Epsilon factors

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1 Motivation: geometric class field theory

In this section we let k be a perfect field and X/k a smooth proper curve (which is geometrically connected). The term *local system* on X can refer to one of the following things:

(a) a lisse ℓ-adic sheaf,

(b) a vector bundle E with a flat connection ∇, if $\mathsf{char}(k) = 0$,

(c) a locally constant sheaf (with respect to the standard topology), if $k = \mathbb{C}$.

Type (b) will also be referred to as a *de Rham* local system, while type (c) is called a *Betti* local system. The reason for this conflation of terminology is that these three distinct notions of local systems are governed by similar formal rules. Therefore it is sometimes possible to transport theorems and proofs almost verbatim. Such is the case for geometric class field theory which we discuss in this subsection.

Theorem 1.1 (Deligne)**.** *Let $\mathcal{L}$ a rank 1 local system on X and ω a non-zero rational 1-form. Then we have an isomorphism*

$$(\det H^*(X, \mathcal{L}))(g-1) \simeq \bigotimes_{x \in X_{cl}} \mathcal{L}_x^{\otimes \nu_x(\omega)}\,.$$

Remark 1.2. *Motivated by this* product formula *we call $(\mathcal{L}_x(\frac{1}{2}))^{\otimes -\nu_x(\omega)}$ the ϵ-line $\mathcal{E}_{\omega,x}(\mathcal{L})$.*

This theorem was proven by Deligne in a letter to Serre in 1974. The letter was published as an appendix to [BE01]. In fact Deligne proves a more general statement which also applies to ℓ-adic rank 1 local systems on affine curves.

1.1 Recollection on geometric class field theory

We denote by $\mathsf{Pic}(X)$ the moduli space of line bundles (of arbitrary degree) on X. We write

$$\mathsf{Pic}(X) = \bigsqcup_{d \in \mathbb{Z}} \mathsf{Pic}^d(X)$$

to denote the decomposition according to degree.

The *Abel-Jacobi* map $\mathsf{AJ}\colon X \longrightarrow \mathsf{Pic}^1(X)$ sends $x \in X$ to $\mathcal{O}(x) \in \mathsf{Pic}^1(X)$. More generally we have a canonical morphism (*Abel map*) from $X^{(d)}$ to $\mathsf{Pic}^d(X)$. It is defined by viewing $X^{(d)}$ as the fine moduli space of effective degree d divisors D on X and sending D to $\mathcal{O}(D)$.

Theorem 1.3 (Geometric class field theory)**.** *Let $\mathcal{L}$ be a rank 1 local system on X, then there exists a rank 1 local system $C_{\mathcal{L}}$ on $\mathsf{Pic}(X)$, such that*

(a) $\mathsf{AJ}^* C_{\mathcal{L}} \simeq \mathcal{L}$,

(b) $C_{\mathcal{L}}$ *is a* character sheaf *on* $\mathsf{Pic}(X)$.

A character sheaf $\mathcal{F}$ on a commutative group scheme G is given by the following definition: let $e\colon \operatorname{Spec} k \longrightarrow G$ be the unit map, and $m\colon G \times G \longrightarrow G$ the multiplication. We call a rank 1 local system $\mathcal{F}/G$ together with a trivialisation of $e^*\mathcal{F}$ and an isomorphism $m^*\mathcal{F} \simeq \mathcal{F} \boxtimes \mathcal{F}$ a precharacter sheaf.

Definition 1.4. *Let $m_{123}\colon G \times G \times G \longrightarrow G$ be the multiplication map for triples. A precharacter sheaf is called a* character sheaf *if the diagram*

$$\begin{array}{ccc} m_{123}^*\mathcal{F} & \longrightarrow & \mathcal{F} \boxtimes m_{23}^*\mathcal{F} \\ \downarrow & & \downarrow \\ m_{12}^*\mathcal{F} \boxtimes \mathcal{F} & \longrightarrow & \mathcal{F} \boxtimes \mathcal{F} \boxtimes \mathcal{F} \end{array}$$

commutes. Furthermore we require that for the natural map $(\mathrm{id} \times e)\colon G \longrightarrow G \times G$ we have a commutative diagram

$$\begin{array}{ccc} (\mathrm{id} \times e)^* m^* \mathcal{F} & \longrightarrow & (\mathrm{id} \times e)^* \mathcal{F} \\ \downarrow & & \downarrow \\ (m \circ (\mathrm{id} \times e))^* \mathcal{F} & \longrightarrow & \mathcal{F}\,. \end{array}$$

In the case of ℓ-adic local system over finite base fields, the function-sheaf dictionary allows us to associate to a local system $\mathcal{F}$ a function $f_{\mathcal{F}}$ on $G(k)$.

Exercise 1.5. *Assume that G is a smooth connected group scheme of finite type over a finite field k.*

(a) If $\mathcal{F}$ is an ℓ-adic precharacter sheaf on G then $f_{\mathcal{F}}\colon G(k) \longrightarrow \bar{\mathbb{Q}}_\ell^\times$ is a character (that is, a group homomorphism).

(b) Show that for every character $\chi\colon G(k) \longrightarrow \bar{\mathbb{Q}}_\ell^\times$ there exists a character sheaf $\mathcal{F}$, such that $f_{\mathcal{F}} = \chi$.[1]

In terms of stalks, a character sheaf $\mathcal{F}$ has the property that for $g, h \in G$ there's a (distinguished) isomorphism

$$\mathcal{L}_g \otimes \mathcal{L}_h \simeq \mathcal{L}_{gh}\,.$$

[1] *Hint:* use the Lang isogeny.

This observation suggests a possible construction of the character sheaf $C_{\mathcal{L}}$ on $\mathsf{Pic}(X)$. We know that $\mathsf{AJ}^* C_{\mathcal{L}}$ should be isomorphic to $\mathcal{L}$. That is, for every divisor x of degree 1 we have an isomorphism $(C_{\mathcal{L}})_x \simeq \mathcal{L}_x$. The character sheaf property now suggest for a divisor $D = \sum_{i=1}^m n_i x_i$ the formula $(C_{\mathcal{L}})_{\mathcal{O}(D)} \simeq \bigotimes_{i=1}^m \mathcal{L}_{x_i}^{\otimes n_i}$.

Lemma 1.6. *Consider the quotient map $\pi\colon X^d \longrightarrow X^{(d)}$. The sheaf $D_{\mathcal{L}} = \pi_*(\mathcal{F}^{\boxtimes d})^{\mathfrak{G}_i}$ on $X^{(d)}$ is a rank 1-local system with the property $\pi^{\Delta} D_{\mathcal{L}} \simeq \mathcal{F}^{\boxtimes d}$.*

Proof. It suffices to show that for every $D \in X^{(d)}$ the stalk $(D_{\mathcal{L}})_D$ is of rank 1. We can apply proper base change to the map π. The fibre $\pi^{-1}(D)$ is a finite set endowed with a transitive $\mathfrak{G}_d$-action. Therefore we have

$$\dim(\pi_* \mathcal{F}^{\boxtimes d})_D^{\mathfrak{G}_d} = 1.$$

which is what we wanted. For the second assertion we observe that we have a natural morphism $\pi^{\Delta}\pi_* \mathcal{F}^{\boxtimes d} \longrightarrow \mathcal{F}^{\boxtimes d}$ which is by construction $\mathfrak{G}_d$-invariant. It factors through the coinvariants of $\pi^{\Delta}\pi_* \mathcal{F}^{\boxtimes d}$ which are equivalent to the invariants of $\pi^{\Delta}\pi_* \mathcal{F}^{\boxtimes d}$, and hence to $\pi^{\Delta} D_{\mathcal{L}}$. □

Next we will show that $D_{\mathcal{L}}$ descends along the morphism $\mathsf{AJ}_d\colon X^{(d)} \longrightarrow \mathsf{Pic}^d(X)$ for $d >> 0$.

Lemma 1.7. *For $d > 2g - 2$ the morphism AJ_d is a projective space fibration (in particular, it is smooth).*

Proof. Recall that AJ_d sends an effective divisor D of degree d of the line bundle $L = \mathcal{O}(D)$. The fibre $\mathsf{AJ}_d^{-1}(L)$ is a projective space $\mathbb{P}H^0(X, L)$. It suffices to show that the dimension of these spaces is constant for $d > 2g - 2$: since $X^{(d)}$ and $\mathsf{Pic}^d(X)$ are smooth, this implies flatness, and since the fibres are smooth, this shows that the morphism itself is a smooth morphism.

Riemann-Roch shows $\chi(L) = h^0(X, L) - h^0(X, \otimes_X \otimes L^{\vee}) = d + 1 - g$. We have $h^0(X, \otimes_X \otimes L^{\vee}) = 0$, since the degree of $\otimes_X \otimes L^{\vee}$ is negative. This proves that the dimension of the fibres is constant. □

Sketch of Theorem 1.3. For $d > 2g - 2$ the Abel maps AJ_d are smooth fibrations in to projective spaces. The latter are simply connected, and hence local systems on $X^{(d)}$ descend automatically to local systems on $\mathsf{Pic}^d(X)$. This yields $C_{\mathcal{L}}$ on components of $\mathsf{Pic}(X)$

We have a commutative diagram

$$\begin{array}{ccc} X^{(d_1)} \times X^{(d_2)} & \xrightarrow{m} & X^{(d_1+d_2)} \\ \downarrow & & \downarrow \\ \mathsf{Pic}^{d_1}(X) \times \mathsf{Pic}^{d_2}(X) & \xrightarrow{m} & \mathsf{Pic}^{d_1+d_2} \end{array}$$

and one can show that $m^* D_{\mathcal{L}} \simeq D_{\mathcal{L}} \boxtimes D_{\mathcal{L}}$. This allows one to check the character sheaf property on the components where $C_{\mathcal{L}}$ is already defined. The rest of the proof is left as an exercise to the reader. □

1.2 The use of character sheaves

For abelian varieties there is an interesting vanishing statement. It will play an important role in the proof of Theorem 1.1.

Lemma 1.8. *Let $\mathcal{F}$ be a non-trivial precharacter sheaf of rank 1 on an abelian variety A. Then $H^*(A_{\bar{k}}, \mathcal{F}) = 0$.*

Proof for ℓ-adic local systems. Using the (powerful) function-sheaf dictionary this case is particularly easy to prove. Recall from the exercise above that $\mathcal{F}$ gives rise to a character $\chi \in G(k)^*$. The function-sheaf dictionary implies that χ is non-trivial. The Frobenius trace on $H^*(A_{\bar{k}}, \mathcal{F})$ is given by $\sum_{g \in G(k)} \chi(g)$ (Grothendieck-Lefschetz). Since the right hand side is a non-trivial character sum, it is zero. □

Proof for Betti local systems. In this case we claim that a non-trivial local system (of rank 1) on a torus T (in the sense of topology) has vanishing cohomology. We consider first the case of a circle $T = S^1$. We have $H^0(T, \mathcal{F}) \neq 0$ if and only if $\mathcal{F}$ is non-trivial. Poincarè duality implies that $H^1(T, \mathcal{F}) = H^0(T, \mathcal{F}^\vee) = 0$.

In general we can write $T = T' \times S^1$. Since locally constant sheaves of rank 1 on a torus T correspond to representations of the fundamental group, we see that $\mathcal{F} = \mathcal{F}' \boxtimes \mathcal{F}''$ where $\mathcal{F}''$ denotes a locally constant sheaf of rank 1 on S^1. We may assume that $\mathcal{F}''$ is non-trivial. Projecting first to T' we obtain that the derived pushforward of $\mathcal{F}''$ is zero (by virtue of proper base change and the above). The projection formula implies that the cohomology $H^*(T, \mathcal{F})$ is vanishes. □

Finally we turn to the general proof which only uses general principles (i.e. derived pushforwards, projection formula, Künneth formula). It is taken from Laumon's [Lau96].

Proof for "all" notions of local systems. As before we use that we have $H^0(A, \mathcal{F}) \neq 0$ if and only if $\mathcal{F}$ is non-trivial. We assume by contradiction that there exists a positive integer n, such that $H^n(A, \mathcal{F}) \neq 0$ and $H^i(A, \mathcal{F}) = 0$ for all $i < n$.

