ECO316: Applied Game Theory
Lecture 8

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Based on materials by Martin J. Osborne

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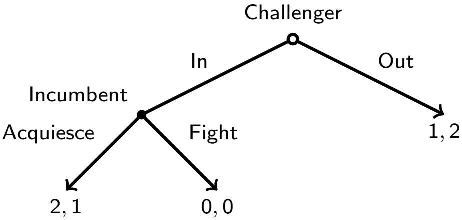
Extensive Games

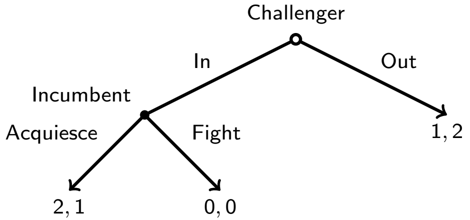
- Strategic game is not a natural model of **sequential choices**
- **Extensive games** explicitly model sequential chocies
- **Example**: Chess

Entry Game



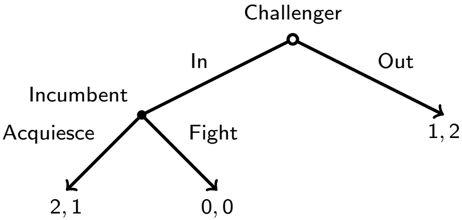
- **Two players**: Challenger and Incumbent
- Small circle denotes **start of game**
- First, Challenger chooses In or Out
  - **In**: Incumbent chooses Acquiesce or Fight, game ends
  - **Out**: Game ends
- Numbers at bottom are **payoffs** (Challenger's payoff first)

Histories



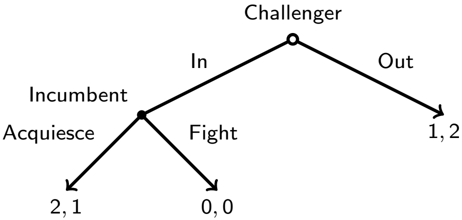
- **History**: Sequence of actions beginning at start of game
- **Entry game**:
  - $\emptyset$ (start of game)
  - In
  - Out
  - (In, Acquiesce)
  - (In, Fight)

Terminal Histories



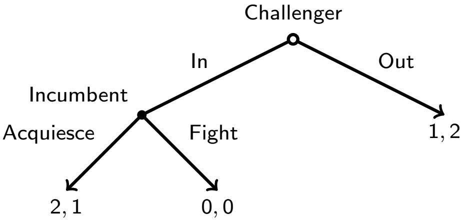
- **Terminal history**: History that reaches end of game
- **Entry game**:
  - Out
  - (In, Acquiesce)
  - (In, Fight)

Player Functions



- **Player function**: Assigns player to each non-terminal history
- **Entry game**:
  - $P(\emptyset) = \text{Challenger}$
  - $P(\text{In}) = \text{Incumbent}$

Preferences



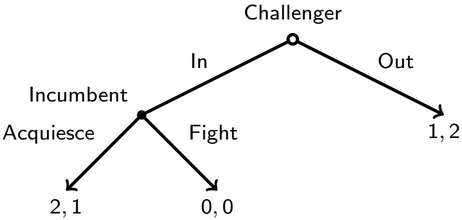
- **Preferences** are over terminal histories, for each player
- **Entry game**:
  - **Challenger**: (In,Acquiesce) $\succ$ Out $\succ$ (In,Fight)
  - **Incumbent**: Out $\succ$ (In,Acquiesce) $\succ$ (In,Fight)

Extensive Games

An **extensive game** is a model consisting of

- a set of **players**
- a set of **terminal histories**
- a **player function**
- **preferences** over terminal histories, for each player

Actions



- **Actions** are defined implicitly by terminal histories
- $A(h)$ denotes actions of player $P(h)$ after history h
- **Entry game**:
  - $A(\emptyset) = \{\text{In}, \text{Out}\}$
  - $A(\text{In}) = \{\text{Acquiesce}, \text{Fight}\}$

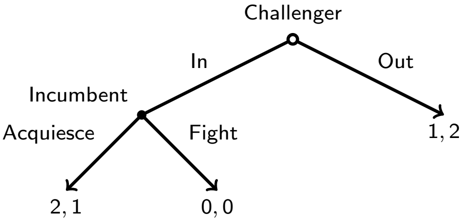
Strategies

A **strategy** of player i in an extensive game is a function that assigns an action in $A(h)$ to **every** non-terminal history h for which $P(h) = i$

To find a player's possible strategies:

