**: any non-negative bid b_i, for each bidder i
- **Signals**: bidder i's signal function is $t_i(v_1, \ldots, v_n) = v_i$
- **Beliefs**: values drawn independently from known distribution
- **Preferences**: bidder i's payoff is
  - **win**: $v_i - \max_{j \neq i} b_j$
  - **tie**: $\frac{v_i - b_i}{m}$ (when m bidders tie)
  - **lose**: 0

Second-Price Sealed-Bid Auction: Analysis



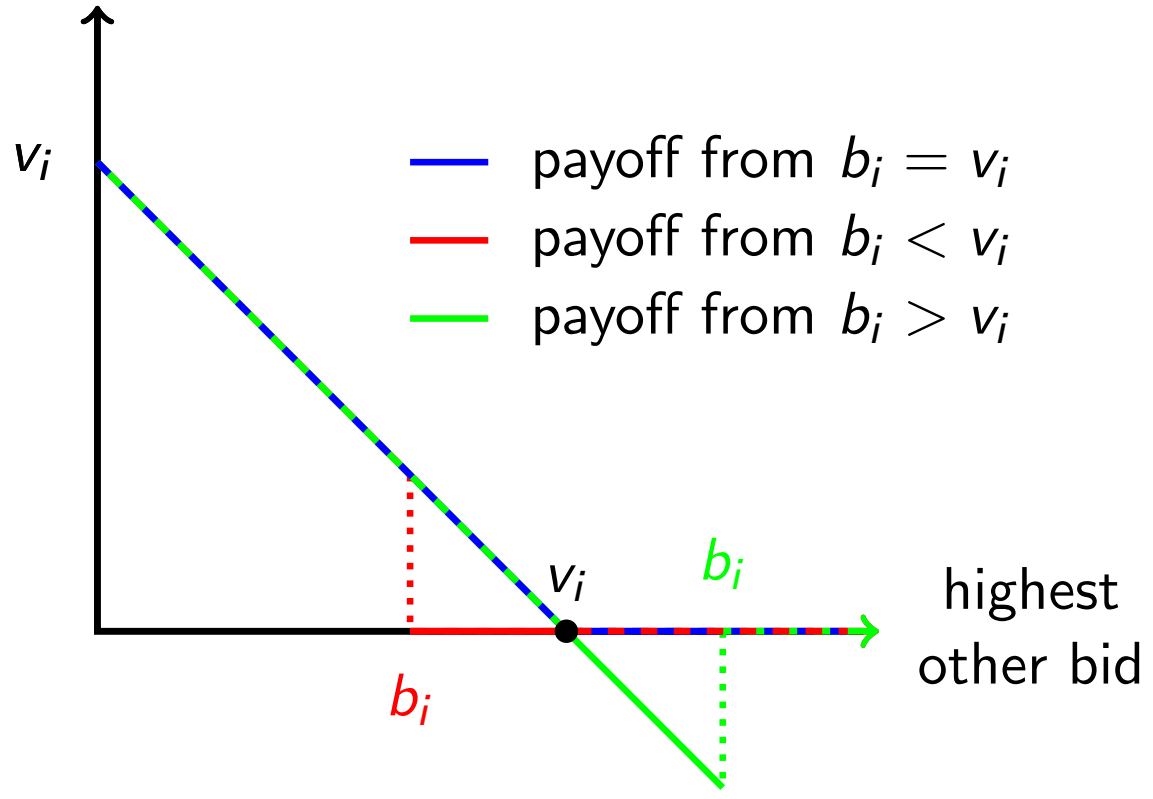
v_i **weakly dominates bid less than** v_i: Bidding less than v_i affects the probability of winning but not the price

Second-Price Sealed-Bid Auction: Analysis



v_i **weakly dominates bid greater than** v_i: Bidding more than v_i makes it possible to win and get a negative payoff

Second-Price Sealed-Bid Auction: Analysis



v_i **weakly dominates all** $b_i \neq v_i$

Second-Price Sealed-Bid Auction: Nash Equilibrium

Because bidding your value weakly dominates every other bid, iterated elimination implies that there is a Nash equilibrium in which every bidder bids her value for the good:

An independent private values second-price sealed-bid auction has a Nash equilibrium in which every player bids her value for the good

- Good is sold to the bidder with the highest value
- Auction has other equilibria too, but we focus on this one

First-Price Sealed-Bid Auction

- **Players**: n bidders
- **States**: all lists of values $(v_1, \ldots, v_n)$
- **Actions**: any non-negative bid b_i, for each bidder i
- **Signals**: bidder i's signal function is $t_i(v_1, \ldots, v_n) = v_i$
- **Beliefs**: values drawn independently from known distribution
- **Preferences**: bidder i's payoff is
  - **win**: $v_i - b_i$
  - **tie**: $\frac{v_i - b_i}{m}$ (when m bidders tie)
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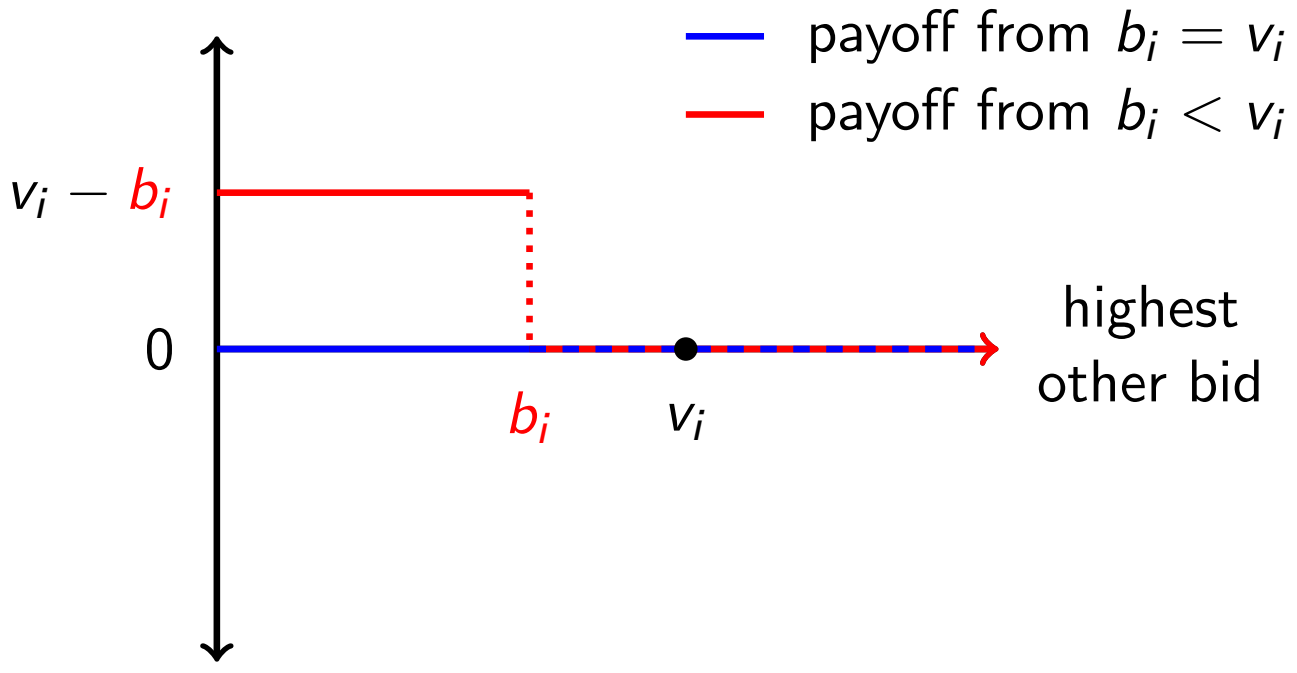
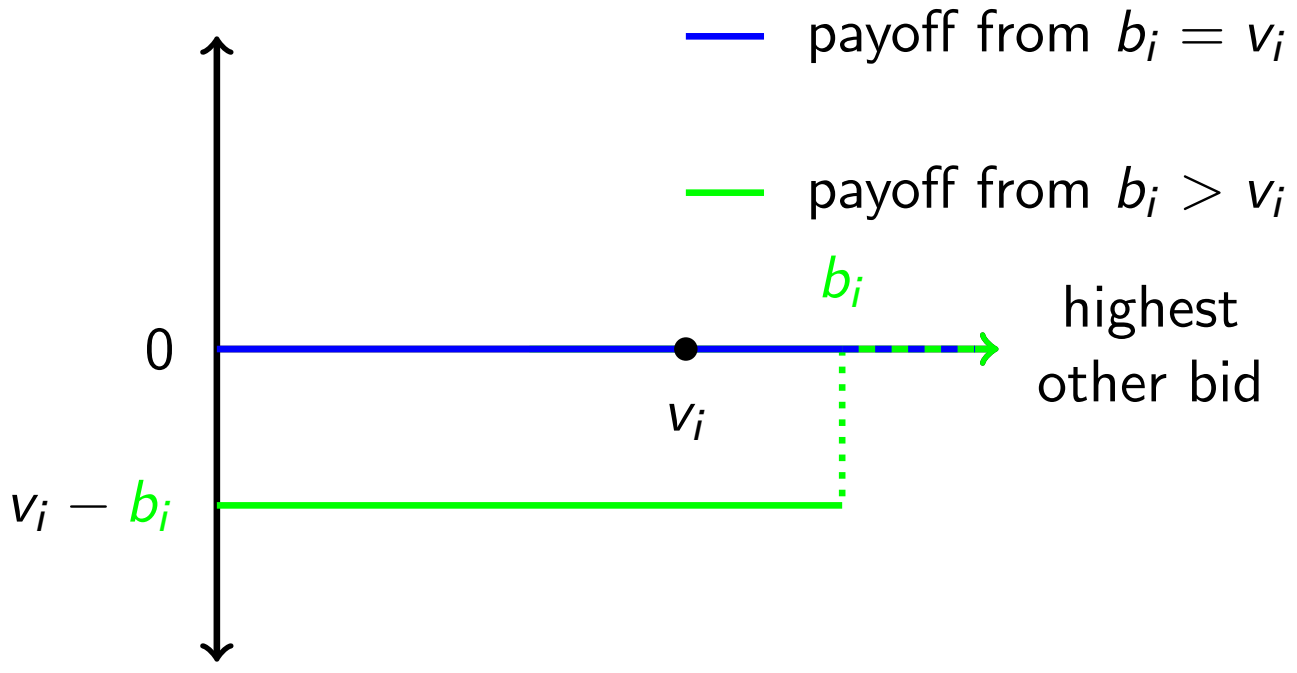
First-Price Sealed-Bid Auction: Analysis