Now we consider the complex $Rm_* m^* \mathcal{F}$.The projection formula yields

$$Rm_* m^* \mathcal{F} \simeq \mathcal{F} \otimes Rm_*(\mathsf{triv}),$$

where triv denotes the trivial rank 1 local system. The morphism $m \colon A \times A \longrightarrow A$ is equivalent to $p_2 \colon A \times A \longrightarrow A$, and hence we obtain

The Künneth formula implies

$$H^n(A, Rm_* m^* \mathcal{F}) = \bigoplus_{i+j=n} H^i(A, \mathsf{triv}) \otimes H^j(A, \mathcal{F}) = H^n(A, \mathcal{F}) \neq 0,$$

where the last equality follows from the assumption $H^j(A, \mathcal{F}) = 0$ for $0 \leq j < n$ On the other hand, using that $m^* \mathcal{F} \simeq \mathcal{F} \boxtimes \mathcal{F}$ we can apply the Künneth formula to obtain

$$H^n(A, Rm_* m^* \mathcal{F}) = H^n(A \times A, \mathcal{F} \boxtimes \mathcal{F}) = \bigoplus_{i+j=n} H^i(A, \mathcal{F}) \otimes H^j(A, \mathcal{F}).$$

Since we assume $H^i(A, \mathcal{F}) = 0$ for all $0 \leq i < n$ we obtain that the right hand side is 0. This is a contradiction. □

We can now turn to the first step in the proof of Deligne's theorem 1.1. To avoid considerations of special cases we will assume that X is a curve of genus g at least 2. The Abel map of degree $2g - 2$ will be essential to the proof. For general degrees d, the fibre $\mathsf{AJ}_d^{-1}(L)$ for $L \in \mathsf{Pic}^d(X)$ can be identified with $\mathbb{P}H^0(X, L)$. A combination of Serre duality and Riemann-Roch shows that $\mathsf{AJ}_{2g-2}^{-1}(L)$ is a projective space of dimension $g - 1$ if $L = \omega_X$ and of dimension $g - 2$ otherwise. In fact, the morphism AJ_{2g-2} is a $\mathbb{P}^{g-2}$-fibration away from $\{\omega_X\} \subset \mathsf{Pic}^{2g-2}(X)$.

Lemma 1.9. *Let $\mathcal{F}$ be the character sheaf $C_{\mathcal{L}}$ associated to a non-trivial rank 1 local system on X. Then we have $H^i(X^{(2g-2)}, \mathsf{AJ}^*_{2g-2}\mathcal{F}) = 0$, if $i \neq 2g-2$ and $H^{2g-2}(X^{(2g-2)}, \mathsf{AJ}^*_{2g-2}\mathcal{F}) = \mathcal{F}_{\omega_X}(g-1)$.*

Proof. We use once again the projection formula to obtain the identity $H^*(X^{(2g-2)}, \mathsf{AJ}^*_{2g-2}\mathcal{F}) = H^*(\mathsf{Pic}^{2g-2}(X), \mathcal{F} \otimes R\,\mathsf{AJ}_{2g-2,*}(\mathsf{triv}))$. Since AJ_{2g-2} is a $\mathbb{P}^{g-2}$-fibration, we obtain for $i < 2g-2$ that $R^i AJ_{2g-2,*}(\mathsf{triv})$ is a trivial local system with fibre $H^i(\mathbb{P}^{g-2}, \mathsf{triv})$. Vanishing of the cohomology of projective space in odd degrees implies that

$$RAJ_{2g-2,*}(\mathsf{triv}) = \left(\bigoplus_{i=0}^{2g-3} H^i(\mathbb{P}^{g-2}) \otimes \mathsf{triv}\right) \oplus R^{2g-2}\,\mathsf{AJ}_{2g-2,*}\,\mathcal{F}\,.$$

The Vanishing Lemma 1.8 yields that the first $2g-2$ summands don't contribute to the cohomology of $H^*(X^{(2g-2)}, \mathsf{AJ}^*_{2g-2}\mathcal{F})$. It remains to deal with the contribution of $R^{2g-2}\,\mathsf{AJ}_{2g-2,*}\,\mathsf{AJ}^*\,\mathcal{F}$. By the proper base change theorem (and the fact that only the fibre over ω_X is $(g-1)$-dimensional), we infer that this is a skyscraper sheaf with support $\{\omega_X\} \subset \mathsf{Pic}^{2g-2}(X)$. Applying proper base change and the projection formula once more, we see that it equals $H^{2g-2}(\mathbb{P}^{g-1}) \otimes \mathcal{F}_{\omega_X}$, that is its cohomology equals $\mathcal{F}_{\omega_X}(g-1)$ as we wanted. □

1.3 The proof of Deligne's theorem

Proof of Theorem 1.1. Let us assume first that $\mathcal{L}$ is a trivial rank 1 local system on the curve X. The cohomology groups $H^i(X, \mathsf{triv})$ is then a equal to triv for $i = 0$ and to $\mathsf{triv}(1)$ for $i = 2$. The determinant of $H^1(X, \mathsf{triv})$ is equal to $H^{2g}(\mathsf{Pic}^0(X)) = \mathsf{triv}(g)$. This shows that we have an isomorphism $\det H^*(X, \mathsf{triv}) \simeq (1-g)$. Since $\mathcal{L}_x = \mathsf{triv}$ for all closed points in x, this computation confirms indeed Deligne's product formula for the case of trivial rank 1 local systems.

Henceforth we may assume without loss of generality that $\mathcal{L}$ is a non-trivial rank 1 local system. This is equivalent to $H^0(X, \mathcal{L}) = 0$, and by virtue of Poincaré duality also $H^2(X, \mathcal{L}) = 0$. These vanishing statements can be used to determined the rank of $H^1(X, \mathcal{L})$:

Lemma 1.10. *If $\mathcal{L}$ is a rank 1 local system on X then $\chi(X, \mathcal{L}) = \chi(X, \mathsf{triv}) = 2 - 2g$.*

Remark 1.11. *I don't know a proof of this formula which works for all notions of local systems at once. In fact I believe that if such a proof exists it can't be entirely obvious, as the ℓ-adic case follows (or rather is a special case) from Grothendieck–Ogg–Shafarevich.*

Proof of the de Rham case. Let $\mathcal{L} = (E, \nabla)$ be a rank 1 local system. The de Rham cohomology of $\mathcal{L}$ is then computed by the hypercohomology of the de Rham complex

$$[E \longrightarrow E \otimes \Omega^1_X].$$

on X. This implies

$$\chi_{dR}(\mathcal{L}) = \chi_{coh}(E) - \chi_{coh}(E \otimes \Omega^1_X).$$

According to the Riemann-Roch formula, the right hand side can be identified with $\deg E + (1-g) - (\deg E + (2g-2) + (1-g)) = 2 - 2g$. □

Over the field of complex numbers Betti local systems correspond to de Rham local systems by means of the Riemann-Hilbert correspondence. It is therefore not necessary to treat the Betti case separately.

The determinant of cohomology which we wish to compute, therefore satisfies:

$$\det(X,\mathcal{L})^{-1} = \bigwedge^{2g-2} H^1(X,\mathcal{L}) = \mathrm{Sym}^{2g-2}\left(H^1(X,\mathcal{L})[-1]\right).$$

Remark 1.12. *The second equality sign might come as a surprise to those unfamiliar with the symmetric product operation for complexes. However that's a fairly standard identification which works as follows: for a vector space V (over a characteristic zero field) we have*

$$\mathrm{Sym}^i V = (V^{\otimes i})^{\mathfrak{G}_i}$$

where $\mathfrak{G}_i$ acts by permuting the factors,

$$\bigwedge^i V = (V^{\otimes i})^{\mathfrak{G}_i,-}$$

where the superscript $^-$ indicates that we take antiinvariants (or invariants for the action twisted by the sign character). Those definitions can also be applied to a complex (using tensor products of complexes instead). Using the usual sign rules one sees immediately that invariants turn into antiinvariants in combination with the shift functor:

$$\mathrm{Sym}^i(V[-1]) = \left(\bigwedge^i V\right)[-i].$$

The classical Künneth formula states

$$H^*(Y_1 \times Y_2, \mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

for local systems $\mathcal{F}_i$ on Y_i. It has the following counterpart for symmetric powers:

Corollary 1.13 (Symmetric power Künneth formula)**.** *Let $\mathcal{L}$ be a local system on X. Then we have an isomorphism $H^*(X^{(d)}, (\mathcal{L}^{\boxtimes d})^{\mathfrak{G}_d}) \simeq \mathrm{Sym}^d(H^*(X,\mathcal{L}))$.*

Once one has observed that taking $\mathfrak{G}_d$-invariants commutes with cohomology, this statement follows from the classical Künneth formula.

By construction of the character sheaf $\mathcal{F} = C_{\mathcal{L}}$, we have $\mathsf{AJ}_d^*\mathcal{F} \simeq (\mathcal{L}^{\boxtimes d})^{\mathfrak{G}_d})$. Lemma 1.9 therefore shows that the only non-vanishing cohomology group of $(\mathcal{L}^{\boxtimes d})^{\mathfrak{G}_d})$ is equal to $(C_{\mathcal{L}})_{\omega_X}(g-1)$. This yields an isomorphism

$$\det(X,\mathcal{L})^{-1} \simeq (C_{\mathcal{L}})_{\omega_X}(g-1).$$

In order to conclude the proof, it remains to produce an isomorphism $(C_{\mathcal{L}}))_{\omega_X} \simeq \bigotimes_{x\in X_{cl}} \mathcal{L}_x^{\otimes \nu_x(\omega)}$. This follows from the following more general assertion:

Lemma 1.14. *For a divisor $D = \sum_{x\in X_{cl}} n_x x$ we have $(C_{\mathcal{L}})_{\mathcal{O}_D} \simeq \bigotimes_{x\in X} \mathcal{L}_x^{\otimes n_x}$.*

Proof. This is the case since $C_{\mathcal{L}}$ is a character sheaf and we have $(C_{\mathcal{L}})_{\mathcal{O}(x)} = \mathcal{L}_x$. □

This concludes the proof of Deligne's formula. □

2 Axiomatics and analogies

2.1 Epsilon factors for ℓ-adic sheaves

Deligne's Theorem 1.1 fits into a broader context. In this subsection we will focus on local systems as lisse ℓ-adic sheaves. The definition given below only applies to this case, as it is anchored to the function-sheaf dictionary. We treat the global and local case separately.

Definition 2.1. *Let k be a finite field and X/k a smooth proper curve. Let $U \subset X$ be an open subset and $\mathcal{L}$ an ℓ-adic local system. We define the global epsilon factor as*

$$\epsilon(U, \mathcal{L}) = Tr(-F, \det(H^*(U, \mathcal{L}))^{-1}) \in \bar{\mathbb{Q}}_\ell^\times.$$

Alternatively one can define $\epsilon(U, \mathcal{L})$ as the constant in the functional equation of $\mathcal{L}$. This makes for sense for complexes of constructible sheaves K on X which extend $\mathcal{L}$:

$$L(X, K, t) = \epsilon(X, K) \cdot t^{a(X,K)} \cdot L(X, \mathbb{D}K, t^{-1})$$

According to the Langlands conjecture (which is known for function fields), an irreducible ℓ-adic local system on $U \subset X$ corresponds to a cuspidal automorphic representation for the function field of X. Furthermore, this correspondence preserves L-functions.

The L-functions of a cuspidal automorphic representation can be written as an infinite product over the places of X (i.e., closed points x of X). This presentation depends on an additional choice, namely an additive Haar measure on the local field F_x which arises in the guise of a formal 1-forms $\omega \in \Omega^1_F$. The functional equation for the global L-function is then obtained by taking a product over local functional equations. The local constants are also known as the local (automorphic) epsilon factors. The Langlands programme therefore suggests that the following formalism should exist.

In the following we denote by F and L local fields of equicharacteristic p, and an auxiliary prime $\ell \neq p$. We also fix an additive character $\psi\colon \mathbb{F}_p \longrightarrow \bar{\mathbb{Q}}_\ell^\times$, and a square root $p^{\frac{1}{2}} \in \bar{\mathbb{Q}}_\ell$.

Definition 2.2 (Epsilon formalism)**.** *An* ϵ-formalims *assings to an isomorphism class of an ℓ-adic local systems $\mathcal{L}$ on* Spec F*, and ω a non-zero formal* 1*-form on F a constant $\epsilon_\omega(\mathcal{L}) \in \bar{\mathbb{Q}}_\ell^\times$, such that the following properties hold:*

(a) For a short exact sequence of local systems $L' \hookrightarrow L \twoheadrightarrow L''$ on Spec F *we have*

$$\epsilon_\omega(\mathcal{L}) = \epsilon_\omega(\mathcal{L}')\epsilon_\omega(\mathcal{L}'').$$

We obtain a well-defined homomorphism $\epsilon_\omega\colon K_0(\mathsf{Loc}_\ell(F)) \longrightarrow \bar{\mathbb{Q}}_\ell^\times$.

