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Nash Equilibrium Seeking By a Gossip-Based Algorithm

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Abstract— **In this paper an asynchronous gossip-based algorithm is proposed for finding a Nash equilibrium of a game in a distributed multi-player network. The algorithm is designed in such a way that the players' actions are updated based on the estimates of the other players' actions which are obtained from the local neighbors. The almost sure convergence proof of the algorithm to a Nash equilibrium is provided under a set of standard assumptions on the cost functions and the communication graph. The effectiveness of the proposed algorithm is demonstrated via simulation.**

I. INTRODUCTION

The problem of finding a Nash equilibrium of a game in a distributed multi-player network has received increasing attention in recent years. Many important real-world applications in wired and wireless networks involve such a setup [1]. In this problem each player pursues the minimization of his cost function myopically by taking a proper action in response to other players' actions. Thus each player requires the full information of all other players' actions in the network. However, in a distributed network this is a stringent requirement. Players have to minimize their cost functions based on the limited local information received from the neighboring players. Peer-to-peer (P2P) and mobile ad-hoc networks are two of the best examples of such networks.

Our goal is to design a locally distributed algorithm to find a Nash equilibrium of a game over a network. All the players share their information locally and update their actions in order to minimize their cost functions according to the limited information. The reason that we are interested in locally distributed algorithms is the players' limited memory, the restricted power source and also the significance of the communication overhead. This could cause an expensive start-up cost and network latencies [2].

Literature review. Our work is related to the literature on Nash games [3], [4], and [5]. Distributed algorithms for computing Nash equilibria have recently received significant attention due to a wide range of applications, to name only a few [6], [5], [7], [8], and [9]. A distributed algorithm is proposed in [10] for a class of generalized games and convergence to the Nash equilibrium is studied for a complete communication graph under data transmission delays and dynamical changes of network topologies. For a two-network zero-sum game, [11] considers a distributed algorithm for Nash equilibrium seeking. Players of each network collaborate with the players of their network but they can only have access to the partial information of the other network. Another type of distributed algorithm is designed in [12], which takes advantage of state definition as an extra variable in the game. It is shown that the resultant game is a state-based potential game for which there exist distributed algorithms that guarantee convergence to an equilibrium point. In [13], an iterative regularization algorithm is studied for capturing the equilibria of a monotone game. The authors in [14] have mainly worked on a distributed modified fictitious play algorithm which converges to a subset of the mixed strategy Nash equilibria.

Gossip-based communication has been widely used in asynchronous algorithms due to simplicity and applicability [15], and [16]. In [17], a gossip-based algorithm has been designed for finding a Nash equilibrium in a special class of games called *aggregative games*. In an aggregative game each player's cost function is coupled through aggregate of actions of all players. The players share the estimates of the aggregate to update their actions. Since the algorithm is designed for aggregative games there is no need to estimate the other players' actions. However, the aggregate of the actions is not enough to update players' actions in a general game.

Contributions. Inspired by [17], we propose an asynchronous gossip-based algorithm for a larger class of games. In this algorithm each player maintains an estimate vector as his guess about the players' actions except itself. Then a communication protocol is designed for sharing the estimates between the local players to update their estimates and actions. Particularly, a player randomly wakes up, selects a neighboring player and exchanges his estimates to update his action. We prove that the algorithm converges almost surely toward a Nash equilibrium of the game under a set of standard assumptions on the cost functions and communication graph. In contrast to [17], in which the players take average of the scalar aggregate estimate including the actions themselves, in our algorithm we exclude their own actions from the estimates. While this exclusion is appropriate in a generalized game context, it precludes exploiting doubly stochastic properties in the gossiping step. However, this is overcome by using an extra intermediary variable.

The remainder of the paper is organized as follows. In Section II, the problem statement and assumptions are provided. Asynchronous gossip-based algorithm is then proposed in Section III. In Section IV, the convergence proof is discussed in detail. Simulation results are presented in Section V to demonstrate the effectiveness of the proposed algorithm and finally, concluding remarks are drawn in Section VI.

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II. Problem Statement

A. Nash Game

Consider a set of N players in a network specified by a graph $G(V, E)$ where $V = \{1, \dots, N\}$ denotes the set of players and $E \subset V \times V$ denotes the set of possible interactions between the players. For $i \in V$, $J_i : \Omega \to \mathbb{R}$ is the cost function of player i where $\Omega = \Omega_1 \times \ldots \times \Omega_i \times \ldots \times \Omega_N \subset \mathbb{R}^N$ is the action set of all players and $\Omega_i \subset \mathbb{R}$ is the action set of player i. The Nash game denoted by $\mathcal{G}(V, \Omega_i, J_i)$ is defined based on the set of players, V, the action set of player i, Ω_i and the cost function of player i, J_i.