- Make a list of all histories in which the player moves
- Choose a possible action for the player at each history
- Repeat until you have exhausted all possible combinations

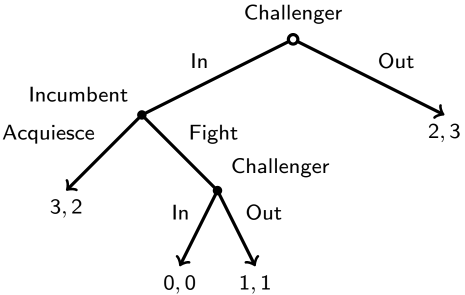
Strategies: Simple Example



Strategies:

- **Challenger**: In, Out (only moves after $\emptyset$)
- **Incumbent**: Acquiesce, Fight (only moves after In)

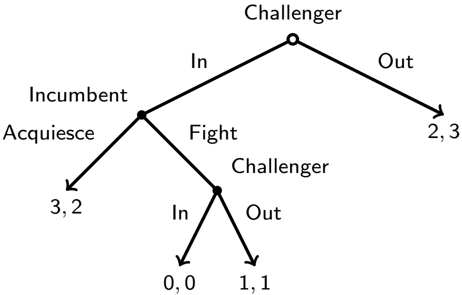
Strategies: Richer Example



Strategies:

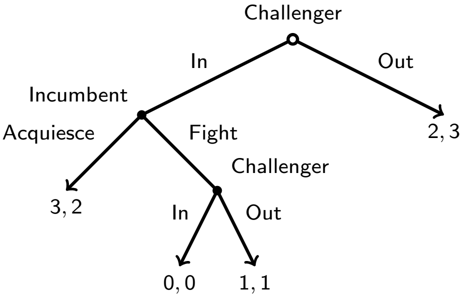
- **Challenger**: (In,In), (In,Out), (Out,In), (Out,Out)
- **Incumbent**: Acquiesce, Fight

Strategies: Interpretation of Actions



- Strategy must have action for each history with move
- **Example**: (Out,In) is a strategy for Challenger even though game ends after Challenger chooses Out at start
  - "In" models how Challenger would behave if she were to reach history (In,Fight) despite her intention to choose Out at start

Strategies: Another Interpretation of Actions



- Challenger thinks about action Incumbent intends to choose
- Challenger thinks about action Incumbent thinks Challenger intends to choose after history (In,Fight)
  - "In" in (Out,In) models Challenger's belief about Incumbent's belief about Challenger's action after history (In,Fight)

Number of Strategies

Extensive games can have a **large number** of possible strategies:

If a player moves after k histories and has m_1 possible actions after the first history, m_2 possible actions after the second, and so on, then she has $m_1 \cdot m_2 \cdots m_k$ possible strategies

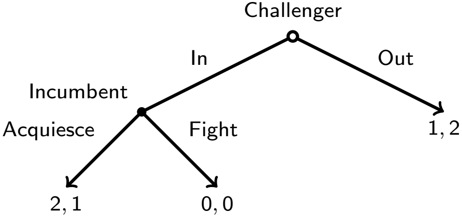
Strategic Form

Every extensive game induces a strategic game defined by

- **Players**: players in extensive game
- **Actions**: strategies in extensive game, for each player
- **Preferences**: preferences over terminal histories that result when players follow strategies associated with action profile

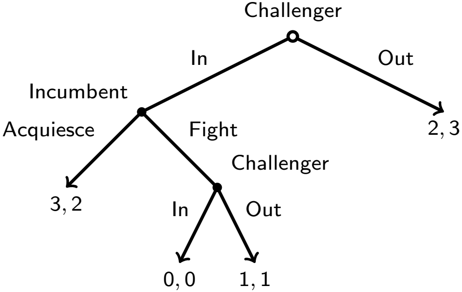
The resulting game is the **strategic form** of the extensive game

Strategic Form: Example



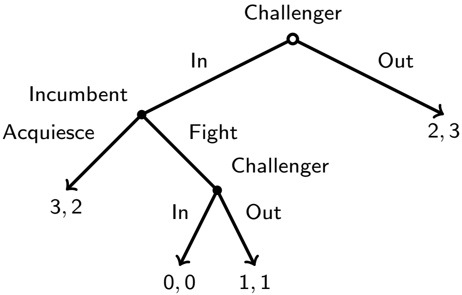
		Incumbent	
		Acquiesce	Fight
Challenger	In	$2, 1$	$0, 0$
	Out	$1, 2$	$1, 2$

