Computer Design of Super-Orthogonal Space-Time Trellis Codes

Michael Bale, Brady Laska, Dustin Dunwell, François Chan, and Hamid Jafarkhani

Abstract— Super-orthogonal space-time trellis codes (SOSTTCs) designed by hand can significantly improve the performance of space-time trellis codes. This paper introduces a new representation of SOSTTCs based on a generator matrix that allows a systematic and exhaustive search of all possible codes. This will verify that some of the known codes are optimal, and provides a means to easily implement encoders and decoders with a large number of states without relying on a graphical representation. New codes with up to 256 states that outperform previously known codes are presented.

Index Terms— Diversity, MIMO (Multiple-Input Multiple Output), space-time coding.

I. INTRODUCTION

SPACE-time trellis coding [1] provides a diversity gain and a coding gain to wireless communications systems employing multiple transmit antennas, thereby improving the error performance or the data rate of these systems. In [2], Jafarkhani and Seshadri introduced a new structure called super-orthogonal space-time trellis codes (SOSTTCs) which can yield an additional coding gain of more than 2 dB while providing the highest possible rate. By concatenating a space-time block coding scheme with an outer trellis code, the diversity gain of the space-time block code is maintained and a coding gain is realized. Since the trellis coding gain is achieved through redundancy, the signal set of the inner code must be expanded to maintain full transmission rate. Note that it is not the signal constellation that is expanded but the set of orthogonal matrices, i.e., the number of available orthogonal matrices is increased. In order to accomplish this, Jafarkhani and Seshadri proposed a parameterized class of space-time block codes.

Beginning with the original space-time block code of Alamouti [3], a class of orthogonal designs or transmission matrices for two transmit antennas was created as follows:

$$C(x_1, x_2, \theta) = \begin{pmatrix} x_1 e^{j\theta} & x_2 \\ -x_2^* e^{j\theta} & x_1^* \end{pmatrix} \qquad (1)$$

where x_1 and x_2 are selected by input bits. The first row corresponds to the symbols transmitted in time slot 1, the second row, to the symbols in time slot 2. The first column corresponds to the symbols transmitted by antenna 1, the second column to the symbols of antenna 2. By varying the rotation angle θ, multiple orthogonal block codes can be constructed, and a super-orthogonal code is formed from the union of these codes. For M-PSK signal constellations, the signals x_1 and x_2 can be represented by $e^{j\frac{2\pi nl}{M}}$, $l = 0, 1, ..., M - 1$ and θ can take on the values $\theta = 2\pi l'/M$, where $l' = 0, 1, ..., M - 1$, without expanding the signal set. To maximize the coding gain, the matrix sets are partitioned in a manner similar to Ungerboeck's method [4], but using the determinant criteria from [1] rather than Euclidian distance. Note that using the trace of the difference matrix would produce the same partitioning.

In addition to Jafarkhani and Seshadri's work [2], Siwamogsatham and Fitz [5], [6], and Ionescu [7] have also independently developed methods to expand the orthogonal matrix set. Siwamogsatham and Fitz's method applies a similar unitary transformation to the original orthogonal design to produce additional sets and Ionescu uses cosets which are equivalent to the rotations of [2]. For the QPSK signal constellation, the A, J, B and K sets of Siwamogsatham and Fitz correspond (respectively) to the $\theta = 0, \pi/2, \pi$, and $3\pi/2$ rotations in [2] and the two cosets in [7], [8] correspond to the $\theta = 0$ and π rotations.

The trellises presented in [2], [5]-[6] are all hand-designed by systematically following set partitioning rules to maximize the determinant of the difference matrix, while those in [7] also attempt to maximize a criterion based on the trace of the difference matrix. The set partitioning rules are similar to Ungerboeck's rules [4] but use the determinant instead of the Euclidean distance. While the hand-designed codes perform well, it is not known if they are optimal. For a large number of states, there are many possible trellis structures (assignments of sets to different states) and assigning these sets by hand may yield a code which is not optimal. This is one of the motivations for our work. Also, hand-designing codes for a large number of states becomes tedious. Furthermore, these codes are described using a graphic representation of the trellis, where all possible outputs for each state are listed [2], [5]-[8]. Hence, implementing the encoder or decoder requires listing the outputs of all the states, which can become quite cumbersome for a large number of states.