Let $x = [x_1, \dots, x_i, \dots, x_N]^T \in \Omega$, with $x_i \in \Omega_i$, denote all players' actions. For simplicity, one can represent $x = (x_i, x_{-i})$ and $\Omega = \Omega_i \times \Omega_{-i}$ where $x_{-i} = [x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N]^T$ and $\Omega_{-i} = \Omega_1 \times \ldots \times \Omega_{i-1} \times \Omega_{i+1} \times \ldots \times \Omega_N$. The cost function J_i depends on all (x_i, x_{-i}). The game is played such that each player i aims to minimize his own cost function myopically to find an optimal action,

$$\begin{aligned} \underset{y_i}{\text{minimize}} \quad & J_i(y_i, x_{-i}) \\ \text{subject to} \quad & (y_i, x_{-i}) \in \Omega. \end{aligned} \tag{1}$$

We assume that the cost function J_i and the action set Ω_i are only available to player i, for each $i \in V$. So the players are required to exchange some information to update their actions.

A pure-strategy Nash equilibrium (or simply a Nash equilibrium) of a game is defined in the following.

Definition 1. *Consider an N-player game where each player i minimizes the cost function $J_i : \Omega \to \mathbb{R}$. A vector $x^* = (x_i^*, x_{-i}^*) \in \Omega$ is called a Nash equilibrium of this game if for every x_{-i}^*, we have:*

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*) \quad \forall x_i \in \Omega_i,\ \forall i \in V. \tag{2}$$

Consider a graph $G_C(V, E_C)$ as a *communication graph* where $E_C \subset E$ specifies the pairs of players that may communicate.

Definition 2. *The set of neighbors of player i in G_C, denoted by $N_C(i)$, is a set consisting of any vertex in $G_C(V, E_C)$ which is connected to vertex i by an edge, i.e., $N_C(i) := \{j \in V | (i, j) \in E_C\}$.*

Assumption 1. *The communication graph $G_C(V, E_C)$ is connected and undirected.*

The connectivity assumption is critical in order to ensure that the information of each player is reached by all other players, frequently.

Our objective is to find an algorithm for computing a Nash equilibrium of $\mathcal{G}(V, \Omega_i, J_i)$ over the communication graph $G_C(V, E_C)$ using only partial or imperfect information. The convergence needs to be proved under some assumptions on the cost functions and the communication graph.

B. Variational Inequality Problem Definition

The computation of a Nash equilibrium can be efficiently done by solving a *variational inequality* [18].

Definition 3. *Given $\Omega \subset \mathbb{R}^N$ and a mapping $F : \Omega \to \mathbb{R}^N$, a variational inequality problem, which is denoted by $VI(\Omega, F)$, is to determine a vector $x \in \Omega$ such that*

$$F(x)^T (y - x) \geq 0, \quad \forall y \in \Omega. \tag{3}$$

The set of solutions of this problem is represented by $SOL(\Omega, F)$.

The following proposition from [18], Proposition 1.4.2, gives a set of necessary and sufficient conditions under which a Nash equilibrium can be obtained by solving the associated variational inequality problem.

Proposition 1. *Let Ω_i be a closed convex subset of $\mathbb{R}$ for $i \in V$. Let also for $i \in V$, function $J_i(y_i, x_{-i})$ be convex and continuously differentiable in y_i for each fixed x_{-i}. Then a tuple $x^* = [x_1^*, \dots, x_i^*, \dots, x_N^*]^T$ is a Nash equilibrium if and only if $x^* \in SOL(\Omega, F)$, where*

$$\Omega = \Omega_1 \times \ldots \times \Omega_i \times \ldots \times \Omega_N, \quad F(x) = \begin{bmatrix} \nabla_{x_1} J_1(x) \\ \vdots \\ \nabla_{x_i} J_i(x) \\ \vdots \\ \nabla_{x_N} J_N(x) \end{bmatrix}.$$

The mapping $F : \Omega \to \mathbb{R}^N$ is called a pseudo-gradient mapping.

Using Proposition 1, one can find a characterization of a Nash equilibrium in a variational inequality problem. The following lemma is from [18], Proposition 1.5.8, page 83.

Lemma 1. *x^* is a Nash equilibrium of the game represented by (1) if and only if*

$$x^* = T_\Omega[x^* - \alpha F(x^*)] \tag{4}$$

for any given $\alpha > 0$ where F is defined in Proposition 1 and $T_\Omega : \mathbb{R}^N \to \Omega$ is an Euclidean projection.

A set of relatively mild conditions is stated in the following under which existence of pure Nash equilibrium is guaranteed.

Assumption 2. *The set Ω_i is non-empty, compact and convex subset of $\mathbb{R}$ for every $i \in V$. The cost function of player i, $J_i : \Omega \to \mathbb{R}$ is a continuously differentiable function for every $i \in V$. Also $J_i(x_i, x_{-i})$ is jointly continuous and convex in x_i for every x_{-i} and $i \in V$.*

By Assumption 2, one can also conclude that there exists $C > 0$ such that for all $i \in V$ and for all $x \in \Omega$, we have

$$\|\nabla_{x_i} J_i(x)\| \leq C. \tag{5}$$

In the following, some other assumptions are made including the monotonicity condition which ensures the uniqueness of Nash equilibrium.