(b) For an element $\mathcal{L}$ of $K_0(\mathsf{Loc}_\ell(L))$ of virtual dimension 0*, and a finite field separable extension L/F we have*

$$\epsilon_\omega(\mathrm{Ind}_F^L \mathcal{L}) = \epsilon_{\omega_L}(\mathcal{L}).$$

(c) (Product formula) Let $U \subset X$ be an open subset of a smooth proper curve as in Definition 2.1. For an ℓ-adic local system on U and a non-zero rational 1*-form ω we have a* product formula *(where only finitely many factors are $\neq 1$)*

$$\epsilon(U, \mathcal{L}) = \prod_{x \in X_{cl}} \epsilon_{\omega_{F_x}}(\mathcal{L}\,|_{F_x}).$$

(d) *If $\mathcal{L}$ is a rank 1 ℓ-adic local system on F, then $\epsilon_\omega(\mathcal{L})$ is the ϵ-factor assigned by Tate's thesis to the multiplicative character $F^\times \longrightarrow \mathrm{Gal}(F)^{\mathsf{ab}} \longrightarrow \bar{\mathbb{Q}}_\ell^\times$ and the additive character $\Psi\colon F \longrightarrow \bar{\mathbb{Q}}_\ell^\times$ which sends $a \in F$ to $\psi(\mathsf{Tr}_{k_F/\mathbb{F}_p} \operatorname{res}(a \cdot \omega))$.*

It was conjectured by Langlands and Deligne that such a formalism should exist. This conjecture was proven by Laumon [Lau87] using ℓ-adic Fourier transform. His proof is based on a local-to-global argument. Every ℓ-adic local system on a local field $k((t))$ can be extended to a ramified local system on $\mathbb{P}^1_k$ with tame ramification at ∞ (by virtue of the Gabber–Katz theorem). Using the product formula, one can therefore reduce the definition of epsilon factors to the tamely ramified case.

Although the Langlands philosophy was essential in arriving at the conjecture of the existence of local epsilon factors, we emphasise that Laumon's proof is independent thereof, and in fact his result is used in Lafforgue's proof of the Langlands conjecture for function fields.

2.2 Epsilon lines

Question 2.3. *Are the ℓ-adic epsilon factors $\epsilon_\omega(\mathcal{L})$ Frobenius eigenvalues of ℓ-adic lines $\mathcal{E}_\omega(\mathcal{L})$?*

Let's try to make this question more precise:

Definition 2.4. *Epsilon lines A formalism of epsilon lines associates to $(F, \mathcal{L}, \omega)$ as in Definition 2.2 a rank 1 vector space over $\bar{\mathbb{Q}}_\ell^\times$ with a $\widehat{\mathbb{Z}}$-action, such that the following properties hold:*

(a) *For a short exact sequence of local systems $L' \hookrightarrow L \twoheadrightarrow L''$ on $\operatorname{Spec} F$ we have an isomorphism*

$$\mathcal{E}_\omega(\mathcal{L}) \simeq \mathcal{E}_\omega(\mathcal{L}') \otimes \mathcal{E}_\omega(\mathcal{L}''),$$

which satisfies a compatibility condition for triples $\mathcal{L}_1 \hookrightarrow \mathcal{L}_2 \hookrightarrow \mathcal{L}_3$.

(b) *For $\mathcal{L}$ of virtual rank[2] 0 and L/F a finite separable extension we have an isomorphism $\mathcal{E}_\omega(\mathrm{Ind}_F^L \mathcal{L}) \simeq \mathcal{E}_{\omega_L}(\mathcal{L})$.*

(c) *For $U \subset X$ an open subset in a smooth proper curve over a finite field k, and $\mathcal{L}/U$ an ℓ-adic local system, and ω a rational 1-form on X we want an isomorphism*

$$\det(H^*(U, \mathcal{L}))^{-1} \simeq \bigotimes_{x \in X_{cl}} \mathcal{E}_{\omega|_{F_x}}(\mathcal{L}|_{F_x}),$$

where almost all factors are trivialised.

Deligne's product formula 1.1 hints at how epsilon lines can be defined in the case of rank 1 local sytems on X. For general ranks the situation is slightly more complicated. There is a persistent sign issue which obstructs the existence of a formalism of ℓ-adic epsilon lines (and in fact for all notions of local systems). We will later see how to resolve this problem by using *graded lines*[3]. This course is devoted to a theorem of Beilinson–Bloch–Esnault and Deligne which states that de Rham epsilon (graded) lines exist. For the ℓ-adic case this is not known. In fact several footnotes scattered throughout the literature suggest that there should be another mysterious obstruction to a formalism of ℓ-adic epsilon lines:

[2] This requires a lot of contemplation, we will provide more details in a future section.

[3] A graded line is pair consisting of a rank 1 vector space and an integer. The integer is used as a device to resolve sign problems, that is, only it congruence class modulo 2 really matters. You could also work with *super lines*.

Remark 2.5 (For Fabio, Lei and Tanya; and everyone who likes gerbes)**.** *The expectation is (see footnote 3 on p. 3 of [BBE02]) that for every cohomology theory there exists a μ_2-gerbe, let's call it the* epsilon gerbe*, such that there is a formalism of twisted (graded) epsilon lines. For de Rham and Betti cohomology this gerbe is trivial, but for ℓ-adic and crystalline cohomology it is non-trivial (if it exists).*

2.3 Geometric analogies

Let us assume for now that we are working with a notion of local systems and a cohomology theory, such that there is a formalism of epsilon lines. We will see later that this is true (at least up to signs) for de Rham and Betti cohomology. The product formula

$$\det(H^*(U,\mathcal{L}))^{-1} \simeq \bigotimes_{x\in X_{cl}} \mathcal{E}_{\omega|_{F_x}}(\mathcal{L}\,|_{F_x})$$

suggests that the determinant of cohomology (which depends on the global structure, or topology of X), can be computed in terms of local contributions depending only on the auxiliary choice of a non-zero rational 1-form. Particularly for the case of ℓ-adic cohomology it is difficult to imagine what the connection between rational 1-forms and cohomology could be.

It turns out that this is just the arithmetic counterpart of an old observation in topology which is known as *Morse theory*. The first accounts can be found in papers by Cayley ("On contour and slope lines") and Maxwell ("On hills and dales") in the 1860s.

2.3.1 Recollection of Morse theory

Let M be a smooth compact manifold (over the reals) and $f\colon M \longrightarrow \mathbb{R}$ a C^2-function on M. A point $m \in M$ is called *critical*, if its a zero of the 1-form df. We assume that f has only finitely many critical points, and that all critical points are non-degenerate (that is, the Hessian is non-degenerate). Furthermore we assume that for every pair of critical points m_1 and m_2 we have $f(m_1) \neq f(m_2)$.

Definition 2.6. *We say that $m \in M$ is a critical point of index γ, if there exists a chart near m, such that f has the formal development*

$$f(x) = f(m) - x_1^2 - \cdots - x_\gamma^2 + x_{\gamma+1} + \cdots + x_n^2 + \cdots .$$

We denote the critical points of index γ by c_γ.

According to the main theorem of Morse theory, there exists a cell decomposition of M, such that there is cell of dimension γ for every critical point of index γ. Therefore we have the following identity:

$$\chi(M) = \sum_{\gamma\in\mathbb{Z}} c_\gamma,$$

which looks like an additive form of the product formula describing the determinant of cohomology:

$$\chi(M) = \sum_{d_m f=0} \mathrm{ind}(m).$$

Furthermore, Morse theory only depends on the exact 1-form df.

2.3.2 Poincaré–Hopf theorem

There is another classical theorem with a striking resemblance to the product formula. As before we let M be a smooth compact manifold.

Definition 2.7. *Let ν be a smooth vector field on $\mathbb{R}^n$ with isolated zeroes, such that $\nu(0) = 0$. We define the index of ν at 0 to be the degree of the induced self-map of a sphere S^{n-1} of very small radius .*

Theorem 2.8 (Poincaré-Hopf)**.** *Let ν be a smooth vector field with finitely many isolated zeroes on M. Then we have*

$$\chi(M) = \sum_{\nu(x)=0} \mathsf{ind}_x(\nu).$$

A particularly interesting case to consider is where ν has no zeroes at all.

Corollary 2.9. *If there exists a smooth vector field ν on M without zeroes, then $\chi(M) = 0$.*

A geometric argument due to Thurston[4]*.* Choose a very fine triangulation of M and place a positive charge at the barycentre of every even-dimensional simplex, and a negative charge at the barycentre of every odd-dimensional simples. The Euler characteristic of M equals the total charge.

For a split second we let the charges flow along the vector field. We take stock for each top-dimensional simplex at a time: all charges now lie in the interior of a top-dimensional simplex. Drawing a picture we convince ourselves that inside every top-dimensional simplex we have charge zero. This proves the assertion. □

3 D-modules

Henceforth we work over base fields k of characteristic 0. Let X be a smooth k-variety.

Definition 3.1. *A* D-module *on X is a quasi-coherent sheaf $\mathcal{F}$ on X with a flat connection ∇. That is, a k-linear map of sheaves $\nabla\colon \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega^1_X$, such that ∇ satisfies the Leibniz rule, and $\nabla^2 = 0$.*[5]

As suggested by the terminology, every smooth variety X carries a sheaf of non-commutative rings D_X (the ring of differential operators), such that D-modules as defined above, correspond to quasi-coherent sheaves of D_X-modules. We will use these two viewpoints interchangeably.

Definition 3.2. *(a) A differential operator on X of order ≤ 0 is a section of $\mathsf{End}_k(\mathcal{O}_X)$ given by multiplication with a local section of $\mathcal{O}_X$.*

(b) A differential operator of order $\leq k$ is a section of $\mathsf{End}_k(\mathcal{O}_X)$, such that for every local section f of $\mathcal{O}_X$ we have that $[q, f]$ is an operator of order $\leq k - 1$.

(c) We denote by $D_X^{\leq k}$ the sheaf of differential operators of order $\leq k$ on X. The union of sheaves $D_X = \bigcup_{k\geq 0} D_X^{\leq k}$ is defined to be the ring of differential operators.

[5] We should really call this a quasi-coherent sheaf of D-modules, but non-quasi-coherent examples will not be of interest to us.

Zariski-locally on X, an order $\leq k$ differential operators q can be expressed as a sum

$$\sum_{i=1}^{m} \partial_{i1} \cdots \partial_{ik} + q',$$

where ∂_{ij} are tangent vector fields, and q' is a differential operator of order $\leq k-1$. This important observation can be neatly packaged as a statement about filtered rings.

Proposition 3.3 (PBW for differential operators)**.** *The associated graded* $\mathsf{gr}^i(D_X) = D_X^{\leq i}/D_X^{\leq i-1}$ *is canonically isomorphic to* $\operatorname{Sym}^i T_X$*. Therefore* $\mathsf{gr}(D_X) \simeq \operatorname{Sym} T_X = \pi_* \mathcal{O}_{T^*X}$*.*

We will return to this in Subsection 3.4 when defining the category of holonomic D-modules.

A good source for the theory of D-modules are the lecture notes [] by Bernstein. Our exposition is indebted to his.

3.1 An example in dimension 1

Let's consider an example of a D-module on the affine line $\mathbb{A}^1$ which is not a vector bundle with a flat connection. This is a purely algebraic problem: we have an $k[x]$-module M, and for the vector field $\partial = \frac{\partial}{\partial x}$ we have an operator

$$\partial \colon M \longrightarrow M$$

which satisfies the Leibniz rule.

Let $s \in M$ be a non-zero element, such that $x \cdot s = 0$. Applying ∂ to the equation, and using the Leibniz rule we obtain

$$s = -x\partial s.$$

We define $s^{(0)} = s$, and recursively $s^{(n)} = \partial s^{(n-1)}$. The equation above shows $xs^{(1)} = -s \neq 0$, which implies three things:

(a) $s^{(1)} \neq 0$,

(b) $xs^{(1)} \neq 0$,

(c) $x^2 s^{(1)} = 0$.

By induction one proves the identities $x^n s^{(n-1)} = 0$ and $x^{n-1} s^{(n-1)} \neq 0$. We conclude that there cannot exist torsion D-modules on $\mathbb{A}^1$ which are finitely generated as $\mathcal{O}$-modules. Using the same method we can prove an interesting generalisation:

Exercise 3.4. *(a) Let X be a curve and $\mathcal{F}$ a D-module on X which is coherent as $\mathcal{O}_X$-module. Then $\mathcal{F}$ is locally free.*[6]

(b) Prove (a) without assuming $\dim X = 1$.[7]

[6] *Hint:* observe that $\mathcal{F}$ splits as direct sum of a torsion-module and a vector bundle. Observe that the torsion part is a D-submodule.