This paper presents a method for performing a computer search of codes. Using a compact matrix notation, similar to that used for space-time trellis codes [9] a computer search is performed to find trellises that maximize both the determinant and trace criteria. The assignment of sets to the states is done

$C(x_1, x_2, 0)$ $S_{000}\ S_{001}\ S_{011}\ S_{010}\ S_{111}\ S_{110}\ S_{101}\ S_{100}$

$C(x_1, x_2, \pi/2)$ $S_{110}\ S_{111}\ S_{100}\ S_{101}\ S_{000}\ S_{001}\ S_{011}\ S_{010}$

$C(x_1, x_2, \pi)$ $S_{001}\ S_{000}\ S_{010}\ S_{011}\ S_{110}\ S_{111}\ S_{100}\ S_{101}$

$C(x_1, x_2, 3\pi/2)$ $S_{111}\ S_{110}\ S_{101}\ S_{100}\ S_{001}\ S_{000}\ S_{010}\ S_{011}$

$C(x_1, x_2, 0)$ $S_{010}\ S_{011}\ S_{001}\ S_{000}\ S_{100}\ S_{101}\ S_{110}\ S_{111}$

$C(x_1, x_2, \pi/2)$ $S_{101}\ S_{100}\ S_{111}\ S_{110}\ S_{010}\ S_{011}\ S_{001}\ S_{000}$

$C(x_1, x_2, \pi)$ $S_{011}\ S_{010}\ S_{000}\ S_{001}\ S_{101}\ S_{100}\ S_{111}\ S_{110}$

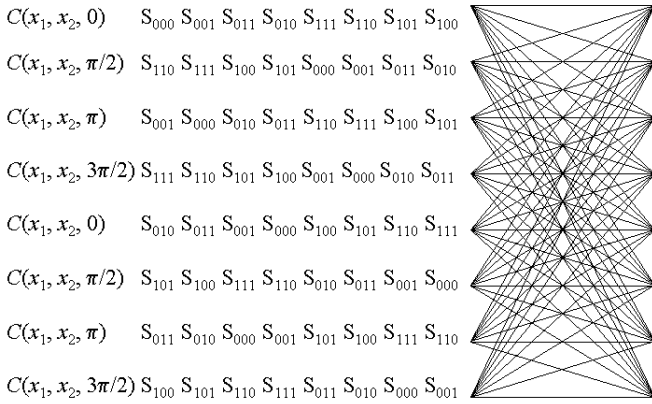$C(x_1, x_2, 3\pi/2)$ $S_{100}\ S_{101}\ S_{110}\ S_{111}\ S_{011}\ S_{010}\ S_{000}\ S_{001}$

Fig. 1.   8-state SOSTTC with 2 parallel branches and a rate of 2 bits/s/Hz using QPSK.

by computer. Simulations are then performed to compare the performance of these new codes to that of the known ones. The matrix representation allows a more compact representation of the code, making the implementation of the encoder and decoder easier.

The organization of the paper is as follows. Section II describes the representation of SOSTTCs using a generator matrix. Section III discusses the details of the code search and Section IV provides simulation results. Section V includes some concluding remarks.

II. MATRIX REPRESENTATION

Maximizing the determinant and trace produced by a SOSTTC has proven effective in the design of optimal codes with a small number of states. However, as the number of states in the code increases, the complexity involved in the design of these codes increases proportionally and it becomes more difficult to produce optimal codes. In order to facilitate this, we show in this section that the code can be represented by a simple generator matrix, allowing for a systematic and exhaustive computer search of all possibilities. This matrix representation is similar to that used for the search of space-time trellis codes [9].

Fig. 1 shows a simple example of the graphical representation of an 8-state QPSK SOSTTC for a rate of 2 bits/s/Hz; this trellis has two parallel branches. Using the set partitioning for QPSK in [2], S_{000} means that x_1, x_2 in the transmission matrix (1) are equal to 0, 0 and 2, 2, respectively for the first and the second parallel branch. It can be seen that as the number of states gets larger, describing the code graphically becomes increasingly cumbersome.