Assumption 3. *The pseudo-gradient vector $F : \Omega \to \mathbb{R}^N$ is strictly monotone on Ω, i.e.,*

$$(F(x) - F(y))^T(x - y) > 0 \quad \forall x, y \in \Omega,\ x \neq y.$$

Assumption 4. *$\nabla_{x_i} J_i(x_i, u)$ is Lipschitz continuous in u, for every fixed $x_i \in \Omega_i$ and for every $i \in V$, that is, for some positive constant L_i and for every $i \in V$, we have*

$$\|\nabla_{x_i} J_i(x_i, u) - \nabla_{x_i} J_i(x_i, z)\| \leq L_i\|u - z\| \quad \forall u, z \in \Omega_{-i}.$$

III. Asynchronous Gossip-based Algorithm

We undertake the problem to compute a Nash equilibrium of $\mathcal{G}(V, \Omega_i, J_i)$ over $G_C(V, E_C)$ using a distributed asynchronous gossip-based algorithm. To better explain the algorithm, we start from the perfect information case in which each player has access to all players' actions at each iteration without any estimation. Given an initial action vector $x(0) = [x_1(0), \ldots, x_N(0)]^T$, the solution of $VI(\Omega, F)$ can be found by a projected gradient-based method with diminishing step size [18]. Since at each iteration, all the other players' actions are available to player i, he can update his action as, $x_i(k+1) = T_{\Omega_i}[x_i(k) - \alpha_{k,i}\nabla_{x_i} J_i(x_i(k), x_{-i}(k))]$, where $T_{\Omega_i} : \mathbb{R} \to \Omega_i$ is an Euclidean projection and $\alpha_{k,i}$ is a diminishing step size, for which the following hold:

$$\sum_{k=1}^{\infty} \alpha_{k,i}^2 < \infty, \qquad \sum_{k=1}^{\infty} \alpha_{k,i} = \infty \quad \forall i \in V. \tag{6}$$

The step size is directly related to the number of updates that a player has made in the asynchronous regime.

Unlike the perfect case, *our interest is in the case of imperfect and partial information. Specifically, we assume that each player i has only access to his cost function J_i and his action x_i but not the other players' actions x_{-i}.*

Assume that, as in a gossip-based algorithm, each player has a local clock which ticks with rate 1 according to a Poisson process. In each time interval only one player is allowed to wake up and select a neighbor to share his information with. Thus the ticking times are independent for each player across G. Similarly, one can assume a central clock which ticks according to a rate N Poisson process [19]. At the k^{th} time interval $T(k), k \geq 0$, where $\{T(k+1) - T(k)\}$ are independent and identically distributed (iid) random variables, a player picks another player from his neighbor set $N_C(.)$ and exchanges the information. Such a player, whose clock ticks at $T(k)$ is represented by $i_k \in V$ and the neighbor picked by i_k at $T(k)$, is denoted by $j_k \in V$.

The algorithm is elaborated in the following steps:

1- ***Initialization Step***

At this step each player i considers an initial temporary estimate vector $\tilde{x}^i(0) = [\tilde{x}_1^i(0), \ldots, \tilde{x}_j^i(0), \ldots, \tilde{x}_N^i(0)]^T \in \Omega$ where $\tilde{x}_j^i(0) \in \Omega_j$ is player i's initial temporary estimate of player j's action.

2- ***Gossiping Step***

At the gossiping step, player i_k wakes up at $T(k)$ and finds a neighbor indexed by j_k. They exchange their temporary estimate vector together. Then players i_k and j_k construct their estimate vectors $\hat{x}^i(k) = [\hat{x}_1^i(k), \ldots, \hat{x}_N^i(k)]^T \in \Omega$, $i \in \{i_k, j_k\}$ by updating their temporary estimate vectors. It is worth mentioning that $\tilde{x}_i^i(k) = x_i(k)$ for all $i \in V$ in every iteration k since the action of player i is known to himself and there is no need to estimate. The estimate vectors are computed through the following terms:

$$\begin{cases} \hat{x}_{i_k}^{i_k}(k) = \tilde{x}_{i_k}^{i_k}(k) \\ \hat{x}_{-i_k}^{i_k}(k) = \frac{\tilde{x}_{-i_k}^{i_k}(k) + \tilde{x}_{-i_k}^{j_k}(k)}{2} \end{cases} \quad \begin{cases} \hat{x}_{j_k}^{j_k}(k) = \tilde{x}_{j_k}^{j_k}(k) \\ \hat{x}_{-j_k}^{j_k}(k) = \frac{\tilde{x}_{-j_k}^{j_k}(k) + \tilde{x}_{-j_k}^{i_k}(k)}{2} \end{cases} \tag{7}$$