[7] *Hint:* convince yourself that the assertion can be reduced to the one-dimensional case.

3.2 Pushforward and pullback

Let $f\colon Y \longrightarrow X$ be a morphism of smooth varieties. There exist derived functors

$$f_+\colon D(\mathsf{Mod}(D_X)) \longrightarrow D(\mathsf{Mod}(D_Y))$$

and

$$f^\Delta\colon D(\mathsf{Mod}(D_X)) \longrightarrow D(\mathsf{Mod}(D_Y)).$$

In terms of the classical theory of flat connections, f^Δ generalises the construction of the pullback connection, while f_+ is best thought of as a generalisation of the Gauss-Manin connection on relative de Rham cohomology.

Definition 3.5. *There is a natural map $D_Y \longrightarrow f^*D_X$ induced by $T_Y \longrightarrow f^*T_X$. A D_X-module pulls back to a f^*D_X-module, and by restriction of scalars along the composition $D_Y \longrightarrow f^*D_X$ we can define a natural D_Y-module structure on the pullback of a D_X-module. We denote the resulting derived functor by f^Δ. If the morphism f is flat, then f^Δ also makes sense on the underived level.*

We observe that there is a diagram of derived functors

$$\begin{array}{ccc} D(\mathsf{Mod}(D_X)) & \xrightarrow{f^\Delta} & D(\mathsf{Mod}(D_Y)) \\ \downarrow & & \downarrow \\ D(\mathsf{Mod}(\mathcal{O}_X)) & \xrightarrow{Lf^*} & D(\mathsf{Mod}(\mathcal{O}_X)) \end{array}$$

which commutes up to natural isomorphisms. The vertical functors are given by forgetting the D-module structure and retaining only the underlying (complex of) quasi-coherent sheaves.

The definition of f_+ is more complicated. The reason for this is that its more naturally defined in terms of right D-modules than left D-modules. One then uses that there is an equivalence between the categories of left and right D-modules. Mathematically speaking this is akin to swapping functions for distributions. But in practical terms one obtains a definition which is more complicated than what one would like. Therefore we will circumvent a rigorous definition of f_+ for as long as possible. Rather than defining f_+ we will describe an compatibility relation derived pushforward of $\mathcal{O}$-modules Rf_* which determines f_+.

Definition 3.6. *We denote by $D_X \otimes_{\mathcal{O}_X}\colon \mathsf{Mod}_{qc}(\mathcal{O}_X) \longrightarrow \mathsf{Mod}(D_X)$ the functor which assigns to a quasi-coherent $\mathcal{O}_X$-module M the free D_X-module generated by it. We let $D_X \otimes^L_{\mathcal{O}_X}$ be its left derived functor.*

D-module pushforward is compatible with this free construction of D-modules. For the purpose of these notes this is going to be the only thing we have to know about f_+.

Proposition 3.7. *Let $f\colon Y \longrightarrow X$ be a morphism of smooth varieties and $\mathcal{F}$ a complex of $\mathcal{O}_Y$-modules on Y. There is a natural isomorphism $f_+(D_Y \otimes^L_{\mathcal{O}_Y} \mathcal{F}) \simeq (Rf_*\mathcal{F}) \otimes^L_{\mathcal{O}_X} D_X$.*

At least locally it is always possible to express a D-module M on Y as a cokernel of a presentation

$$D_Y^{\oplus I} \longrightarrow D_Y^{\oplus J} \longrightarrow M,$$

that is, of two D-modules in the image of the functor $\otimes^L_{\mathcal{O}_Y}$. This is the reason why the proposition above can be used (in combination with a Čech resolution) to compute f_+ for arbitrary D_Y-modules.

Remark 3.8. *We are not claiming that f^Δ and f_+ are adjoint functors (that might be the reason for the unfamiliar notation). This will happen under circumstances to be discussed later.*

3.3 Kashiwara's theorem and consequences

There are a few places where the theory of D-modules behaves unexpectedly different from the theory of quasi-coherent sheaves of $\mathcal{O}$-modules. Kashiwara's theorem is an instance of this phenomenon. It shows that D-modules do not see nilpotent thickenings of subvarieties.

Theorem 3.9 (Kashiwara)**.** *Let X and Y be a smooth variety and $Y \hookrightarrow X$ be a closed immersion. The full subcategory of D-modules on X with set-theoretic support Y is equivalent to the category of D-modules on Y:*

$$\mathsf{Mod}(D_Y) \cong \mathsf{Mod}_Y(D_X).$$

This result has several far-reaching applications. For instance it allows one to define D-modules on a singular variety Y by plunging Y (locally) into a smooth variety X and considering the full subcategory of D-modules on X which are set-theoretically supported on Y.

A second application is the following consequence about D-modules on projective spaces.

Corollary 3.10 (Beilinson–Bernstein)**.** *Let X be a smooth affine variety and M be a D-module on $\mathbb{P}^n_X$. Then $M \neq 0$ if and only if $\mathsf{Hom}_D(D_{\mathbb{P}^n}, M) \neq 0$. (In other words, the category $\mathsf{Mod}(D_{\mathbb{P}^n})$ is globally generated by $D_{\mathbb{P}^n}$.) Furthermore, the functor $\mathsf{Hom}_D(D_{\mathbb{P}^n}, -)$ is exact.*

Remark 3.11. *The analogous assertion for quasi-coherent $\mathcal{O}$-modules is never true for non-affine schemes. One says that $\mathbb{P}^n$ is D-affine in reference to this phenomenon. Beilinson–Bernstein have shown that flag varieties are D-affine, other examples are probably not known.*

Proof of Corollary 3.13. Let $\pi\colon \mathbb{A}^{n+1}\setminus\{0\} \longrightarrow \mathbb{P}^n$ the canonical $\mathbb{G}_m$-invariant morphism. We denote by $j\colon \mathbb{A}^{n+1}\setminus\{0\} \hookrightarrow \mathbb{A}^{n+1}$.

For every D-module M on $\mathbb{P}^n$, the pullback $\pi^\Delta M$ is a $\mathbb{G}_m$-equivariant sheaf on $\mathbb{A}^{n+1}\setminus 0$. We can express the functor $\mathsf{Hom}_D(D_{\mathbb{P}^n}, -)$ as the composition

$$(j_* \circ \pi^\Delta M)^{\mathbb{G}_m}.$$

The functor $(-)^{\mathbb{G}_m}$ is exact, and so is π^Δ. It remains to analyse the functor j_*.

Let $M' \hookrightarrow M \twoheadrightarrow M''$ be a short exact sequence of D-modules on $\mathbb{P}^n$, such that $j_*\pi^\Delta M \longrightarrow j_*\pi^\Delta M''$ is not surjective. Its cokernel N defines a D-module on $\mathbb{A}^{n+1}$ with set-theoretic support $\{0\}$. Kashiwara's Theorem 3.9 implies that N is isomorphic to i_+W where W is a vector space, and $i\colon \{0\} \hookrightarrow \mathbb{A}^{n+1}$.

Claim 3.12. *The $\mathbb{G}_m$-equivariant D-module N corresponds to a graded $k[x_0, \ldots, x_n]$-module where all elements are negatively-graded.*

Proof. The Euler vector field $e = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$ is an infinitesimal generator of the $\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$. Since N is a $\mathbb{G}_m$-equivariant D-module we have that $s \in N_i$ if and only if $es = is$. Using computations similar to those in 3.1, one can prove that $es = is$ and $e(\partial_j s) = (i-1)s$. We conclude that the positively-graded part of i_+W is zero. □

We conclude that $N^{\mathbb{G}_m} = 0$, and hence exactness of the functor $(j_*\pi_*)^{\mathbb{G}_m}$. It remains to check conservativity, that is, that $\mathsf{Hom}_D(D_{\mathbb{P}^n}, M) = 0$ implies $M = 0$.

If $M \neq 0$ there must be a minimal non-negative integer $n \in \mathbb{Z}$, such that $(j_*\pi^\Delta M)^n \neq 0$. Let $m \in j_*\pi^\Delta M$ be a section of weight n. Using the Euler vector field defined earlier, we see that $em = nm$.

Assume $n > 0$: since $e = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$ there exists an $i = 0, \ldots, n$, such that $\partial_i m \neq 0$. We have $e(\partial_i m) = (n-1)(\partial_i m)$ which contradicts the minimality assumption of m. □

Corollary 3.13. *The category of D-modules on $\mathbb{P}^n_X$ (where X is as before a smooth affine variety) is equivalent to the category of R-modules where $R = \Gamma(D_{\mathbb{P}^n_X})$.*

Proof. We have shown that $D_{\mathbb{P}^n_X}$ is projective generator. By virtue of Morita theory we obtain what we wanted. □

We won't stop here. There's another corollary of this corollary coming up.

Definition 3.14. *A D-module is called coherent, if its Zariski-locally finitely presented as a sheaf of D-modules. A complex of D-modules is called coherent, it its cohomology sheaves are coherent.*

Corollary 3.15. *Let $f\colon Y \longrightarrow X$ be a projective morphism. Then $f_+\colon D(\mathsf{Mod}(D_Y)) \longrightarrow D(\mathsf{Mod}(D_X))$ preserves coherent complexes.*

Proof. It suffices to check the assertion for the case of closed immersions and the projection $Y = \mathbb{P}^n_X \longrightarrow X$ where X is a smooth affine variety.

If $f\colon Y \longrightarrow X$ is a closed immersion of affine varieties we can use the identity

$$f_+(D_Y) \simeq f_+(\mathcal{O}_Y \otimes_{\mathcal{O}_Y} D_Y) \simeq f_*(\mathcal{O}_Y) \otimes^L_{\mathcal{O}_X} D_X D_X$$

to deduce that f_+ sends coherent objects to coherent objects.

Henceforth we assume that f is a trivial $\mathbb{P}^n$-fibration over a smooth affine variety. Let $R = \Gamma(D_Y)$. An R-module N is finitely presented if and only if the functor $\mathsf{Hom}_R(N, -)$ preserves filtered colimits. Similarly, (for any smooth variety) Y, we have that a D_Y-module M is coherent if and only if the functor $\mathsf{Hom}_D(M, -)$ preserves filtered colimits. By virtue of Corollary 3.13 we see that a coherent D-module on $\mathbb{P}^n_X$ is globally finitely presented.

It suffices therefore to prove $f_*(D_{\mathbb{P}^n_X}) \subset D_{coh}(\mathsf{Mod}(D_X))$. This follows from the compatibility condition of f_+ with derived pushforward of $\mathcal{O}$-modules (see Proposition 3.7) $f_+(D_Y) \simeq D_X \otimes_{\mathcal{O}_X} Rf_*(\mathcal{O}_{\mathbb{P}^n_X}) \simeq D_X$. □

3.4 Holonomic D-modules

Holonomic D-modules form a particularly nice subcategory. Their definition is closely tied to the order filtration on the sheaf of rings D_X.

Definition 3.16. *Let M be a coherent D-module on X. A* good *filtration on M is a structure of a filtered module $(M, M^{\leq i})$ with respect to the filtered ring $(D, D^{\leq i})$, such that the pieces $M^{\leq i}$ are coherent.*

Good filtrations can be constructed by hand Zariski-locally for every coherent D-module. The associated graded $\mathsf{gr}(M)$ is a module with respect to $\pi_* \mathcal{O}_{T^*X}$ and therefore gives rise to a coherent sheaf on T^*X.

Theorem 3.17 (Bernstein)**.** *Let M be a coherent D-module with a good filtration. The underlying (reduced) closed subset of T^*X given by the support of $\mathsf{gr}(M)$ is independent of the chosen filtration. It is a co-isotropic subset of T^*X (i.e. its dimension is $\geq \dim X$).*

This reduced subset of T^*X is also known as the singular support of M. Note that this subset is well-defined globally without assuming the existence of global good filtrations. The estimate $\dim \operatorname{supp} M \geq \dim X$ is also known as the *Bernstein inequality*. The case where $\dim \operatorname{supp} M = \dim X$ corresponds to holonomic D-modules.

Definition 3.18. *A coherent D-module M on X is holonomic if its singular support satisfies* $\dim \operatorname{supp} M = \dim X$.

The singular supports of holonomic D-modules are $\mathbb{G}_m$-equivariant Lagrangian subsets of T^*X.