In general, the generator matrix $\mathbf{G}$ of a full-rate [2] SOSTTC for n_T transmit antennas is of the form r rows by n_T columns, where r is determined by the sum of the number of input bits m at each trellis level and the number of bits s needed to represent each state. Note that in this example each trellis level corresponds to a transmission matrix (1) and hence, a trellis level corresponds to two time slots. s is given by

$$s = \log_2 N$$

where N is the number of states. For an M-PSK or M-QAM constellation with the highest possible rate, m is given by

$$m = 2\log_2 M$$

As an example, an 8-state code requires three bits to represent each state. For QPSK with a rate of 2 bits/s/Hz and 2 transmit antennas, four bits are input at each trellis level. Hence, the generator matrix has 7 rows and 2 columns, as shown below

$$\mathbf{G} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{13} & a_{14} \end{pmatrix} \tag{2}$$

where $a_i = 0, 1, 2$ or 3, and $i = 1, 2, ..., 14$. For an M-PSK modulation, $a_i = 0, 1, ..., M-1$.

Let the n-bit information sequence to be transmitted be $\mathbf{u} = (u_1, ..., u_n)$, where $u_i = 0$ or 1. The number of bits influencing the transmission matrix (1) of a given trellis level is equal to the number of rows of $\mathbf{G}$ or $(m + s)$. The m bits come from the information sequence $\mathbf{u}$ and the s bits represent the current state. Let these $(m + s)$ bits be represented by a vector $\mathbf{u}^l$, where l is the trellis level. When $\mathbf{u}^l$ is multiplied (modulo 4 or in the general case, modulo M) by $\mathbf{G}$, we obtain two symbols x_1 and x_2 which are then mapped to a transmission matrix using the mapping scheme described by (1). The rotation θ depends on the current state and is determined in advance. As an example, if two rotations are used, all transitions originating from an odd state are assigned $\theta = 0$, and those from an even state, $\theta = \pi$. Hence, all possible outputs or transmission matrices of the SOSTTC can be obtained from $\mathbf{u}^l\mathbf{G}$. There is no need to list all possible outputs from each state, making the implementation of the encoder and decoder much easier.

The difference between this new representation of SOSTTCs and Baro et al.'s representation of space-time trellis codes [9] is that the two symbols x_1 and x_2 are first mapped to the transmission matrix (1) whereas in [9] the two symbols are simply transmitted. Furthermore, in this paper, the optimal set-partitioning of [2] is used and the two symbols select the subset that will be assigned to a given branch. Hence, the assignment of sets to the branches or states is done by computer search and the computer search is used to determine which generator matrix $\mathbf{G}$ follows the set-partitioning rules. No set-partitioning rule is used for space-time trellis codes.

If the number of transmit antennas exceeds two and the code is not full-rate the number of columns of $\mathbf{G}$ is no longer equal to n_T but to the number of symbols used in the transmission matrix. As an example, for the rate-3/4 SOSTTC [2], three symbols x_1, x_2 and x_3 are used and hence, $\mathbf{G}$ has three columns. The generator matrix for the code in Fig. 1 is

$$\mathbf{G^T} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 2 & 3 \\ 2 & 3 & 0 & 2 & 3 & 0 & 2 \end{pmatrix}$$

In this example of an 8-state code with two parallel branches, the bits of $\mathbf{u}^l$ can be considered as follows

$$\mathbf{u}^l = \begin{bmatrix} \underbrace{\text{Next State}}_{u_7\ u_6\ u_5} & \underbrace{\text{Parallel Branch}}_{u_4} & \underbrace{\text{Current State}}_{u_3\ u_2\ u_1} \end{bmatrix} \tag{3}$$

Upon moving to the next transition, four new input bits are shifted into the $\mathbf{u}^l$ vector. The next state is determined
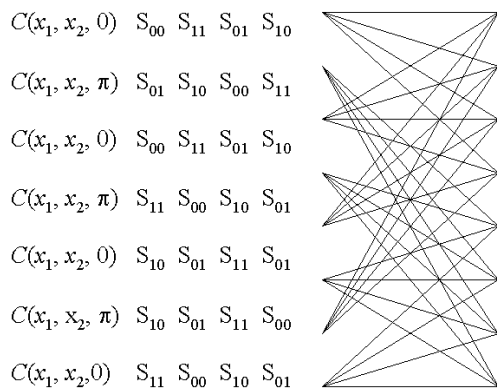
Fig. 2. 8-state SOSTTC with 4 parallel branches and a rate of 2 bits/s/Hz using QPSK.