and

$$\hat{x}^i(k) = \tilde{x}^i(k), \quad \forall i \notin \{i_k, j_k\}. \tag{8}$$

3- ***Local Step***

All the players are ready to update their actions, after obtaining their estimate vectors. Due to the imperfect information available to player i, he uses $\hat{x}^i(k)$ as his estimate of all other players' actions, and updates his action as follows:

$$x_i(k+1) = \begin{cases} T_{\Omega_i}[x_i(k) - \alpha_{k,i}\nabla_{x_i} J_i(x_i(k), \hat{x}_{-i}^i(k))], & \text{if } i \in \{i_k, j_k\} \\ x_i(k), & \text{otherwise.} \end{cases} \tag{9}$$

Note that a player i, $i \notin \{i_k, j_k\}$, who is not involved in the communication at $T(k)$, keeps his action unchanged for the next iteration. By running (9), player i computes his action for the next iteration and uses this information to update his temporary estimate vector by the following equation.

$$\tilde{x}^i(k+1) = \hat{x}^i(k) + \Big(x_i(k+1) - x_i(k)\Big)e_i, \quad \forall i \in V, \tag{10}$$

where e_i is a unit vector in $\mathbb{R}^N$ whose i-th element is one and the others are zero.

At this point, every player updates his temporary estimate vector and ready to begin the new iteration by running step 2 again.

IV. Convergence Proof

In this section we prove convergence of the algorithm under Assumptions 1-4. Consider a memory in which the history of the decision making is recorded. We define $\mathcal{M}_k$ to denote the *sigma-field* generated by the history up to time k with $\mathcal{M}_0 = \{\tilde{x}^i(0),\ i \in V\}$.

$$\mathcal{M}_k = \mathcal{M}_0 \cup \Big\{(i_l, j_l) : 1 \leq l \leq k\Big\} \quad \forall k \geq 1.$$

In the proof we take advantage of a well-known result on supermartingale convergence which is provided in the following lemma from [20] (Chapter 2.2 Lemma 11).

Lemma 2. *Let V_k, u_k, β_k and ζ_k be non-negative random variables adapted to σ-algebra $\mathcal{M}_k$. If almost surely $\sum_{k=0}^{\infty} u_k < \infty$, $\sum_{k=0}^{\infty} \beta_k < \infty$, and $\mathbb{E}[V_{k+1}|\mathcal{M}_k] \leq (1 + u_k)V_k - \zeta_k + \beta_k$ for all $k \geq 0$, then almost surely V_k converges and $\sum_{k=0}^{\infty} \zeta_k < \infty$.*

The convergence proof is developed in two parts. First, we prove almost sure convergence of the temporary estimate vector $\tilde{x}^i, \forall i \in V$ to an average consensus which is shown to be the average of the temporary estimate vectors. Then, we prove convergence of the players' actions toward the Nash equilibrium, almost surely.

For convenience, we rewrite the algorithm by defining an intermediary variable $\bar{x} = [\bar{x}^1, \ldots, \bar{x}^i, \ldots, \bar{x}^N]^T \in \Omega^N$ with $\bar{x}^i \in \Omega$, where

$$\bar{x}(k) = (W(k) \otimes I_N)\tilde{x}(k). \tag{11}$$

In (11), $\tilde{x}(k) = [\tilde{x}^1(k), \ldots, \tilde{x}^N(k)]^T \in \Omega^N$ is the overall temporary estimate at $T(k)$ and $W(k) = I_N - \frac{1}{2}(e_{i_k} - e_{j_k})(e_{i_k} - e_{j_k})^T$ is a doubly stochastic weight matrix.

We rewrite the algorithm as the following:

1) Each player i chooses an initial temporary estimate vector $\tilde{x}^i(0) = [\tilde{x}^i_1(0), \ldots, \tilde{x}^i_N(0)]^T$.
2) The gossiping rule is $\bar{x}(k) = (W(k) \otimes I_N)\tilde{x}(k)$.
3) Each player i executes the following updating protocol.

$$x_i(k+1) = \begin{cases} T_{\Omega_i}[x_i(k) - \alpha_{k,i}\nabla_{x_i} J_i(x_i(k), \bar{x}^i_{-i}(k))], & \text{if } i \in \{i_k, j_k\} \\ x_i(k), & \text{otherwise,} \end{cases} \tag{12}$$

$$\tilde{x}^i(k+1) = \bar{x}^i(k) + \Big(x_i(k+1) - \bar{x}^i_i(k)\Big)e_i, \ \forall i \in V. \tag{13}$$

A. Convergence of Temporary Estimates to An Average Consensus

In this section we prove that the temporary estimate vector of each player i converges almost surely toward a consensus under Assumptions 1-2. The consensus point is shown to be the average of all temporary estimates. Let $Z(k)$ be the average of all temporary estimates at $T(k)$, i.e.,

$$Z(k) = \frac{1}{N}(\mathbf{1}_N^T \otimes I_N)\tilde{x}(k). \tag{14}$$

The following theorem is the main result of this section, on the convergence of $\tilde{x}^i(k)$ to $Z(k) \in \Omega$, for all $i \in V$.