Lemma 3.19. *A coherent D-module is $\mathcal{O}$-coherent if and only if its singular support is contained in the zero section $X \hookrightarrow T^*X$.*

Proof. Let $(M^{\leq i})_{i\in\mathbf{N}}$ be a filtration on M, such that the associated graded $\mathsf{gr}(M)$ is a coherent $\pi_*\mathcal{O}_{T^*X}$-module with set-theoretic support $X \hookrightarrow T^*X$. This implies that $\mathsf{gr}(M) = \bigoplus_{i=1}^{\infty} M^{\leq i}/M^{\leq i-1}$ is $\mathcal{O}_X$-coherent, and therefore there exists $n_0 \in \mathbf{N}$, such that $M^{\leq n} = M^{\leq n-1}$ for $n > n_0$. Since the filtered piec $M^{\leq n_0}$ is $\mathcal{O}$-coherent by the defining assumption on good filtrations, we conclude that $M = M^{\leq n_0}$ is $\mathcal{O}$-coherent. □

We have seen in Exercise 3.4 that $\mathcal{O}$-coherent D-modules are vector bundles with a flat connection. Since singular supports of holonomic D-modules are $\mathbb{G}_m$-invariant Lagrangian subsets of T^*X, they are generically contained in the zero section $X \hookrightarrow T^*X$. We conclude the following:

Corollary 3.20. *Holonomic D-modules are generically vector bundles with a flat connection.*

For this reason, holonomic D-modules on X give a particularly nice "compactification" of a local system defined over an open subset of X. Their singular support is measuring where and to which degree, the holonomic D-module deviates from being a local system.

Example 3.21. *Consider the Dirac Delta D-module M on $\mathbb{A}^1$ of Subsection 3.1. We identify $T^*\mathbb{A}^1$ with $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$. The singular support of M is then given by $\{x = 0\} \subset \mathbb{A}^2$.*

Indeed we have seen that M was generated by a single section s. As a k-vector space one can write $M = k[\partial] \otimes_k \langle s\rangle$. A good filtration on M is given by $M^{\leq n} = \langle \partial^{n-1}s, \dots, s\rangle$. The induced associated graded yields the polynomial $k[y]$ where we identify $y = \partial$.

4 Deligne lattices, irregularity and de Rham epsilon factors

4.1 Regular and irregular connections

In this subsection we denote by k a field of characteristic zero. We denote by X a smooth proper curve over k and $j\colon U \hookrightarrow X$ an affine open subset. Its complement will be referred to as $S = X \setminus U$. Let $\mathcal{E} = (E, \nabla)$ be a flat connection on U.

Since X is a curve, the vector bundle E extends to X. In general we cannot expect ∇ to extend from U to X. In such a case we say that the connection ∇ is singular near S. Not all singularities are born equal.

Definition 4.1. *(a) By abuse of notation we view S as an effective reduced divisor on X. The invertible sheaf $\Omega^1_X(S)$ will be referred to as the sheaf of log-differential forms on X.*

(b) We say that $\mathcal{E}$ has regular singularities *at S, if there exits an extension L of E to X, such that $\nabla(L) \subset L \otimes \Omega^1_X(S)$.*

In concrete terms this amounts to the existence of a local presentation of ∇ as $d + \omega$ where ω is a section of $\Omega^1_X(S)$. That is, ω has poles of order ≤ 1. The extension L/X of E/U in (b) is not unique.

Theorem 4.2 (Deligne)**.** *Let $k \subset \mathbb{C}$, and $\tau\colon \mathbb{C}/2\pi i\,\mathbb{Z} \longrightarrow \mathbb{C}$ a set-theoretic section of $\mathbb{C} \longrightarrow \mathbb{C}/2\pi i\,\mathbb{Z}$. There exits a unique extension L of E to X, such that the residues of ∇ along S are contained in the image of τ. Furthermore, the complex of sheaves*

$$[L \longrightarrow L \otimes \Omega^1_X(S)]$$

is quasi-isomorphic to the de Rham complex

$$[j_*E \longrightarrow j_*(E \otimes \Omega^1_U)].$$

A connection with singularities along S which isn't regular is called *irregular*. There is a natural generalisation of Deligne's theorem on regular connections.

Definition 4.3. *A pair of good lattices for $\mathcal{E}$ is an ordered pair of locally free subsheaves $M \subset N \subset j_*E$, such that*

(1) $M|_U = N|_U = E|_U$,

(2) $\nabla(M) \subset N \otimes \Omega^1_X(S)$,

(3) For every effective divisor D with $\mathsf{supp}\ D = S$*, the inclusion of complexes*

$$[M \longrightarrow N \otimes \Omega^1_X(S)] \longrightarrow [M(D) \longrightarrow N(D) \otimes \Omega^1_X]$$

is a quasi-isomorphism.

By taking the colimit over the filtered set of divisors D with support S, it follows from (3) that

$$[M \longrightarrow N \otimes \Omega^1_X(S)] \longrightarrow [j_*(E|_U) \longrightarrow j_*(E|_U) \otimes \Omega^1_X] \tag{1}$$

is a quasi-isomorphism.

Theorem 4.4 (Deligne)**.** *Good lattices exist.*

The proof of this result is deferred to Subsection 4.6. In the meantime we will discuss special cases and consequences. At first we focus on the application for de Rham epsilon factors.

4.2 Graded lines and determinants

Let k be a field (the characteristic 0 assumption won't be necessary in this subsection). We denote by $\mathsf{Pic}(k)$ the groupoid of rank 1 k-vector spaces. That is, it is the category whose objects are vector spaces of rank 1 and morphisms are invertible linear maps between them. At face value, that's a rather boring category.

Lemma 4.5. *The category $\mathsf{Pic}(k)$ is equivalent to the groupoid $Bk^\times$, that is, the category with precisely one object $\bullet$, and $\mathsf{Hom}(\bullet,\bullet) = k^\times$.*

The Tensor product $\otimes = \otimes_k$ defines a symmetric monoidal structure on $\mathsf{Pic}(k)$. In terms of our simple toy model above, $\otimes$ is given by $\bullet \otimes \bullet = \bullet$ and

$$\begin{array}{ccc} \mathsf{Hom}(\bullet,\bullet) \times \mathsf{Hom}(\bullet,\bullet) & \xrightarrow{\otimes} & \mathsf{Hom}(\bullet,\bullet) \\ \downarrow & & \downarrow \\ k^\times \times k^\times & \xrightarrow{\text{mult}} & k^\times. \end{array}$$

The symmetry constraint is an additional structure. We have to specify an isomorphism

$$s\colon \bullet = \bullet_1 \otimes \bullet_2 \simeq \bullet_2 \otimes \bullet_1 = \bullet.$$

In order to be consistent with the natural isomorphism $V \otimes_k W \simeq W \otimes_k V$ which exists for k-vector spaces, we choose $s = \mathrm{id}_\bullet$.

Let Vect_k denote the groupoid of all finite-dimensional k-vector spaces. That is, we discard all non-isomorphisms from the category of finite-dimensional k-vector spaces and linear maps. The direct sum operation of vector spaces $\oplus$, defines a symmetric monoidal structure on Vect_k. Furthermore we have a monoidal functor $\det\colon \mathsf{Vect}_k \longrightarrow \mathsf{Pic}(k)$ which sends V to its top exterior power $\bigwedge^{\mathrm{top}} V$.

The formulation that det is a monoidal functor actually disguises an extra structure. For every pair of finite vector spaces V, W we have a natural isomorphism

$$c_{V,W}\colon \det V \otimes \det W \simeq \det(V \oplus W)$$

which is given by $(v_1 \wedge \cdots \wedge v_n) \otimes (w_1 \wedge \cdots \wedge w_m) \mapsto v_1 \wedge \cdots \wedge v_n) \wedge (w_1 \wedge \cdots \wedge w_m)$. These isomorphisms satisfy a coherence condition for triples of vector spaces.

Warning 4.6. *The functor* $\det\colon \mathsf{Vect}_k \longrightarrow \mathsf{Pic}(k)$ *is not* symmetric monoidal *functor. What does this mean? For every pair of vector spaces* V, W *we can ask if the following diagram commutes*

$$\begin{array}{ccc} \det V \otimes \det W & \longrightarrow & \det(V \oplus W) \\ \downarrow & & \downarrow \\ \det W \otimes \det V & \longrightarrow & \det(W \oplus V), \end{array}$$

which compares the commutativity constraint for tensor products with the one of direct sums. It turns out that the diagram commutes only up to the sign $(-1)^{mn}$ *where* m, n *are the dimensions of* V *respectively* W*.*

In order to fix this we would like to *twist* the commutativity constraint of lines by the sign $(-1)^{mn}$. This can be done by remembering the ranks m, n.

Definition 4.7. *(a) A graded line is a pair* (L, n) *where* $L \in \mathsf{Pic}(k)$ *and* $n \in \mathbb{Z}$*. We denote the category of graded lines by* $\mathsf{Pic}^{gr}(k)$*. Formally it is defines to be* $\mathsf{Pic}^{gr}(k) = \mathsf{Pic}(k) \times \mathbb{Z}$*.*

(b) The monoidal structure on $\mathsf{Pic}^{gr}(k)$ *is defined factorwise:*

$$(L_1, n_1) \otimes (L_2, n_2) = (L_1 \otimes L_2, n_1 + n_2).$$

(c) The symmetry constraint $c_{12}\colon (L_1, n_1) \otimes (L_2, n_2 \longrightarrow (L_2, n_2) \otimes (L_1, n_1)$ *is defined by multiplying the factorwise symmetry constraint with* $(-1)^{n_1 n_2}$*.*

The groupoid of graded lines is a product of the two gropoids $\mathbb{Z}$ (viewed as a discrete category, all all morphisms are identities) and $\mathsf{Pic}(k)$. This is also true for the monoidal structure. But, as a symmetry monoidal category, $\mathsf{Pic}^{gr}(k)$ is not a product.

Definition 4.8. *We let* $\det^{\mathrm{gr}}\colon \mathsf{Vect}_k \longrightarrow \mathsf{Pic}^{gr}(k)$ *be the symmetric monoidal functor* $V \mapsto (\det V, \mathrm{rk}\, V)$*.*

The definition of Pic^{gr} as a symmetric monoidal category is rigged for the functor above to respect the symmetry constraint. Graded determinants are a device which takes care of sign problems which would otherwise be inevitable.

The graded determinant is multiplicative with respect to short exact sequences. This follows by combining additivity of rank with respect to short exact sequences and multiplicativity of top exterior powers.

Lemma 4.9. *Let $0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$ be a short exact sequence of finite-dimensional k-vector spaces. We have an isomorphism $\mathsf{det}^{\mathrm{gr}}(V_2) \simeq \mathsf{det}^{\mathrm{gr}}(V_1) \otimes \mathsf{det}^{\mathrm{gr}}(V_3)$.*

4.3 Lattices and relative determiants

Let X be a k-curve (not necessarily smooth or proper) and $j\colon U \hookrightarrow X$ an open subset. We fix a vector bundle E on U. We will also allow X to be a trait, that is, isomorphic to $\operatorname{Spec} k'[[t]]$ where k'/k is a finite field extension.

Definition 4.10. *A U-*lattice *L is a locally free subsheaf $L \subset j_*E$, such that $L|_U = E$.*

If X is proper then we can define $\mathsf{det}^{\mathrm{gr}}(X, L) = \mathsf{det}^{\mathrm{gr}}(H^*(X, L)) = \bigotimes_{i=0}^{1}(\mathsf{det}^{\mathrm{gr}}(H^i(X, L))^{(-1)^i})$. If X is not proper, the cohomology groups of L will not be finitely generated and hence the graded determinant is not well-defined. Let L_1, L_2 be two lattices, then "there is a way to define" $\mathsf{det}^{\mathrm{gr}}(X, L_1) \otimes (\mathsf{det}^{\mathrm{gr}}(X, L_2))^{-1}$ even if X is not proper.

Lemma-Definition 4.11. *For a pair of lattices $L_1, L_2 \subset j_*\mathcal{E}$ we define a graded line*

$$\mathsf{det}^{\mathrm{gr}}(L_1 : L_2) = \mathsf{det}^{\mathrm{gr}}(\Gamma(L_1/L)) \otimes (\mathsf{det}^{\mathrm{gr}}(\Gamma(L_2/L)))^{-1},$$

where L is an arbitrarily chosen lattice satisfying $L \subset L_1 \cap L_2$. This graded line only depends on (L_1, L_2) up to a unique isomorphism.