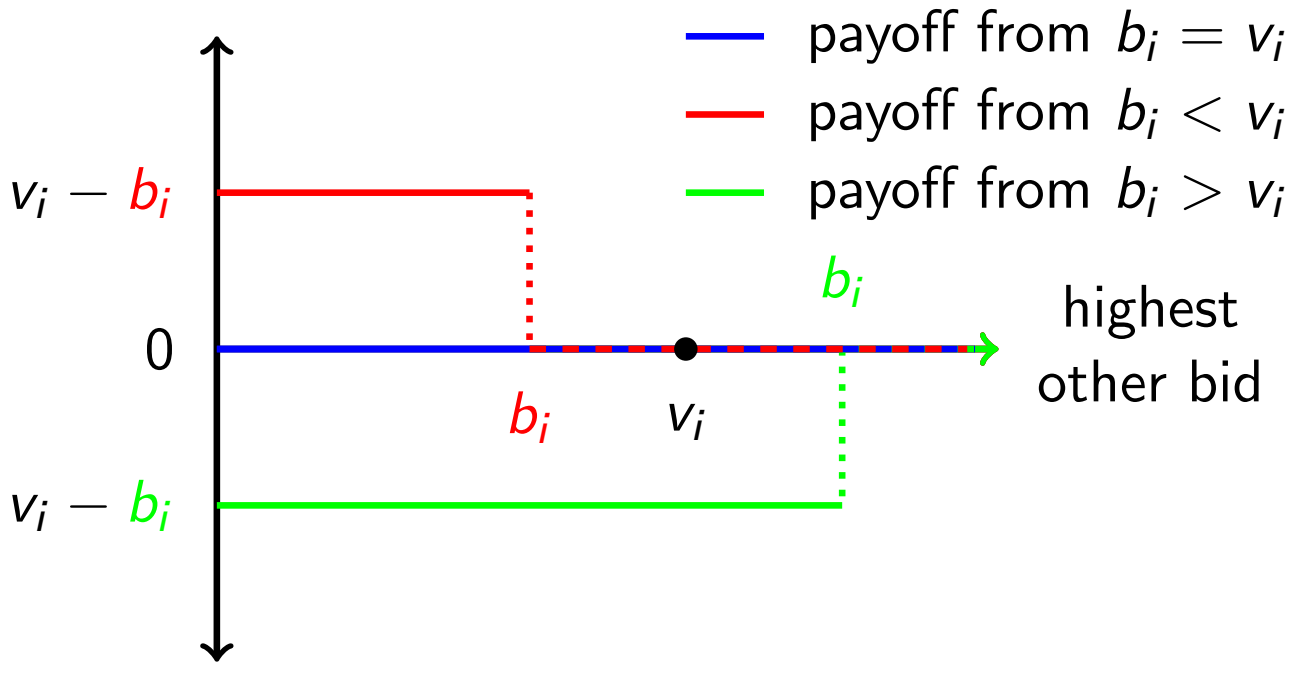
v_i is weakly dominated by bid less than v_i