from the current state and the input bits by using a lookup table. Note that in the previous example only the bits $u_7 u_6 u_5$, labeled 'Next State' in (3), are required to determine the next state. Bit u_4 specifies the parallel branch and the next state does not depend on its value. At the next trellis level, four new input bits are shifted into $\mathbf{u}^l$ and the three bits representing the next state that have been obtained from the lookup table become the current state bits of $\mathbf{u}^l$. In general, these three bits are not equal to $u_7 u_6 u_5$ except in the special case where there is a possible transition from a state to any other state as in Fig. 1. The description of the bits of $\mathbf{u}^l$ given in (3) is different when the number of parallel branches changes. For example, if an 8-state SOSTTC with 4 parallel branches is desired as shown in Fig. 2, a vector $\mathbf{u}^l$ with the same number of bits can be used but bits $u_5 u_4$ now represent the parallel branch and in the lookup table, the next state depends on the current state and bits $u_7 u_6$. Fig. 2 shows the SOSTTC with 4 parallel branches and 2 rotations obtained using

$$\mathbf{G}^T = \begin{pmatrix} 3 & 3 & 2 & 0 & 3 & 3 & 1 \\ 1 & 2 & 0 & 2 & 0 & 3 & 3 \end{pmatrix}$$

By combining the approach of matrix multiplication with a lookup table, every possible 8-state trellis shape and permutation can be generated and tested. The lookup table is rather compact compared to what would be required to describe the whole trellis and can be determined in advance. As an example, it can be seen from Fig. 2 that the next states are simply assigned in a regular fashion: the next states of state 0 are the first four states, those of state 1 are the next four and so on.

This idea can easily be extended to generate a code with any number of states or antennas for any signal constellation by increasing r and the size of the vector $\mathbf{u}^l$ and changing the number of columns to the number of symbols used in the transmission matrix. Hence, the matrix representation presented earlier is an efficient and compact way to describe a SOSTTC. Furthermore, it allows a computer design of a SOSTTC with a large number of states by performing an exhaustive or random search of all possible generator matrices $\mathbf{G}$ and selecting the matrix $\mathbf{G}$ which yields the best characteristics.

III. CODE SEARCH

An exhaustive search among all possible 4^{14} or 268,435,456 matrices $\mathbf{G}$ for 8-state SOSTTCs has yielded thousands of codes with good characteristics (largest determinant and trace) which exhibit several improvements in both the minimum determinant and minimum trace compared to the codes in [5], [7]. There has been no SOSTTC which presented the largest determinant but not the largest trace. Many of these codes are equivalent, i.e., they generate the same outputs. The greater the number of parallel branches in the trellis, the more duplicate codes exist. For example, for 2-parallel branch trellises, a code whose generator matrix starts with [2,3], i.e., $[a_1, a_2] = [2, 3]$, is a duplicate of a code that starts with [0,1] when the remaining elements of $\mathbf{G}$ of the first code are equal to those of the second one; if the product $\mathbf{u}^l \mathbf{G}$ results in (x_1, x_2) for the code starting with [0,1], the output with $[a_1, a_2] = [2, 3]$ would be identical if $u_7 = 0$ and $(x_1 + 2, x_2 + 2)$ if $u_7 = 1$. Hence, the transmission matrices of their parallel branches are simply switched as can be seen from Jafarkhani and Seshadri's set-partitioning for QPSK [2]. Likewise, codes that begin with [2,0], [2,1], [2,2], [3,0], [3,1], [3,2] and [3,3] are duplicates of codes that begin with [0,2], [0,3], [0,0], [1,2], [1,3], [1,0] and [1,1] respectively, i.e., the codes that begin with the pairs of indexes (S_{000}) [2] are duplicates. Similarly, for 4-parallel branch trellises, all codes that begin with S_{00} are duplicates. The matrices that produce known duplicates were not considered, but there still exist some duplicate codes. The duplicate ones have been determined using a computer search and eliminated. The remaining codes have been simulated at a signal-to-noise ratio of 14 dB since the error probabilities of space-time codes usually do not crossover. In fact since all of the codes provide full diversity, the SNR-BER curve is a line at high SNRs (in a logarithmic scale) and therefore one point in the SNR-BER plane is good enough for comparison.

Exhaustive searches have also been performed for 16 and 32-state codes. All possible numbers of parallel branches and rotations have been considered. For 64, 128 and 256 states, an exhaustive search would have been prohibitively complex and extensive random searches have been conducted instead. The number of codes examined range from tens of million for 64 states to more than 1.4 billion codes for 256 states. Thousands of codes with good determinant and trace are then simulated. The new codes with the best error performance are presented in Table I. Some of these best codes, indicated by * in Table I, do not have the largest minimum determinant and trace but they yield the best error performance because the number of pairs of codewords with this determinant may be smaller. As expected, the best codes with 64 and more states have no parallel branch since parallel transitions would limit the largest determinant or trace that these codes can achieve. The performance of the new codes is presented in Section IV.