Proof. It suffices to show that $\mathsf{det}^{\mathrm{gr}}(L_1 : L_2)$ is well-defined, that is, independent of the choice of L. For $L, L' \subset L_1 \cap L_2$ we may choose $L'' \subset L \cap L'$. We have $\mathsf{det}^{\mathrm{gr}}(\Gamma(L_1/L'') \simeq \mathsf{det}^{\mathrm{gr}}(\Gamma(L_1/L) \otimes \mathsf{det}^{\mathrm{gr}}(\Gamma(L/L'')$, and similarly $\mathsf{det}^{\mathrm{gr}}(\Gamma(L_2/L'') \simeq \mathsf{det}^{\mathrm{gr}}(\Gamma(L_2/L) \otimes \mathsf{det}^{\mathrm{gr}}(\Gamma(L/L'')$. This induces an isomorphism

$$\mathsf{det}^{\mathrm{gr}}(L_1 : L_2) = \mathsf{det}^{\mathrm{gr}}(\Gamma(L_1/L)) \otimes (\mathsf{det}^{\mathrm{gr}}(\Gamma(L_2/L)))^{-1} \simeq \mathsf{det}^{\mathrm{gr}}(\Gamma(L_1/L'')) \otimes (\mathsf{det}^{\mathrm{gr}}(\Gamma(L_2/L'')))^{-1}.$$

Reversing the role of L and L' we obtain an isomorphism

$$\mathsf{det}^{\mathrm{gr}}(L_1 : L_2) = \mathsf{det}^{\mathrm{gr}}(\Gamma(L_1/L)) \otimes (\mathsf{det}^{\mathrm{gr}}(\Gamma(L_2/L)))^{-1} \simeq \mathsf{det}^{\mathrm{gr}}(\Gamma(L_1/L')) \otimes (\mathsf{det}^{\mathrm{gr}}(\Gamma(L_2/L')))^{-1}.$$

The verification that this isomorphism is independent of the choice of L'' is left to the reader. □

The multiplicativity of graded determinants with respect to short exact sequences yields the following transitivity property whose proof we leave to the reader.

Lemma 4.12. *For every triple of lattices $L_1, L_2, L_3 \subset j_*\mathcal{E}$ we have an isomorphism $t_{123}\colon \mathsf{det}^{\mathrm{gr}}(L_1 : L_2) \otimes \mathsf{det}^{\mathrm{gr}}(L_2 : L_3) \xrightarrow{\simeq} \mathsf{det}^{\mathrm{gr}}(L_1 : L_3)$, such that for every quadruple $L_1, L_2, L_3, L_4 \subset j_*\mathcal{E}$ we get*

a commutative diagram

$$\begin{array}{ccc} \mathsf{det}^{\mathrm{gr}}(L_1 : L_2) \otimes \mathsf{det}^{\mathrm{gr}}(L_2 : L_3) \otimes \mathsf{det}^{\mathrm{gr}}(L_3 : L_4) & \xrightarrow{t_{123}\otimes \mathrm{id}} & \mathsf{det}^{\mathrm{gr}}(L_1 : L_3) \otimes \mathsf{det}^{\mathrm{gr}}(L_3 : L_4) \\ \downarrow{\scriptstyle \mathrm{id}\otimes t_{234}} & & \downarrow{\scriptstyle t_{134}} \\ \mathsf{det}^{\mathrm{gr}}(L_1 : L_2) \otimes \mathsf{det}^{\mathrm{gr}}(L_2 : L_4) & \xrightarrow{\qquad t_{124} \qquad} & \mathsf{det}^{\mathrm{gr}}(L_1 : L_4) \end{array}$$

of graded lines.

Another important aspect of relatives determinants is their factorisation property. We denote by $S = X \setminus U$ the compliment of X.

Lemma 4.13. *For two lattices L_1, L_2 we have*

$$\mathsf{det}^{\mathrm{gr}}(X, L_1 : L_1) \simeq \bigotimes_{x\in S} \det(V_x, L_1 : L_2),$$

where $V_x \subset X$ intersects S precisely in x.

Proof. Let $N \subset L_1 \cap L_2$ be a lattice. The coherent sheaf L_1/N can be expressed as a sum of the stalks

$$L_1/N = \oplus_{x\in S}(L_1/N)_x,$$

and similarly for L_2/N. We therefore have

$$\mathsf{det}^{\mathrm{gr}}(\Gamma(X, L_1/N)) \simeq \bigotimes_{x\in S} \mathsf{det}^{\mathrm{gr}}(\Gamma(X, (L_1/N)_x)).$$

This yields the required factorisation format. □

Proposition 4.14. *Let X be proper, then for a pair of lattices L_1, L_2 we have an isomorphism*

$$\mathsf{det}^{\mathrm{gr}}(X, L_1 : L_2) \simeq \mathsf{det}^{\mathrm{gr}}(X, L_1) \otimes \mathsf{det}^{\mathrm{gr}}(X, L_2)^{-1}.$$

Proof. There exists a lattice $N \subset L_1 \cap L_2$. By virtue of Lemma 4.12 it suffices to prove this for $N \subset L_2$. Therefore we may assume $L_1 \subset L_2$ without loss of generality.

The short exact sequence of sheaves

$$0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_2/L_1 \longrightarrow 0$$

yields a long exact sequence of sheaf cohomology groups. Since X is proper, this is an exact sequence of finite-dimensional k-vector spaces. Applying $\mathsf{det}^{\mathrm{gr}}$ we obtain an isomorphism

$$\mathsf{det}^{\mathrm{gr}}(H^*(X, L_2)) \simeq \mathsf{det}^{\mathrm{gr}}(H^*(X, L_1)) \otimes \mathsf{det}^{\mathrm{gr}}(H^*(X, L_2/L_1)).$$

This concludes the argument. □

4.4 De Rham epsilon factors

Let X be a k-curve (or a trait), $U \subset X$ and $\mathcal{E} = (E, \nabla)$ a flat connection on U. We denote by (L, N) a pair of good lattices for (E, ∇) as in Definition 4.3, and by ν a 1-form on U without zeroes.

Lemma-Definition 4.15. *We define*

$$\tilde{\varepsilon}_\nu(X, E, \nabla) = \mathsf{det}^{\mathrm{gr}}(L : \nu^{-1}(N \otimes \Omega^1_X(S))).$$

Proof. In order for this to be a well-defined graded line we have to show independence from the chosen pair of good lattices. Recall that for any effective divisor D supported on S, $(L(D), N(D))$ is as well a pair of good lattices. Since this family of good lattices exhausts j_*E, ever pair of good lattices is eventually contained in $(L(D), N(D))$ for a sufficiently big divisor D. We can therefore assume that we have a nested pair of good lattices $(L_1, N_1) \subset (L_2, N_2)$.

Inclusion of the complexes $[L_1 \longrightarrow N_1 \otimes \Omega^1_X(\log S)] \longrightarrow [L_2 \longrightarrow N_2 \otimes \Omega^1_X(\log S)]$ is a quasi-isomorphism. This follows from $[L_1 \longrightarrow N_1 \otimes \Omega^1_X(\log S)] \longrightarrow [j_*E \longrightarrow j_*(E \otimes \Omega^1_X)]$ being quasi-isomorphisms for $i = 1, 2$. We conclude that $[L_2/L_1 \longrightarrow N_2/N_1 \otimes \Omega^1_X(\log S)]$ is acyclic. Taking global sections we conclude

$$\mathsf{det}^{\mathrm{gr}}(L_2 : \nu^{-1}(N_2 \otimes \Omega^1_X(\log S))) \simeq \mathsf{det}^{\mathrm{gr}}(L_1 : \nu^{-1}N_1) \otimes \mathsf{det}^{\mathrm{gr}}(\Gamma[L_2/L_1 \longrightarrow N_2/N_1 \otimes \Omega^1_X(\log S)]),$$

but the additional factor on the right is (by virtue of acyclicity) isomorphic to the trivialised k-line k. □

Definition 4.16. *Let $V_x \subset X$ be an open subset (or a trait) which intersects S precisely in x. We define*

$$\varepsilon_{\nu,x}(E, \nabla) = \tilde{\varepsilon}_\nu(V_x, E, \nabla)$$

Theorem 4.17 (Product formula, Beilinson–Bloch–Esnault)**.** *Let X be proper and smooth. We have an isomorphism of graded lines*

$$\bigotimes_{x \in S} \varepsilon_{\nu,x}(E, \nabla) \simeq H^*_{dR}(U, E, \nabla).$$

Proof. We choose a global pair of good lattices (L, N) for $\mathcal{E} = (E, \nabla)$. By virtue of the definition of good lattices we have

$$\mathsf{det}^{\mathrm{gr}}(H^*_{dR}(U, \mathcal{E})) \simeq \mathsf{det}^{\mathrm{gr}}(H^*(X, [L \longrightarrow N \otimes \Omega^1_X(\log S)])) \simeq \mathsf{det}^{\mathrm{gr}}(H^*(X, L)) \otimes \mathsf{det}^{\mathrm{gr}}(X, N \otimes \Omega^1_X(\log S))^{-1}.$$

According to Proposition 4.14, the right hand side is isomorphic to

$$\det(X, N : \nu^{-1}(N \otimes \Omega^1_X(\log S))).$$

Lemma 4.13 yields and isomorphism with

$$\bigotimes_{x \in S} \det(V_x, N : \nu^{-1}(N \otimes \Omega^1_X(\log S))) \simeq \bigotimes_{x \in S} \varepsilon_{\nu_x}(\mathcal{E}).$$

This concludes the proof. □

4.5 Epsilon factors for holonomic D-modules

In this subsection we discuss a variation of the product formula 4.17 for holonomic D-modules on smooth proper curves. The starting point is the following general fact. We assume k to be a field of characteristic 0.

Proposition 4.18. *Let X be a k-variety and $i\colon Z \hookrightarrow X$ a closed subset with open complement $U \subset X$. For a holonomic D-module M we have a distinguished triangle*

$$R\Gamma_{dR,Z}(X,M) \longrightarrow R\Gamma_{dR}(X,M) \longrightarrow R\Gamma(U,M) \longrightarrow R\Gamma_Z(X,M)[1].$$

In particular we have a long exact sequence of finite-dimensional vector spaces

$$\cdots \longrightarrow H^i_{dR,Z}(X,M) \longrightarrow H^i_{dR}(X,M) \longrightarrow H^i_{dR}(U,M) \longrightarrow H^{i+1}_{dR,Z}(X,M) \longrightarrow \cdots .$$

This motivates the following definition.

Definition 4.19. *Let V be a curve (or a trait), M a holonomic D-module on X. For a non-zero rational 1-form ν on X we choose an open subset $U \subset V$, such that $M|_U = \mathcal{E}$ is a flat connection and $\omega|_U$ is nowhere vanishing. We then define*

$$\varepsilon_\nu(V,M) = \varepsilon_\nu(V,\mathcal{E}) \otimes \mathsf{det}^{\mathrm{gr}}(H^*_{dR,X\setminus U}(M)).$$

For $x \in V$ we can define a local factor $\varepsilon_{\nu,x}(M)$ using Lemma 4.13.

We leave it as an exercise to the reader to check that this definition is independent of the choice of U. We obtain the following product formula as a consequence of Theorem 4.17.

Corollary 4.20 (Product formula)**.** *Let X be a smooth proper curve, M a holonomic D-module on X, and ν a non-zero rational 1-form. We then have*

$$\mathsf{det}^{\mathrm{gr}}(H^*_{dR}(X,M)) \simeq \bigotimes_{x\in S} \epsilon_{x,\nu}(M),$$

for any finite closed subset $X^{sing} \subset S \subset X$, such that $M|_(X \setminus S)$ is a flat connection and S contains all zeroes and poles of ν.

Proof. We write $U = X \setminus S$, and $\mathcal{E} = M|_U$. By virtue of Theorem 4.17 we have

$$\mathsf{det}^{\mathrm{gr}}(H^*_{dR}(U,\mathcal{E})) \simeq \bigotimes_{x\in S} \varepsilon_{x,\nu}(\mathcal{E}).$$

We have $\mathsf{det}^{\mathrm{gr}}(H^*_{dR}(X,M)) \simeq \mathsf{det}^{\mathrm{gr}}(U,M|_U) \otimes \mathsf{det}^{\mathrm{gr}}(H^*_{dR,S}(X,M))$, and the factor $\mathsf{det}^{\mathrm{gr}}(H^*_{dR,S}(X,M))$ decomposes as a tensor product $\bigotimes_{x\in S} \mathsf{det}^{\mathrm{gr}}(H^*_{dR,\{x\}}(X,M))$. Re-arranging terms we obtain the asserted identity. □

Exercise 4.21. *Develop a formalism for de Rham epsilon factors for holonomic D-modules on singular curves.*

4.6 Good lattices exist

We now turn to the proof of the existence of good lattices in the local case (the global version is deduced as a corollary using formal glueing). The strategy is to produce good lattices more or less explicitly in the case where the connection (E, ∇) is regular, and for irregular singularities of rank 1. We then combine these two cases using the Levelt–Turrittin decomposition.[8]

Lemma 4.22. *Let $\nabla = d+\omega$ be a formal connection on $k((t))^{\oplus r}$ where ω has a regular singularity. We assume $k \subset \mathbb{C}$ and assume that the real parts of the eigenvalues of the residue belong to the interval $(-1, 0]$. Then there exists a good lattice pair (L, L) for ∇.*

The second case we consider is a rank 1 example of a flat connection.