First-Price Sealed-Bid Auction: Analysis



v_i weakly dominates bid greater than v_i

First-Price Sealed-Bid Auction: Analysis



$b_i < v_i$ **weakly dominates** v_i **which weakly dominates** $b_i > v_i$

First-Price Sealed-Bid Auction: Example

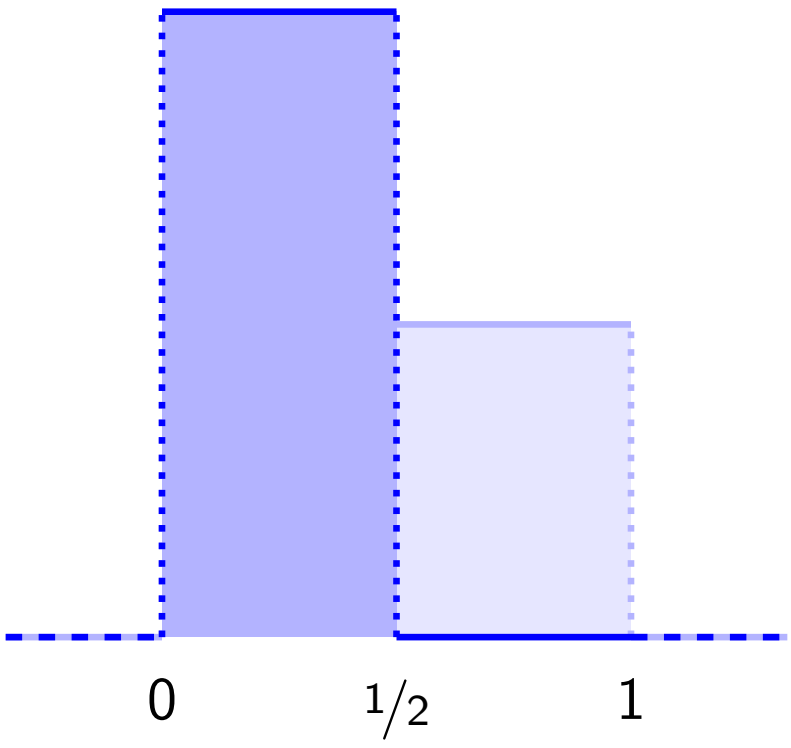
Example:

- Two bidders
- Distribution of values is uniform between 0 and 1

First-Price Sealed-Bid Auction: Example

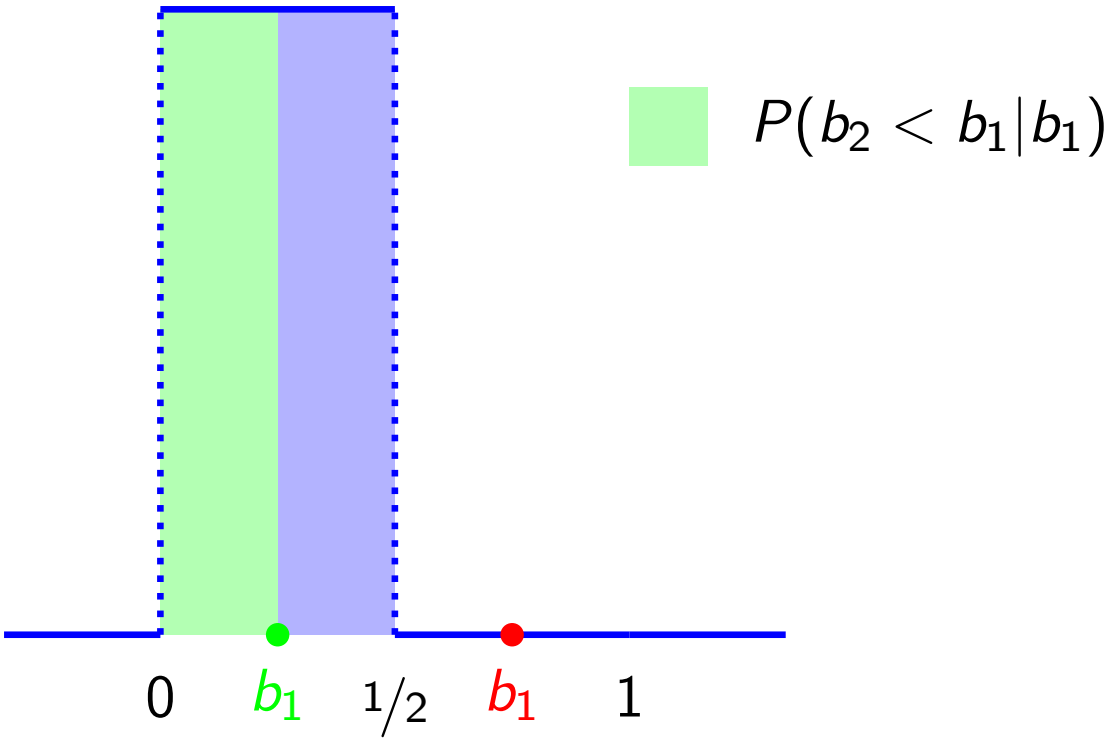
Nash equilibrium where bidders bid half their values. Argument:

- If bidder 2 bids half her value then bidder 1 believes that bidder 2's bid has a uniform distribution between 0 and $1/2$
- Bidder 1's expected payoff is maximized at half her value