Strategic Form: Example



		Incumbent	
		Acquiesce	Fight
Challenger	(In,In)	3, 2	0, 0
	(In,Out)	3, 2	1, 1
	(Out,In)	2, 3	2, 3
	(Out,Out)	2, 3	2, 3

Reduced Strategic Form: Example



Can remove duplicate actions to obtain a **reduced strategic form**:

Incumbent

Challenger		Acquiesce	Fight
	(In,In)	3, 2	0, 0
	(In,Out)	3, 2	1, 1
	X	2, 3	2, 3

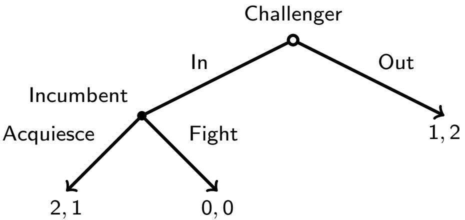
Nash Equilibrium

A **Nash equilibrium of an extensive game** is a Nash equilibrium of the strategic form of the extensive game

To find all Nash equilibria of an extensive game:

- Construct the strategic form of the extensive game
- Find all Nash equilibria of the resulting strategic form game

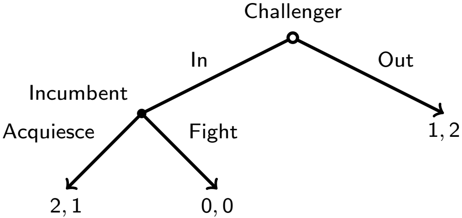
Nash Equilibrium: Example



		Incumbent	
		Acquiesce	Fight
Challenger	In	2, 1	0, 0
	Out	1, 2	1, 2

Nash Equilibria: (In,Acquiesce) and (Out,Fight)

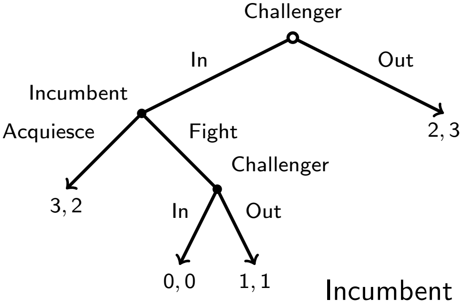
Nash Equilibrium: Example



Nash Equilibria:

- (In,Acquiesce):
  - Challenger uses In ⇒ Acquiesce optimal for Incumbent
  - Incumbent uses Acquiesce ⇒ In optimal for Challenger
- (Out,Fight):
  - Challenger uses Out ⇒ Any action optimal for Incumbent
  - Incumbent uses Fight ⇒ Out optimal for Challenger

Nash Equilibrium: Example

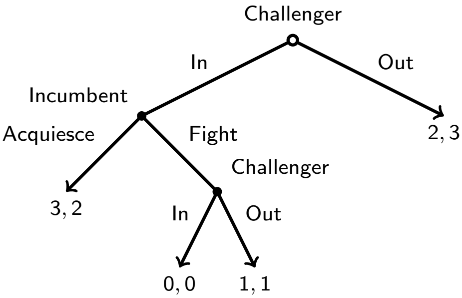


Incumbent

		Acquiesce	Fight
Challenger	(In,In)	3, 2	0, 0
	(In,Out)	3, 2	1, 1
	(Out,In)	2, 3	2, 3
	(Out,Out)	2, 3	2, 3

Nash Equilibria: {(In, In), Acquiesce}, {(In, Out), Acquiesce}, {(Out, In), Fight} {(Out, Out), Fight}

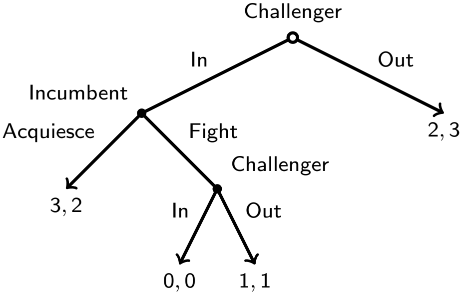
Nash Equilibrium: Example



Nash Equilibria:

- $\{(\text{In}, \text{In}), \text{Acquiesce}\}$:
  - Challenger uses (In,In) $\Rightarrow$ Acquiesce optimal for Incumbent
  - Incumbent uses Acquiesce $\Rightarrow$ $(\text{In}, \cdot)$ optimal for Challenger
- $\{(\text{In}, \text{Out}), \text{Acquiesce}\}$:
  - Challenger uses (In,Out) $\Rightarrow$ Acquiesce optimal for Incumbent
  - Incumbent uses Acquiesce $\Rightarrow$ $(\text{In}, \cdot)$ optimal for Challenger