Theorem 1. *Let $\tilde{x}(k)$ be the overall temporary estimate vector and $Z(k)$ be its average at $T(k)$ as in* (14). *Let also $\alpha_{k,max} = \max_{i\in V} \alpha_{k,i}$. Then under Assumptions 1-2, almost surely*

i) $\sum_{k=0}^{\infty} \alpha_{k,max}\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\| < \infty$,
ii) $\sum_{k=0}^{\infty} \|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|^2 < \infty$.

Proof of Part i). In the convergence proof, we repeatedly use Lemma 2 to show that a term is absolutely summable.

The proof follows by deriving an upper bound for $\mathbb{E}\Big[\|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\|\Big|\mathcal{M}_{k-1}\Big]$ and applying Lemma 2 to the expression.

From (13), (11), (14), the doubly stochastic property of $W(k)$ and $\Big[(W(k) - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T W(k)) \otimes I_N\Big](\mathbf{1}_N \otimes I_N)Z(k) = 0$, it follows:

$$\begin{aligned} &\mathbb{E}\Big[\|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\|\Big|\mathcal{M}_{k-1}\Big] \\ &\leq \underbrace{\mathbb{E}\Big[\|Q(k)(\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k))\|\Big|\mathcal{M}_{k-1}\Big]}_{\text{Term 1}} \\ &+ \underbrace{\mathbb{E}\Big[\|R\mu(k+1)\|\Big|\mathcal{M}_{k-1}\Big]}_{\text{Term 2}}, \end{aligned} \tag{15}$$

where $\mu(k+1) = [(x_1(k+1) - \bar{x}^1_1(k))e_1, \ldots, (x_N(k+1) - \bar{x}^N_N(k))e_N]^T$, $Q(k) = (W(k) - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T W(k)) \otimes I_N$ and $R = (I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T) \otimes I_N$. Let $\gamma = \mathbb{E}\Big[\|Q(k)\|^2\Big|\mathcal{M}_{k-1}\Big]$. Lemma 2 in [21] yields that

$$\text{Term 1} \leq \sqrt{\gamma}\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|. \tag{16}$$

To bound Term 2, we use (12), the non-expansive property of projection, $\|R\| = 1$ and $x_i(k+1) = x_i(k) = \bar{x}^i_i(k)$ for $i \notin \{i_k, j_k\}$.

$$\text{Term 2} \leq \sqrt{2}\sum_{i\in\{i_k,j_k\}} \|x_i(k) - \bar{x}^i_i(k)\| + 2\alpha_{k,\max}C. \tag{17}$$

Next we show that the first term in the RHS of (17) is absolutely summable. By (11), $\|x_i(k) - \bar{x}^i_i(k)\| = \frac{1}{2}\|\tilde{x}^i_i(k) - \tilde{x}^j_i(k)\|$ for $i, j \in \{i_k, j_k\}$, $i \neq j$. Using (12), (13), $\alpha_{k+1,i} < \alpha_{k,i}$ and the non-expansive property of projection, for $i, j \in \{i_k, j_k\}$, $i \neq j$ we obtain

$$\begin{aligned} &\alpha_{k+1,i}\mathbb{E}\Big[\|\tilde{x}^i_i(k+1) - \tilde{x}^j_i(k+1)\|\Big|\mathcal{M}_{k-1}\Big] \\ &\leq \alpha_{k,i}\|\tilde{x}^i_i(k) - \tilde{x}^j_i(k)\| - \frac{\alpha_{k,i}}{2}\|\tilde{x}^i_i(k) - \tilde{x}^j_i(k)\| \\ &+ \alpha_{k,i}^2\|\nabla_{x_i} J_i(x_i(k), \bar{x}^i_{-i}(k))\|, \end{aligned} \tag{18}$$

To apply Lemma 2, we denote $V_k = \alpha_{k,i}\|\tilde{x}^i_i(k) - \tilde{x}^j_i(k)\|$, $u_k = 0$, $\beta_k = \alpha_{k,i}^2\|\nabla_{x_i} J_i(x_i(k), \bar{x}^i_{-i}(k))\|$, $\zeta_k = \frac{\alpha_{k,i}}{2}\|\tilde{x}^i_i(k) - \tilde{x}^j_i(k)\|$. According to Lemma 2 and taking (6) and (5) into account, one can conclude

$$\begin{cases} \sum_{k=0}^{\infty} \alpha_{k,i}\|\tilde{x}^i_i(k) - \tilde{x}^j_i(k)\| < \infty, \\ \sum_{k=0}^{\infty} \alpha_{k,i}\|x_i(k) - \bar{x}^i_i(k)\| < \infty. \end{cases} \tag{19}$$

Substituting (16) and (17) in (15), and then multiplying the LHS and RHS of the resulting inequality by $\alpha_{k+1,\max}$ and $\alpha_{k,\max}$, respectively and using the same idea as in (18), Lemma 2 completes the proof.