First-Price Sealed-Bid Auction: Example

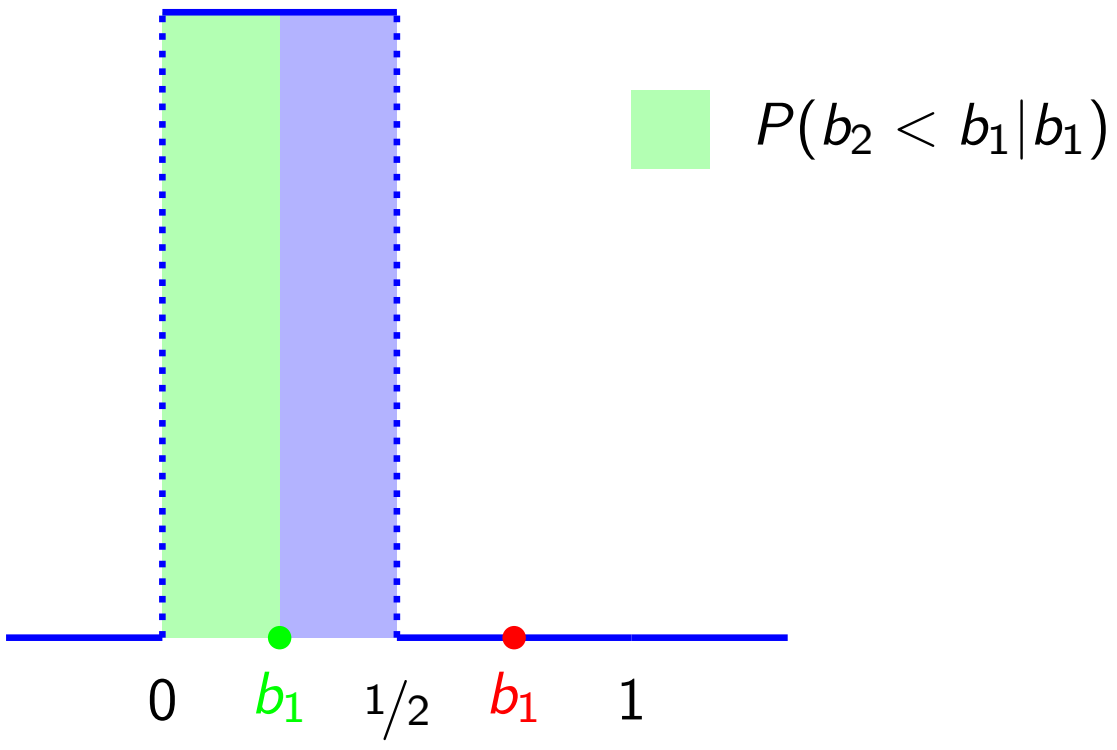
- If bidder 2 bids half her value, she will bid no more than $1/2$
- $b_1 > 1/2$: bidder 1 wins with certainty
- $b_1 \leq 1/2$: bidder 1 wins with probability $P(b_1 < b_2 | b_1) = 2b_1$



First-Price Sealed-Bid Auction: Example

Player 1's payoff given player 2's strategy:

$$\begin{cases} 2b_1(v_1 - b_1), & \text{if } 0 \leq b_1 \leq 1/2, \\ v_1 - b_1, & \text{if } b_1 > 1/2 \end{cases}$$

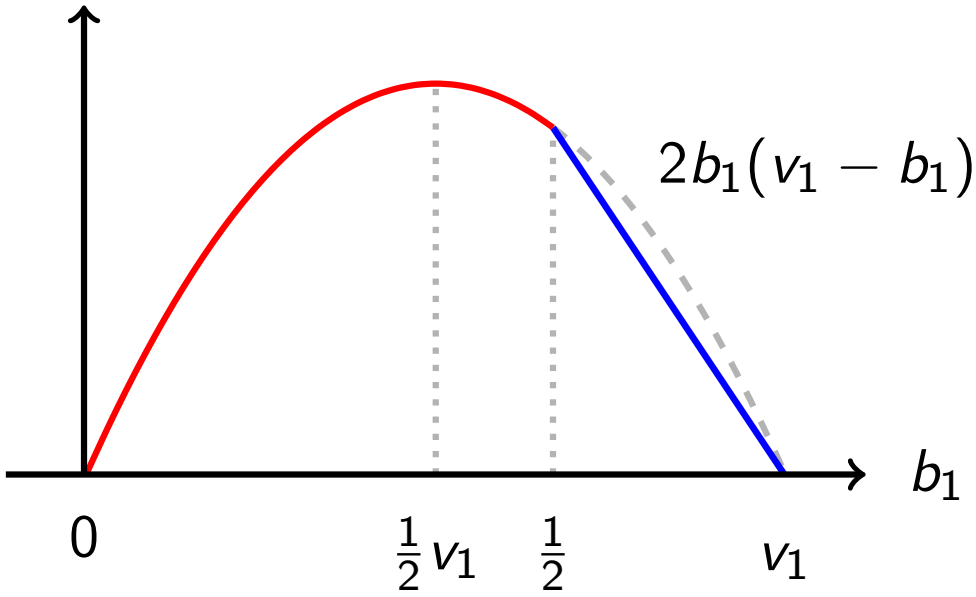


First-Price Sealed-Bid Auction: Example

Player 1's payoff given player 2's strategy:

$$\begin{cases} 2b_1(v_1 - b_1), & \text{if } 0 \le b_1 \le 1/2, \\ v_1 - b_1, & \text{if } b_1 > 1/2 \end{cases}$$