Lemma 4.23. *Let $\omega \in k((t))dt$ be a 1-form of valuation $n \leq -2$. Then $(k[[t]], t^{-n+1}k[[t]])$ is a good lattice pair.*

Proof. Let $\omega = \left(\sum_{i=n}^{\infty} a_i t^i\right) dt$. We write $\sum_{i=0}^{\infty} b_i t^i$ to denote an element of $k[[t]]$. The connection $d + \omega$ maps it to

$$\left(\sum_{i=1}^{\infty} i b_i t^i\right) \frac{dt}{t} + \left(\sum_{i=n+1}^{\infty} \Big(\sum_{j+\ell=i} a_j b_\ell\Big) t^i\right) \frac{dt}{t}.$$

Since $a_{-n} \neq 0$ it is easy to see that $\ker(d+\omega) = \operatorname{coker}(d+\omega) = 0$. This implies right away that $(k[[t]], t^{-n+1}k[[t]])$ is a good lattice pair. □

The general case is implied by the following result due to Levelt–Turrittin about flat connections on $k((t))$. It can be compared with the existence of a Jordan normal form for matrices. In order to emphasise this analogy we begin by stating the existence of Jodan normal forms for non-algebraically closed fields.

Theorem 4.24. *Let F be a field and $A\colon F^r \longrightarrow F^r$ a linear operator. Then there exists a finite field extension F'/F, such that $A_{F'} = A \otimes_F \mathrm{id}_{F'}$ is a direct sum of operators $D_\lambda + N_{(i)}$, where $D_\lambda = \lambda \mathbf{1}$ is a diagonal matrix with diagonal entry $\lambda \in F'$, and $N_{(i)}$ is an $i \times i$ nilpotent Jordan block matrix.*

The version stated above has the downside that we have to extend scalars from F to F'. If we prefer to retain the base field we enlarge F' (if necessary) to obtain a finite Galois extension, and define

$$\bigoplus_{\sigma \in \mathrm{Gal}(F'/F)} (D_{\sigma(\lambda)} + N_{(i)}).$$

Since this is a Galois-invariant operator, it descends to an irreducible F-linear operator. The direct sum indexed by the Galois orbits of the operators $(D_\lambda + N_{(i)})$ defines a decomposition into irreducible F-linear operators of A.

The Levelt–Turritin decomposition provides a fairly similar statement for differential operators on fields $k((t))$. We begin by defining the basic building blocks which will take the role of Jordan blocks.

Definition 4.25. *We denote by $\mathcal{N}_{(i)} = (k((t))^{\oplus i}, \nabla_{(i)})$ the regular connection given by $d + N_{(i)} \frac{dt}{t}$, where $N_{(i)}$ is an $i \times i$ Jordan block.*

[8] This argument was explained to us by H. Esnault.

We denote by $F = k((t))$. A finite field extension F'/F is always of the shape $k'((s))$ for a finite field extension k'/k. We have a canonical isomorphism $\Omega^1_F \otimes_F F' \simeq \Omega^1_{F'}$. This gives rise to a sort of pushforward operation $\mathrm{Ind}_F^{F'}$ for flat connections on F' to flat connections on F.

Theorem 4.26 (Levelt–Turrittin)**.** *Let $F = k((t))$ and $\mathcal{E}$ a formal flat connection over F. There exists a finite field extension, such that we have*

(a) rank 1 connections $\mathcal{L}_1, \ldots, \mathcal{L}_m$ on F',

(b) positive integers $n_1, \ldots, n_m$,

(c) an isomorphism $\mathcal{E} \simeq \mathrm{Ind}_F^{F'} \left(\bigoplus_{i=1}^m \mathcal{L}_i \otimes \mathcal{N}\rfloor_{(n_i)} \right)$.

We refer the reader to Levelt's article [Lev75] for a proof of this result. The existence of pairs of good lattices (in the local case) is a consequence of the following lemma.

Lemma 4.27. *Good lattice pairs are compatible with $\mathrm{Ind}_F^{F'}$.*

Proof. Let (E, ∇) be a formal connection on F', and let (M, N) be a pair of $\mathcal{O}_{F'}$-submodules of E which are a good lattice pair for ∇. We then view M and N as $\mathcal{O}_F$-modules by restricting scalars along the inclusion $\mathcal{O}_F \subset \mathcal{O}_{F'}$. Using the definition of the induction functor, it is easy to see that the complex

$$[M \longrightarrow N \otimes \Omega^1_{\mathcal{O}_F}(S)]$$

is quasi-isomorphic to the de Rham complex of $\mathrm{Ind}_F^{F'}(E, \nabla)$ (which is the the same complex as the de Rham complex for (E, ∇)). Here we make use of the fact that log-differential forms are well-behaved with respect to the field extension F'/F. That is, we have $(\Omega^1_F(S))_{F'} \simeq \Omega^1_{F'}(S)$. □

We can now prove Deligne's theorem on the existence of good lattice pairs.

Proof of Theorem 4.4. By the Levelt–Turrittin decomposition it suffices to produce a good lattice for a flat connection of the form $\mathrm{Ind}_F^{F'}(\mathcal{L} \otimes \mathcal{N}_{(n)})$ where $\mathcal{L}$ is a rank 1-connection. Recall that $\mathcal{N}$ is endowed with a good lattice by virtue of the presentation $(\mathcal{O}^{\oplus n}, d + \frac{N}{t} dt)$.

Claim 4.28. *Let (L, M) be a good lattice pair for $\mathcal{L}$ on F'. Then, $(L^{\oplus n}, M^{\oplus n})$ is a good lattice pair for $(\mathcal{L} \otimes \mathcal{N}_{(n)})$.*

The claim can be checked by induction on n by using the fact that we have short exact sequences $\mathcal{N}_{(n)} \hookrightarrow \mathcal{N}_{(n+1)} \twoheadrightarrow (\mathcal{O}, d)$. The existence of good lattices follows then from Lemma 4.27 by pushing forward the good lattice pair on F'. □

5 Central extensions

This section provides us with the tools to study the variation of de Rham epsilon lines $\varepsilon_\nu(\mathcal{E})$ in dependence of ν. We will see below that for an invertible rational function f we have

$$\varepsilon_{f\nu}(\mathcal{E}) \simeq \varepsilon_\nu(\mathcal{E}) \otimes \det(f, E),$$

where $\det(f, E)$ is the so-called (graded) determinant line of f on E. These determinants line stem from a central extension of the formal loop group $L\,\mathbb{G}_m$ by $\mathbb{G}_m$ which we will construct in two complimentary ways in Subsection 5.3 and 5.4. The second construction seems to mimic the definition of de Rham epsilon lines, the second construction is representation-theoretic in nature.

5.1 Preliminaries on central extension

Let G be a group and A an abelian group.

Definition 5.1. *A* central extension *of G by A is given by a short exact sequence of groups*

$$1 \longrightarrow A \longrightarrow \widehat{G} \xrightarrow{p} G \longrightarrow 1, \tag{2}$$

such that $A \subset Z(\widehat{G})$.

The set of isomorphism classes of central extensions of G by A carries a natural structure of an abelian group. It is isomorphic to degree two group cohomology $H^2(G, A)$ (see Remark 5.7 below, for an explanation of this fact).

We denote by $P(G)$ the set of commuting pairs of elements in G.

Lemma-Definition 5.2. *Let $f, g \in P(G)$, and $\widetilde{f} \in p^{-1}(f)$, $\widetilde{g} \in p^{-1}(g)$. Then the commutator $[\widetilde{f}, \widetilde{g}] = \widetilde{f}\widetilde{g}\widetilde{f}^{-1}\widetilde{g}^{-1}$ belongs to A and depends only on f and g. We denote the resulting function $P(G) \longrightarrow A$ by $(f, g) \mapsto \langle f, g\rangle$.*

Proof. Since $p([\widetilde{f}, \widetilde{g}]) = [f, g] = e$ we have $[\widetilde{f}, \widetilde{g}] \in A$. Two choices for elements in $p^{-1}(f)$ differ by an element in A. Since $A \subset Z(\widehat{G})$ we see that the commutator $[\widetilde{f}, \widetilde{g}]$ depends indeed only on (f, g). □

Lemma 5.3. *For $f, g, h \in P(G)$ we have identities*

(a) $\langle e, g\rangle = 1$

(b) $\langle g, g\rangle = 1$

(c) $\langle fg, h\rangle = \langle f, h\rangle \cdot \langle g, h\rangle$.

Proof. The proof is left to the reader, (a) and (b) follow right from the definitions, the third equation is slightly less obvious. □

If G itself is abelian, then the construction above gives rise to a homomorphism

$$H^2(G, A) \longrightarrow \mathsf{Hom}(\bigwedge^2 G, A).$$

By definition, the kernel of this map consists of central extensions, such that $\widehat{G}$ is abelian. We obtain a short exact sequence

$$0 \longrightarrow \mathsf{Ext}^1(G, A) \longrightarrow H^2(G, A) \longrightarrow \mathsf{Hom}(\bigwedge^2 G, A) \longrightarrow 0.$$

In Prasad's notes [Pra] this sequence is deduced from abstract properties of group (co)homology.

Exercise 5.4. *Let G be a cyclic group. Show that $H^2(G, A) \simeq \mathsf{Ext}^1(G, A)$.*

There is an alternative viewpoint on central extensions. For every $g \in G$, a central extension as in (2) associates an A-torsor $L_g = p^{-1}(g)$. This defines a functor from the discrete category G (all morphisms are identities) to the category of A-torsors BA.

Definition 5.5. *Let A be an abelian group, we denote by Tors_A the symmetric monoidal groupoid of A-torsors. That is, as a category its objects are A-torsors (non-empty sets with a faithful and transitive A-action), and morphisms are A-maps (automatically isomorphisms). The symmetric monoidal structure is given by $L \otimes_A M = (L \times_A M)/_{anti} A$ with the obvious symmetry constraint.*

The category Tors_A is rather modest. It is equivalent to the category BA consisting of a unique object $\bullet$, such that $\mathsf{Hom}(\bullet, \bullet) = A$. Unravelling the symmetric monoidal structure for this model of the category BA, we see that $\bullet \otimes_A \bullet = \bullet$, and on the level of morphisms, $a \otimes_A b = ab$ is given by the group operation on A. The symmetry constraint arises naturally from the commutativity of A. All of this should be reminiscent from the discussion of the category of lines $\mathsf{Pic}(k)$ in Subsection 4.2 (which is indeed equivalent to $Bk^\times$).

For $(g, h) \in G$ we have a natural map

$$m \colon L_g \times L_h \longrightarrow L_{gh}$$

given by multiplication in $\widehat{G}$. This map satisfies $m(ax, y) = m(x, ay)$ for $x \in L_g$, $y \in L_h$ and $a \in A$. Therefore we obtain a morphism of A-torsors

$$L_g \otimes_A L_h \longrightarrow L_{gh},$$

where we use the suggestive shorthand $L_g \otimes L_h$ to denote $(L_g \times L_h)/_{\text{anti}} A$ with A acting anti-diagonally. A morphism of torsors is automatically an isomorphism. This shows that we have produced an isomorphism $m_{g,h} \colon L_g \otimes_A L_h \simeq L_{gh}$.