Nash Equilibrium: Example



Nash Equilibria:

- $\{(\text{Out}, \text{Out}), \text{Fight}\}$:
  - Challenger uses (Out,Out) $\Rightarrow$ Any action opt. for Incumbent
  - Incumbent uses Fight $\Rightarrow$ $(\text{Out}, \cdot)$ optimal for Challenger
- $\{(\text{Out}, \text{In}), \text{Fight}\}$:
  - Challenger uses (Out,In) $\Rightarrow$ Any action optimal for Incumbent
  - Incumbent uses Fight $\Rightarrow$ $(\text{Out}, \cdot)$ optimal for Challenger

Subgame Perfect Equilibrium

Nash equilibrium:

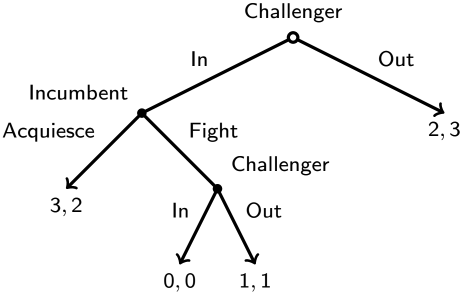
- Each player's strategy optimal given other players' strategies
- Actions only need to be optimal at histories that are reached

Subgame perfect equilibrium (SPE):

- Stronger notion of equilibrium
- Each player's action optimal at **every** history

Subgames

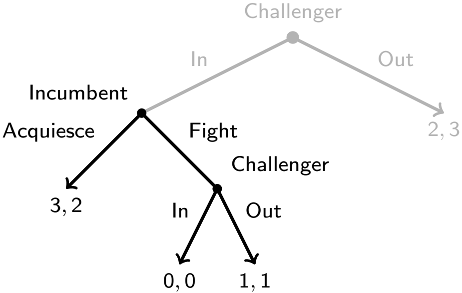
Subgame: Game following a non-terminal history
Number of Subgames: Number of non-terminal histories



Subgame after $\emptyset$

Subgames

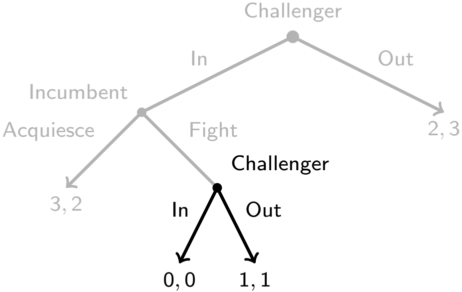
Subgame: Game following a non-terminal history
Number of Subgames: Number of non-terminal histories



Subgame after In

Subgames

Subgame: Game following a non-terminal history
Number of Subgames: Number of non-terminal histories



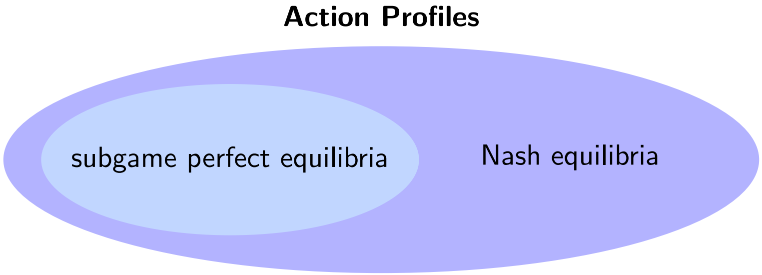
Subgame after (In,Fight)

Subgame Perfect Equilibrium

A strategy profile is a **subgame perfect equilibrium** (SPE) of an extensive game if every player i's strategy is optimal given the other players' strategies, in every subgame in which player i moves

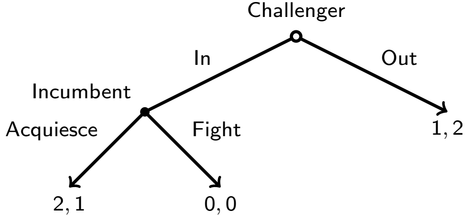
Strategy profile: A list of strategies with one strategy per player

Nash Equilibrium and Subgame Perfect Equilibrium



- Every subgame perfect equilibrium is a Nash equilibrium
- Not every Nash equilibrium is a subgame perfect equilibrium

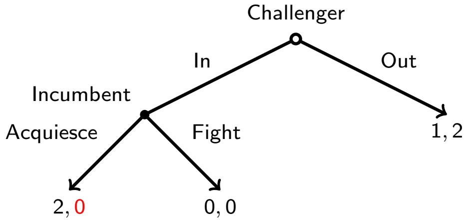
Subgame Perfect Equilibrium: Example



Nash Equilibria:

- **(In,Acquiesce) is an SPE**:
  - In is optimal for Challenger at start given Acquiesce
  - Acquiesce is optimal for Incumbent in subgame after In
- **(Out,Fight) is not an SPE**:
  - Out is optimal for Challenger at start given Fight
  - Fight is not optimal for Incumbent in subgame after In
  - Fight is a **non-credible threat**

Subgame Perfect Equilibrium: Another Example



Nash Equilibria:

- **(In,Acquiesce) is an SPE**:
  - In is optimal for Challenger at start given Acquiesce
  - Any action optimal for Incumbent in subgame after In
- **(Out,Fight) is an SPE**:
  - Out is optimal for Challenger at start given Fight
  - Any action optimal for Incumbent in subgame after In

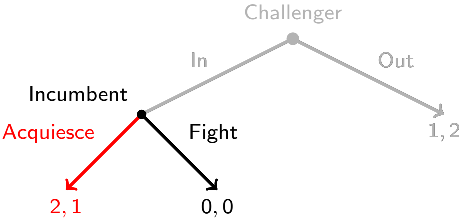
How to Find a Subgame Perfect Equilibrium

- **Finite horizon**: Length of longest terminal history is finite
- Find an SPE using **backward induction** if finite horizon game

Backward induction:

- Find optimal actions in every subgame of **length one**
- Fix one optimal action in each subgame of length one
- Find optimal actions in every subgame of **length two**
- Fix one optimal action in each subgame of length two
- Find optimal actions in every subgame of **length three**...

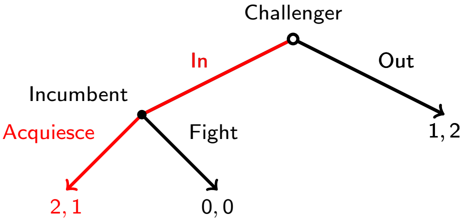
Backward Induction: Example



Backward induction:

- **Subgames of length one**:
  - Subgame after In: Acquiesce is optimal for Incumbent

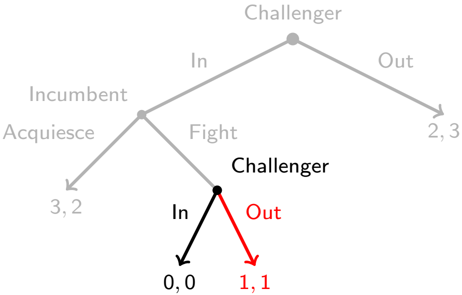
Backward Induction: Example



Backward induction:

- **Subgames of length one**:
  - Subgame after In: Acquiesce is optimal for Incumbent
- **Subgames of length two**:
  - Subgame after $\emptyset$: In is optimal for Challenger given Acquiesce
- **SPE**: (In,Acquiesce)

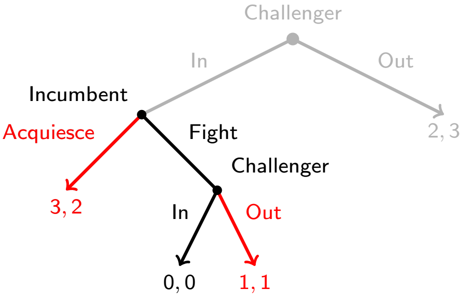
Backward Induction: Another Example



Backward induction:

- **Subgames of length one**:
  - Subgame after (In,Fight): Out is optimal for Challenger

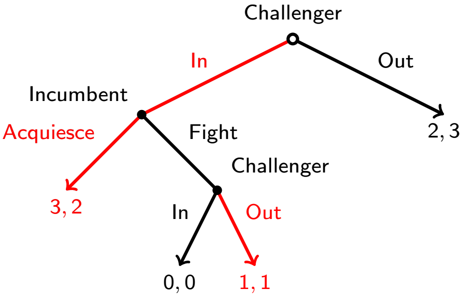
Backward Induction: Another Example



Backward induction:

- **Subgames of length one**:
  - Subgame after (In,Fight): Out is optimal for Challenger
- **Subgames of length two**:
  - Subgame after In: Acquiesce is optimal given Out

Backward Induction: Another Example



Backward induction:

- **Subgames of length three**:
  - Subgame after $\emptyset$: In is optimal given Out and Acquiesce
- **SPE**: $\{(\text{In}, \text{Out}), \text{Acquiesce}\}$