Proof of Part ii. The proof follows by finding an upper bound for $\mathbb{E}\Big[\|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\|^2\Big|\mathcal{M}_{k-1}\Big]$. Using (15), (16), (17) and $\mathbb{E}[\|x\|]^2 \leq \mathbb{E}[\|x\|^2]$, we obtain

$$\begin{aligned} &\mathbb{E}\Big[\|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\|^2\Big|\mathcal{M}_{k-1}\Big] \\ &\leq \gamma\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|^2 \\ &+ 2\sum_{i\in\{i_k,j_k\}} \|x_i(k) - \bar{x}^i_i(k)\|^2 + 4\alpha_{k,\max}^2 C^2 \\ &+ 2\sqrt{\gamma}\|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\| \\ &.\Big(\sqrt{2}\sum_{i\in\{i_k,j_k\}} \|x_i(k) - \bar{x}^i_i(k)\| + 2\alpha_{k,\max}C\Big). \end{aligned} \tag{20}$$

We need first to find an upper bound for some terms in (20). From (18) we can write

$$\begin{aligned} &\mathbb{E}\Big[\|\tilde{x}^i_i(k+1) - \tilde{x}^j_i(k+1)\|^2\Big|\mathcal{M}_{k-1}\Big] \\ &\leq \|\tilde{x}^i_i(k) - \tilde{x}^j_i(k)\|^2 - \frac{3}{4}\|\tilde{x}^i_i(k) - \tilde{x}^j_i(k)\|^2 \\ &+ \alpha_{k,i}^2 C^2 + \alpha_{k,i}C\|\tilde{x}^i_i(k) - \tilde{x}^j_i(k)\|, \end{aligned} \tag{21}$$

where (5) was again used. By Lemma 2, (19) and (6), it follows that

$$\sum_{k=0}^{\infty}\|\tilde{x}_i^i(k)-\tilde{x}_i^j(k)\|^2<\infty \rightarrow \sum_{k=0}^{\infty}\|x_i(k)-\bar{x}_i^i(k)\|^2<\infty. \quad (22)$$

By (18), (15) and (5), and after some manipulation we obtain the following:

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{i\in\{i_k,j_k\}}\|x_i(k+1)-\bar{x}_i^i(k+1)\|\Big)\\ &.\Big(\|\tilde{x}(k+1)-(\mathbf{1}_N\otimes I_N)Z(k+1)\|\Big)\Big|\mathcal{M}_{k-1}\Big]\\ &\le\frac{\sqrt{\gamma}}{2}\Big(\sum_{i\in\{i_k,j_k\}}\|x_i(k)-\bar{x}_i^i(k)\|\Big)\|\tilde{x}(k)-(\mathbf{1}_N\otimes I_N)Z(k)\|\\ &+\frac{\sqrt{2}}{2}\Big(\sum_{i\in\{i_k,j_k\}}\|x_i(k)-\bar{x}_i^i(k)\|\Big)^2\\ &+\Big(1+\sqrt{2}\Big)\alpha_{k,\max}C\sum_{i\in\{i_k,j_k\}}\|x_i(k)-\bar{x}_i^i(k)\|\\ &+\sqrt{\gamma}\alpha_{k,\max}C\|\tilde{x}(k)-(\mathbf{1}_N\otimes I_N)Z(k)\|+2\alpha_{k,\max}^2C^2. \quad (23)\end{aligned}$$

As in (18), we can split the coefficient of the first term in the RHS of (23) as $\frac{\sqrt{\gamma}}{2}=1-(1-\frac{\sqrt{\gamma}}{2})$. According to Lemma 2, (22), (19), Part i of Theorem 1 and (6), we have

$$\sum_{k=0}^{\infty}\Big(\sum_{i\in\{i_k,j_k\}}\|x_i(k)-\bar{x}_i^i(k)\|\Big)\Big(\|\tilde{x}(k)-(\mathbf{1}_N\otimes I_N)Z(k)\|\Big)<\infty. \quad (24)$$

Now we are ready to find an upper bound for $\mathbb{E}[\|\tilde{x}(k+1)-(\mathbf{1}_N\otimes I_N)Z(k+1)\|^2|\mathcal{M}_{k-1}]$ in (20). According to Lemma 2 and also by (22),(6) and (24), one can conclude $\sum_{k=0}^{\infty}\|\tilde{x}(k)-(\mathbf{1}_N\otimes I_N)Z(k)\|^2<\infty$. ■

Corollary 1. *For the players' actions $x(k)$, the following terms hold almost surely under Assumptions 1-2.*

i) $\sum_{k=0}^{\infty}\alpha_{k,max}\|x(k)-Z(k)\|<\infty$,
ii) $\sum_{k=0}^{\infty}\|x(k)-Z(k)\|^2<\infty$.