IV. SIMULATION RESULTS

Maximum-likelihood decoding is employed at the receiver end to determine the transmitted signal. The decision metric was derived using the same method as in [10], and using Jafarkhani and Seshadri's [2] transmission matrix given by (1). The orthogonality of the transmission matrices makes it

TABLE I
BEST NEW QPSK SUPER-ORTHOGONAL CODES

Number of States	Number of Parallel Branches	Generator Matrix		Minimum Determinant, Trace
8	8	$\mathbf{G}^T =$	$\begin{pmatrix} 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$	80, 20 (4 transitions)
16	4	$\mathbf{G}^T =$	$\begin{pmatrix} 1 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$	52, 16 (3 transitions)
32	2	$\mathbf{G}^T =$	$\begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 3 \end{pmatrix}$	28, 12 (2 transitions) 56, 16 (3 transitions)
64	0	$\mathbf{G}^T =$	$\begin{pmatrix} 3 & 2 & 2 & 3 & 2 & 1 & 2 & 3 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 & 0 & 1 & 2 & 3 & 1 \end{pmatrix}$	100, 20 (2 transitions) 32*, 12* (3 transitions)
128	0	$\mathbf{G}^T =$	$\begin{pmatrix} 0 & 3 & 0 & 2 & 2 & 1 & 3 & 3 & 2 & 2 & 3 \\ 2 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 2 & 3 & 0 \end{pmatrix}$	100*, 20* (2 transitions)
256	0	$\mathbf{G}^T =$	$\begin{pmatrix} 1 & 2 & 0 & 0 & 2 & 3 & 2 & 0 & 2 & 0 & 2 & 3 \\ 0 & 3 & 1 & 2 & 0 & 3 & 3 & 2 & 2 & 1 & 0 & 2 \end{pmatrix}$	100*, 20* (3 transitions)

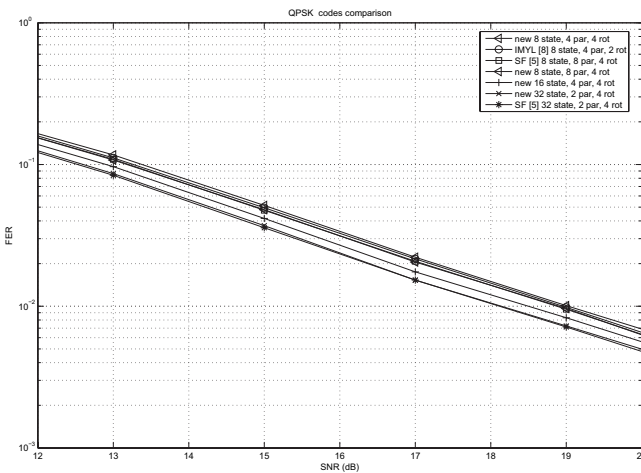*Codes with larger minimum determinant and trace exist but their error performance is poorer.

Fig. 3.   Comparison of the frame error rates of 8-state and 32-state codes - 'rot' (rotation angle) represents the number of values of θ in (1).

Fig. 4.   Performance of the new best QPSK codes.

possible to simplify the decoding of SOSTTCs. In fact, as shown in [12], the decoding complexity of SOSTTCs is much lower than that of the original space-time trellis codes. The decoding is described in detail in [11] and [12].

Computer simulations have been performed to determine the performance of our codes and select the best ones. Each frame consists of 130 transmissions out of each transmit antenna and one receive antenna is used.

