1.) Let $\bar{k}$ be an algebraically closed field, and n a positive integer which is invertible in $\bar{k}$.[1] In Exercise 1.54 we verified that the map

$$\phi_n \colon \mathbb{G}_{m,\bar{k}} \to \mathbb{G}_{m,\bar{k}}$$

given by the ring homomorphism $\bar{k}[t,t^{-1}] \to \bar{k}[t,t^{-1}]$ sending t to t^n, is étale. Let us denote by μ_n the group of n-th roots of unity, that is, $\lambda \in \bar{k}$ satisfying

$$\lambda^n = 1.$$

(a) Show that the map $\phi_n \colon \mathbb{G}_{m,\bar{k}} \to \mathbb{G}_{m,\bar{k}}$ is a μ_n-torsor.

(b) Produce an explicit isomorphism $\mathbb{G}_{m,\bar{k}} \times_{\phi_n, \mathbb{G}_{m,\bar{k}}, \phi_n} \mathbb{G}_{m,\bar{k}} \simeq \mathbb{G}_{m,\bar{k}} \times \mu_n$ using the definition of fibre products in terms of tensor products.

2.) Let $\bar{k}$ and n be as above, and X a smooth $\bar{k}$-variety.

(a) Show that the map $\phi_n \colon \mathbb{G}_{m,\bar{k}} \to \mathbb{G}_{m,\bar{k}}$ induces a short exact sequence

$$0 \to \underline{\mu}_n \to \underline{\mathbb{G}}_{m,\bar{k}} \to \underline{\mathbb{G}}_{m,\bar{k}} \to 0$$

of sheaves on the small-étale site $(X)_{\text{ét}}$.

(b) Next week we will prove that for $X = \mathbb{A}^1_{\bar{k}}$ and $X = \mathbb{G}_{m,\bar{k}}$ one has $H^1(X, \underline{\mathbb{G}}_{m,\bar{k}}) = 0$. Deduce from this the statements

$$H^1(\mathbb{A}^1_{\bar{k}}, \underline{\mu}_n) = 0$$

and

$$H^1(\mathbb{G}^1_{m,\bar{k}}, \underline{\mu_n}) = \mathbb{Z}/n\mathbb{Z}.$$

Due on Tuesday, October 30th

[1] That is, n is coprime to the characteristic p of $\bar{k}$.