Proof. The proof follows directly from Theorem 1. ■

The following Lemma is crucial to prove the convergence of the algorithm to the Nash equilibrium.

Lemma 3. *Let $\tilde{x}(k)$ and $Z(k)$ be as in Theorem 1. Then for $\bar{x}(k)=(W(k)\otimes I_N)\tilde{x}(k)$, (11), the following holds almost surely under Assumptions 1-2.*

$$\sum_{k=0}^{\infty}\mathbb{E}\Big[\|\bar{x}(k)-(\mathbf{1}_N\otimes I_N)Z(k)\|^2\Big|\mathcal{M}_{k-1}\Big]<\infty. \quad (25)$$

Proof. The proof follows from Theorem 1. ■

B. Convergence of Players Actions to the Nash Equilibrium

In this section we present the convergence proof of the players' actions $x(k)$ to x^*. We prove that once the temporary estimate vectors reach the consensus subspace, they move toward the Nash equilibrium.

Theorem 2. *Let $x(k)$ and x^* be the players' actions and the Nash equilibrium of $\mathcal{G}$, respectively. Under Assumptions 1-4, the sequence $\{x(k)\}$ generated by the algorithm converges to x^*, almost surely.*

Proof. We aim to show that $\|x_i(k)-x_i^*\|$ approaches zero as the number of iterations goes to infinity. By (12), Lemma 1 and the non-expansive property of projection, we arrive at the following inequality for $i\in\{i_k,j_k\}$.

$$\begin{aligned}&\|x_i(k+1)-x_i^*\|^2\le\|x_i(k)-x_i^*\|^2\\ &+\alpha_{k,i}^2\Big\|\nabla_{x_i}J_i(x_i(k),\bar{x}_{-i}^i(k))-\nabla_{x_i}J_i(x_i^*,x_{-i}^*)\Big\|^2-2\alpha_{k,i}\\ &.\Big(\nabla_{x_i}J_i(x_i(k),\bar{x}_{-i}^i(k))-\nabla_{x_i}J_i(x_i^*,x_{-i}^*)\Big)^T(x_i(k)-x_i^*). (26)\end{aligned}$$

We try to bring in the temporary estimate average $Z(k)$ by adding and subtracting $\nabla_{x_i}J_i(x_i(k),Z_{-i}(k))$ from the inner product term of (26). Moreover, we need to add and subtract $\nabla_{x_i}J_i(x_i(k),x_{-i}(k))$ from the inner product term of (26) and use strict monotonicity property (Assumption 3). Then (26) becomes

$$\begin{aligned}&\|x_i(k+1)-x_i^*\|^2\le\|x_i(k)-x_i^*\|^2\\ &+\alpha_{k,i}^2\Big\|\nabla_{x_i}J_i(x_i(k),\bar{x}_{-i}^i(k))-\nabla_{x_i}J_i(x_i^*,x_{-i}^*)\Big\|^2-2\alpha_{k,i}\\ &.\Big[\Big(\nabla_{x_i}J_i(x_i(k),\bar{x}_{-i}^i(k))-\nabla_{x_i}J_i(x_i(k),Z_{-i}(k))\Big)^T\\ &+\Big(\nabla_{x_i}J_i(x_i(k),Z_{-i}(k))-\nabla_{x_i}J_i(x_i(k),x_{-i}(k))\Big)^T\\ &+\Big(\nabla_{x_i}J_i(x_i(k),x_{-i}(k))-\nabla_{x_i}J_i(x_i^*,x_{-i}^*)\Big)^T\Big](x_i(k)-x_i^*). (27)\end{aligned}$$

Using (5) and $\pm2a^Tb\le\|a\|^2+\|b\|^2$, one can find an upper bound for the second, third and the fourth term in the RHS of (27). The resulting inequality holds only for $i\in\{i_k,j_k\}$. When $i\notin\{i_k,j_k\}$, $x_i(k+1)=x_i(k)$ and $\|x_i(k+1)-x_i^*\|^2=\|x_i(k)-x_i^*\|^2$. One can combine these two cases together assuming that for all $i\in V$ player i updates his action with a given probability p_i. Let $p_{\max}=\max_{i\in V}p_i$ and $p_{\min}=\min_{i\in V}p_i$. Let also $\alpha_{k,\min}=\min_{i\in V}\alpha_{k,i}$. By summing the resulting inequality over all $i\in V$ and using Assumption 4, we arrive at