Case 1: $v_1 > 1/2$

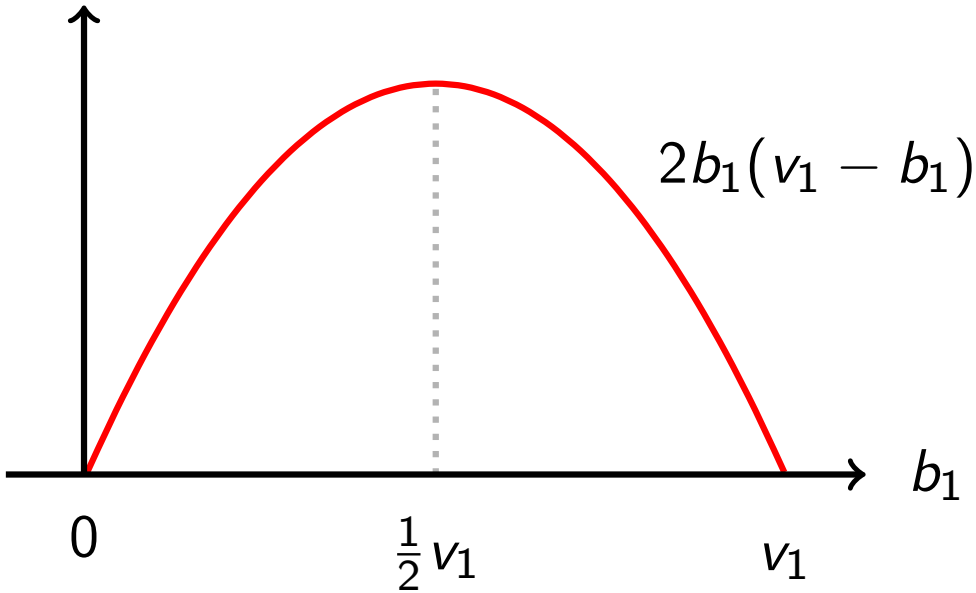


First-Price Sealed-Bid Auction: Example

Player 1's payoff given player 2's strategy:

$$\begin{cases} 2b_1(v_1 - b_1), & \text{if } 0 \leq b_1 \leq 1/2, \\ v_1 - b_1, & \text{if } b_1 > 1/2 \end{cases}$$

Case 2: $v_1 < 1/2$



First-Price Sealed-Bid Auction: Example

Summary:

- If bidder 2's strategy is to bid $v_2/2$, for each value v_2, then bidder 1's **best response** is to bid $v_1/2$, for each value v_1
- If bidder 1's strategy is to bid $v_1/2$, for each value v_1, then bidder 2's **best response** is to bid $v_2/2$, for each value v_2

Nash equilibrium:

- Each bidder bids half her value
- Good is sold to the bidder with the highest value

First-Price Sealed-Bid Auction: Example

Interpretation of equilibrium bids:

- Bidder i wins if and only if $0 \leq v_j \leq v_i$
- Expected value $\mathbb{E}[v_j | v_j \leq v_i]$ of v_j given $v_j \leq v_i$ is $v_i/2$
- Bidder i's bid is this expected value
- Bid expectation of other bidder's value conditional on winning

First-Price Sealed-Bid Auction: Generalizing the Example

Generalization:

- n bidders
- Any distribution of values
- Bid expectation of highest other value conditional on winning:

$$\mathbb{E}\left[X|X \leq v_i\right] \text{ in which } X = \max_{j \neq i} v_j$$

Comparison of Second-Price and First-Price Auctions

Second-price sealed-bid auction:

- Bidder with value v bids v
- Winner is bidder with highest value v^*
- Winner's expected payment is $\mathbb{E}[X|X < v^*]$

First-price sealed-bid auction:

- Bidder with value v bids $\mathbb{E}[X|X < v]$
- Winner is bidder with highest value v^*
- Winner pays $\mathbb{E}[X|X < v^*]$

Implication: Revenue Equivalence Theorem

Revenue equivalence: Under our assumptions, second-price and first-price auctions yield the same expected revenue

Note: One assumption is risk-neutrality, another is independence