1.) Let $k = \mathbb{F}_q$ be a finite field, and E/k an elliptic curve.

(a) Prove the formula

$$\mathsf{res}_{T=1} Z(E,T) = \frac{\#E(k)}{q-1}.$$

(b) Recall that $N_r = \#E(\mathbb{F}_{q^r})$. We define $N'_r = N_r - (q^r + 1)$. Show that there exists a recursive relation

$$N'_{r+2} + x \cdot N'_{r+1} + y \cdot N'_r = z.$$

Conclude that the values of N_1 and N_2 completely determine the zeta function $Z(E,T)$.

2.) This time we consider a complex elliptic curve $E/\mathbb{C}$. Let $\pi\colon E' \to E$ be a finite covering space. Show that E' has a natural structure of a complex manifold (to be precise, a Riemann surface), such that:

(a) the map π is a holomorphic map between complex manifolds,

(b) the complex manifold E' is an elliptic curve,

(c) there exists a positive integer n, such that we have a holomorphic map $E \to E'$, such that the diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ & {\scriptstyle [n]}\searrow & \downarrow \\ & & E \end{array}$$

commutes. Here, we denote by $[n]\colon E \to E$ the map sending $x \in E$ to nx.

Due on Tuesday, October 23rd