The performance of our new codes is compared to that of some known codes from [5], [7] in Fig. 3. The 8-state codes yield little or no improvement on already published codes [5], [7]. This would indicate that the known 8-state codes are optimal. This may be due to the fact that the known codes and the new codes with 4 and 8 parallel branches have the same determinant for parallel branches. The performance of the known 16-state codes is almost similar to that of the 8-state codes as reported in [5], [7] while our new 16-state codes offers more than 0.4 dB improvement over the best 8-state code. This shows the advantage of a computer design. The
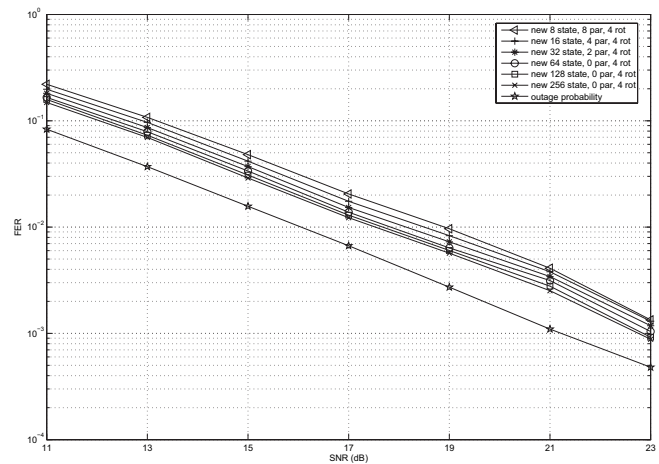
exhaustive search for 32-state codes did not yield a better code as can be seen from Fig. 3, which means that the known code [5] was already very good.

Simulations have shown that, for a given number of parallel branches, 4 rotations consistently outperform 2 rotations. The error performance of the best new codes is shown in Fig. 4. We can see that the performance improves with an increasing number of states and the 256-state code is about 1.5 dB away from the outage capacity.

V. CONCLUSIONS

A matrix representation of super-orthogonal space-time trellis codes has been presented in this paper. This representation allows a more compact description of SOSTTCs which does not require the listing of all possible outputs from each state and therefore, simplifies the implementation of encoders and decoders for codes with a large number of states. Furthermore, this representation can be used for a computer search of good SOSTTCs. New codes which match or outperform previously known codes have been presented. This computer search can

also be extended to super-quasi-orthogonal space-time trellis codes [13].

REFERENCES

[1] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance analysis and code construction," *IEEE Trans. Inf. Theory*, vol. 44, pp. 744-765, Mar. 1998.

[2] H. Jafarkhani and N. Seshadri, "Super-orthogonal space-time trellis codes," *IEEE Trans. Inf. Theory*, vol. 49, pp. 937-950, Apr. 2003.

[3] S. M. Alamouti, "A simple transmitter diversity scheme for wireless communications," *IEEE J. Sel. Areas Commun.*, vol. 16, pp. 1451-1458, Oct. 1998.

[4] G. Ungerboeck, "Channel coding for multilevel/phase signals," *IEEE Trans. Inf. Theory*, vol. 28, pp. 55-67, Jan. 1982.

[5] S. Siwamogsatham and M. P. Fitz, "Improved high-rate space-time codes via orthogonality and set partitioning," in *Proc. IEEE Wireless Communications and Networking Conf. (WCNC) 2002*.

[6] S. Siwamogsatham and M. P. Fitz, "Improved high-rate space-time codes via concatenation of expanded orthogonal block code and M-TCM," in *Proc. IEEE Int. Conf. Commun. (ICC) 2002*, vol. 1, pp. 636-640.

[7] D. M. Ionescu, "On space-time code design," *IEEE Trans. Wireless Commun.*, vol. 2, pp. 20-28, Jan. 2003.

[8] D. M. Ionescu, K. K. Mukkavilli, Z. Yan, and J. Lilleberg, "Improved 8- and 16-state space-time codes for 4PSK with two transmit antennas," *IEEE Commun. Lett.*, vol. 5, pp. 301-305, July 2001.

[9] S. Baro, G. Bauch, and A. Hansmann, "Improved codes for space-time trellis coded modulation," *IEEE Commun. Lett.*, vol. 4, pp. 20-22, Jan. 2000.

[10] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block coding for wireless communications: Performance results," *IEEE J. Sel. Areas Commun.*, vol. 17, pp. 451-460, Mar. 1999.

[11] B. Laska, D. Dunwell, F. Chan, and H. Jafarkhani, "Computer design of super-orthogonal space-time trellis codes," in *Proc. IEEE CCECE (Canadian Conf. on Elec. and Comp. Eng.) 2004*.

[12] H. Jafarkhani, *Space-Time Coding: Theory and Practice*. Cambridge University Press, 2005.

[13] H. Jafarkhani and N. Hassanpour, "Super-quasi-orthogonal space-time trellis codes for four transmit antennas," *IEEE Trans. Wireless Commun.*, vol. 4, pp. 215-227, Jan. 2005.