$$\begin{aligned}&\mathbb{E}\Big[\|x(k+1)-x^*\|^2\Big|\mathcal{M}_{k-1}\Big]\\ &\le(1+2p_{\max}\alpha_{k,\max}^2)\|x_i(k)-x_i^*\|^2+4C^2p_{\max}\alpha_{k,\max}^2\\ &+p_{\max}L^2\sum_{i\in V}\mathbb{E}\Big[\Big\|\bar{x}_{-i}^i(k)-Z_{-i}(k)\Big\|^2\Big|\mathcal{M}_{k-1}\Big]\\ &+p_{\max}L^2\sum_{i\in V}\Big\|Z_{-i}(k)-x_{-i}(k)\Big\|^2\\ &-2p_{\min}\alpha_{k,\min}\Big(F(x(k))-F(x^*)\Big)^T(x(k)-x^*), \quad (28)\end{aligned}$$

where $L=\max_{i\in V}L_i$ (Assumption 4) and F is the pseudo-gradient mapping defined in Proposition 1. By applying Lemma 2 to (28) and using (6), Lemma 3 and Corollary 1, we verify that $\|x(k)-x^*\|^2$ converges to some positive finite random variable and also $\sum_{k=0}^{\infty}2p_{\min}\alpha_{k,\min}(F(x(k))-F(x^*))^T(x(k)-x^*)<\infty$. Then we achieve the following results:

1) $\|x(k)-x^*\|^2$ converges almost surely,
2) $2p_{\min}\sum_{k=0}^{\infty}\alpha_{k,\min}\Big(F(x(k))-F(x^*)\Big)^T(x(k)-x^*)<\infty$.

To complete the proof it only remains to show that $\|x(k)-x^*\|\rightarrow0$. Since the action set of all players Ω is compact

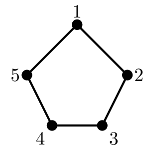

Fig. 1. Communication graph G_C where each node shows a firm.

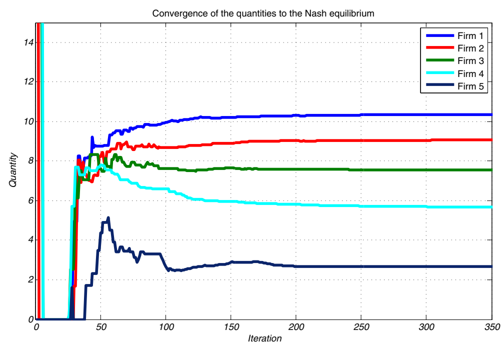

Fig. 2. Quantities converge to the Nash equilibrium.

by Assumption 2, the sequence $x(k)$ has limit points in Ω. Furthermore, since $\sum_{k=0}^{\infty} \alpha_{k,\min} = \infty$, by the second result $(F(x(k)) - F(x^*))^T (x(k) - x^*) \to 0$ along a subsequence of $x(k)$, say $x(k_l)$. On the other hand, by Assumption 3, strict monotonicity of F implies that $x(k_l) \to x^*$. Moreover since $\|x(k) - x^*\|^2$ converges to a positive finite random variable almost surely, one can conclude that $x(k) \to x^*$, almost surely. ∎

V. Simulation Results

For the simulation purpose we consider a quadratic model from classical economic. This economic pattern models N producers involved in the production of a homogeneous commodity. The quantity produced by firm i is denoted by x_i for $i = 1, \ldots, N$. Let u_i denote the cost function of producing the commodity by firm i which is a function of x_i, and let f denote the demand price which is a function of $\sum_{i=1}^{N} x_i^2$. The total cost function of firm i, can be expressed as $J_i(x_i, x_{-i}) = u_i(x_i) - x_i f(\sum_{i=1}^{N} x_i^2)$.

In the following we investigate our gossip based algorithm via a numerical example. Consider a model consists of five firms ($N = 5$), each with a production cost function of the form $u_i(x_i) = c_i x_i$ where $c_i = 100 + 50(i-1)$ for $i = 1, \ldots, N$. The demand price function is given by $f(\sum_{i=1}^{5} x_i^2) = 600 - \sum_{i=1}^{5} x_i^2$. Let the communication graph G_C is defined as Fig. 1. We verify the effectiveness of our gossip-based algorithm through this example over $X_i = [0, 100]$ for all $i \in V$. Fig. 2 represents the convergence of the quantities produced by firms to the Nash equilibrium ($x^* = [10.35, 9.06, 7.56, 5.67, 2.67]^T$). The maximum error of the convergence after 350 iterations is $error_{\max} = \max_{i \in V} \|x_i - x_i^*\| = 0.016$.

VI. Conclusions

An asynchronous gossip-based algorithm is proposed over a distributed multi-player network. A connected and undirected communication graph is considered for the interactions between the players. At each iteration, players maintain estimate vectors of the other players' actions and share them with the local players to update their estimates and actions. We proved that the algorithm converges almost surely to a Nash equilibrium of the game under a set of assumptions on the cost functions and communication graph.

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