Lemma 5.6. *A central extension of the group G by A gives rise to a monoidal functor $G \longrightarrow \mathsf{Tors}_A$.*

Proof. We have already constructed isomorphisms $m_{g,h} \colon L_g \otimes_A L_h \longrightarrow L_{gh}$ which correspond to the A-bilinear multiplication map $L_g \times L_h \longrightarrow L_{gh}$. For this reason, associativity of multiplication in $\widehat{G}$ yields immediately that for a triple g_1, g_2, g_3 of elements in G we have a commutative diagram

$$\begin{array}{ccc} L_{g_1} \otimes_A L_{g_2} \otimes_A L_{g_3} & \xrightarrow{m_{g_1,g_2} \otimes \mathrm{id}} & L_{g_1 g_2} \otimes_A L_{g_3} \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes m_{g_2,g_3}} & & \Big\downarrow{\scriptstyle m_{g_1 g_2, g_3}} \\ L_{g_1} \otimes_A L_{g_2 g_3} & \longrightarrow & L_{g_1 g_2 g_3} \end{array}$$

of isomorphims. Furthermore, for the unit $e \in G$ we have a trivialisation of A-torsors $L_e \simeq p^{-1}(e) \simeq \ker p \simeq A$. For $g \in G$, the map

$$m_{e,g} \colon L_e \otimes_A L_g \simeq A \otimes_A L_g \simeq L_g$$

agrees with the canonical map $A \otimes_A L_g \simeq L_g$ induced by the A-action $A \times L_g \longrightarrow L_g$ (which is evidently A-bilinear). This concludes the verification that the functor $G \longrightarrow BA$ is monoidal. □

Remark 5.7. *A monoidal morphism $G \longrightarrow BA$ corresponds to a morphism of 2-groupoids $BG \longrightarrow B^2 A$ (this follows right from the definitions). Taking geometric realisations of the associated nerves we obtain a map from the classifying space of G to the degree two Eilenberg–MacLane space $K(A, 2)$. That is, by virtue of the topological definition of group cohomology as cohomology of the classifying space of G, an element of $H^2(G, A)$.*

The commutator pairing $\langle g, h\rangle$ of a central extension also has a natural interpretation in terms of this picture. If G is abelian, it measures the obstruction to

$$G \longrightarrow BA$$

being a symmetric monoidal functor. This is the content of the following proposition. We denote by $c_{L,M}\colon L \otimes_A M \simeq M \otimes_A L$ the symmetry constraint for A-torsors.

Proposition 5.8. *For every commuting pair of elemements $(g, h) \in P(G)$ we have a commutative diagram*

$$\begin{array}{ccc} L_{gh} & \longrightarrow & L_g \otimes_A L_h \\ {\scriptstyle \langle g,h\rangle^{-1}\cdot \mathrm{id}}\downarrow & & \downarrow{\scriptstyle c_{L_g,L_h}} \\ L_{hg} & \longrightarrow & L_h \otimes_A L_g \end{array}$$

of A-torsors.

Proof. We fix $\widetilde{g} \in p^{-1}(g) = L_g$ and $\widetilde{h} \in p^{-1}(h) = L_h$. As A-torsors we have $L_g = A \cdot \widetilde{g}$ and $L_h = A \cdot \widetilde{h}$. Furthermore we can use $A \cdot \widetilde{g}\widetilde{h} = L_{gh}$. Tracing through the maps of the commutative diagram above, we obtain

$$\widetilde{g}\widetilde{h} \mapsto \widetilde{g} \otimes \widetilde{h} \mapsto \widetilde{h} \otimes \widetilde{g} \mapsto \widetilde{h}\widetilde{g}.$$

Viewing this as a self-map of the A-torsor L_{gh} we see that it is given by multiplication with $\widetilde{h}\widetilde{g}(\widetilde{g}\widetilde{h})^{-1} = \langle g, h\rangle^{-1}$. □

This lengthy discussion about central extensions finally leads us to a generalised notion of central extensions where abelian groups can be replaced by so-called Picard groupoids.

Definition 5.9. *A* Picard groupoid *is a groupoid endowed with a symmetric monoidal structure $\otimes$ which is group-like, that is, the induced monoid structure on $\pi_0(\mathcal{P})$ (the set of isomorphism classes) is a group. We denote by $\pi_1(\mathcal{P})$ the group of automorphisms of the unit object $e \in \mathcal{P}$.*

A priori the group $\pi_1(\mathcal{P})$ could be non-commutative. Using the Eckmann–Hilton trick one can prove that it is an abelian group.

Definition 5.10. *Let G be a group and $\mathcal{P}$ a Picard groupoid. A central extension of G by $\mathcal{P}$ is a monoidal functor $F\colon G \longrightarrow \mathcal{P}$. For $(g, h) \in P(G)$ a commuting pair of elements we define $\langle g, h\rangle \in \pi_1(\mathcal{P})$, such that the diagram*

$$\begin{array}{ccc} F(gh) & \longrightarrow & F(g) \otimes_A F(h) \\ {\scriptstyle \langle g,h\rangle^{-1}\cdot \mathrm{id}}\downarrow & & \downarrow{\scriptstyle c_{F(g),F(h)}} \\ F(hg) & \longrightarrow & F(h) \otimes_A F(g) \end{array}$$

commutes.

We leave it to the reader to check that the pairing $P(G) \longrightarrow \pi_1(\mathcal{P})$ satisfies the properties

(a) $\langle g, h\rangle = \langle h, g\rangle$,

(b) $\langle g_1 g_2, h\rangle = \langle g_1, h\rangle \cdot \langle g_2, h\rangle$,

(c) $\langle g, e\rangle = \langle e, g\rangle = 1$.

Warning 5.11. *It is no longer true that $\langle g, g\rangle = 1$ for all $g \in G$.*

For the Picard groupoid of graded lines $\mathsf{Pic}^{\mathrm{gr}}(k)$, a central extension $G \longrightarrow \mathsf{Pic}^{\mathrm{gr}}(k)$ is called a *graded central extension*. We have monoidal functors $\mathsf{Pic}^{\mathrm{gr}}(k) \longrightarrow Bk^{\times}$ and $\mathsf{Pic}^{\mathrm{gr}}(k) \longrightarrow \mathbb{Z}$. Therefore we see that a graded central extension of G corresponds to a pair $(\widehat{G}, v)$, where

$$1 \longrightarrow k^{\times} \longrightarrow \widehat{G} \xrightarrow{p} G \longrightarrow 1$$

is a central extension, and $v\colon G \longrightarrow \mathbb{Z}$ is a group homomorphism. For $f, g \in P(G)$ we compute the graded commutator pairing to be

$$\langle f, g\rangle = (-1)^{v(f)v(g)}[\tilde{f}, \tilde{g}]$$

where $\tilde{f} \in p^{-1}(f)$ and $\tilde{g} \in p^{-1}(g)$.

5.2 Tate vector spaces

After having discussed central extensions in theory it is time to take a look at examples "arising in nature". A common source of these natural examples are related to a theory of infinite-dimensional linear algebra: the study of Tate vector spaces. Tate vector spaces were introduced by Lefschetz under the name of linearly locally compact vector spaces. Below we will discuss directly the more general notion of Tate R-modules introduced by Drinfeld.

Definition 5.12. *Let R be a commutative ring which we endow with the discrete topology. A* discrete *Tate R-module is defined to be a projective R-module V with the discrete topology. A* compact *Tate R-module is the topological dual of a discrete R-module. A topological R-module V is called an elementary Tate R-module if there exists a clopen submodule $L \subset V$, such that L is a compact Tate module and V/L is a discrete Tate module.*

The canonical example of a Tate R-module is $R((t))$ with the t-adic topology. It decomposes as a direct sum

$$R((t)) \simeq R[[t]] \oplus t^{-t}R[t^{-1}].$$

The factor on the left hand side is the topological dual of $t^{-1}R[t^{-1}]dt$ with the pairing being the residue pairing.

Definition 5.13. *A Tate R-module is a topological R-module which is a topological direct summand of an elementary Tate R-module.*

The category of Tate R-modules is interesting in its own sake. For instance one can show that its Grothendieck group is isomorphic to $K_{-1}(R)$. We will not use this, and focus mostly on elementary Tate R-modules. Drinfeld shows that every Tate R-module is étale-locally (even Nisnevich-locally) elementary.

Lemma 5.14. *(a) An elementary Tate R-module is of finite rank if and only if it is simultaneously compact and discrete.*

(b) The kernel of a surjective morphism of discrete Tate R-modules is a discrete Tate R-module.

Proof. It suffices to prove this assertion Zariski-locally on $\operatorname{Spec} R$. We may assume that R is a local ring. Let D be a discrete projective R-module, such that its topological dual $D^\vee$ is discrete. Since projective modules on local rings are free, we may choose an isomorphism $D \simeq R^{\oplus I}$ where I is a set. We then have $D^\vee \simeq R^I$ (endowed with the product topology). But since R has at least two elements, the topological space R^I is discrete if and only if I is finite. This proves assertion (a). Assertion (b) is obvious, as a surjective morphism of projective R-modules has a splitting, and direct summands of projective R-modules are projective. □

Definition 5.15. *Let V be a Tate R-module. A lattice in V is a compact Tate submodule $L \subset V$, such that V/L is projective.*

By definition, a lattice exists if and only if V is elementary. Unless $\dim V = 0$ there is always more than one lattice. Two lattices are *commensurable* in the sense that they differ by a finite-dimensional projective R-module:

Lemma 5.16. *Let $L_1 \subset L_2 \subset V$ be two lattices inside a Tate R-module V. Then L_2/L_1 is a projective R-module of finite rank.*

Proof. The quotient L_2/L_1 is the kernel of the morphism $V/L_1 \twoheadrightarrow V/L_2$. Therefore it is a discrete Tate R-module by part (b) of Lemma 5.14. On the other hand, L_2/L_1 is the cokernel of an injective map of Tate R-modules. By duality we obtain that L_2/L_1 is also a compact Tate R-module. We infer that L_2/L_1 is a finite rank projective R-module by using Lemma 5.14. □

Proposition 5.17. *For every pair of lattices $L_1, L_2 \subset V$ there exists a lattice $L \subset L_1 \cap L_2$.*

We omit the proof of this proposition, it will be added to a future version of these notes.

5.3 The Tate extension

Let V be an elementary Tate object. In this section we construct a central extension (of fppf group sheaves)

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widehat{\mathrm{Aut}}(V) \longrightarrow \mathrm{Aut}(V) \longrightarrow 1$$

In fact we will produce a *graded central extension*, that is, a central extension of $\mathrm{Aut}(V)$ by the Picard groupoid $\mathsf{Pic}^{\mathrm{gr}}(R)$ of graded lines (in Subsection 4.2 we only defined $\mathsf{Pic}^{\mathrm{gr}}(k)$ where k is a field, mutatis mutandis one obtains the definition for general commutative rings).

Lemma-Definition 5.18. *Given two lattices $L_1, L_2 \subset V$ we define $\mathsf{det}^{\mathrm{gr}}(L_1 : L_2)$ by choosing a lattice $L \subset L_1 \cap L_2$ and defining*

$$\mathsf{det}^{\mathrm{gr}}(L_1 : L_2) \simeq \mathsf{det}^{\mathrm{gr}}(L_1/L) \otimes \mathsf{det}^{\mathrm{gr}}(L_2/L)^{-1}.$$

We leave it to the reader to check that this definition is independent of L up to a natural isomorphism. Using this independence of the auxiliary lattice, one observes that for an automorphism $g \in \mathrm{Aut}(V)$, and a lattice $L \subset V$, the graded line $\mathsf{det}^{\mathrm{gr}}(gL : L)$ depends only on g up to a canonical isomorphism. Indeed, if L' is another lattice, then $\mathsf{det}^{\mathrm{gr}}(gL : L) \simeq \mathsf{det}^{\mathrm{gr}}(gL' : L') \otimes \mathsf{det}^{\mathrm{gr}}(gL : gL') \otimes \mathsf{det}^{\mathrm{gr}}(L' : L)$. The automorphism g induces an isomorphism $\mathsf{det}^{\mathrm{gr}}(gL : gL') \simeq \mathsf{det}^{\mathrm{gr}}(L : L') \simeq \mathsf{det}^{\mathrm{gr}}(L' : L)^{-1}$.

Lemma-Definition 5.19. *We define a monoidal functor* $F_L\colon \mathrm{Aut}(V) \longrightarrow \mathsf{Pic}^{\mathrm{gr}}(R)$ *as follows: choose a lattice* $L \subset V$*. The functor sends* $g \in \mathrm{Aut}(V) \longrightarrow \mathsf{det}^{\mathrm{gr}}(g^{-1}L : L)$*. For* $g = e$ *we have an isomorphism* $F_L(e) = \mathsf{det}^{\mathrm{gr}}(eL : L) = \mathsf{det}^{\mathrm{gr}}(L : L) \simeq \mathbf{1}$*. For* $g, h \in \mathrm{Aut}(V)$ *we specify the isomorphism*

$$\mathsf{det}^{\mathrm{gr}}((gh)^{-1}L : L) \simeq \mathsf{det}^{\mathrm{gr}}(h^{-1}g^{-1}L : g^{-1}L) \otimes \mathsf{det}^{\mathrm{gr}}(g^{-1}L : L) \simeq \mathsf{det}^{\mathrm{gr}}(g^{-1}L : L) \otimes \mathsf{det}^{\mathrm{gr}}(h^{-1}L : L).$$

The easy verification that this defines a monoidal functor is left to the reader. As we have seen earlier, monoidal functors $\mathrm{Aut}(V) \longrightarrow \mathsf{Pic}^{\mathrm{gr}}(R)$ correspond to functors of 2-groupoids

$$B\,\mathrm{Aut}(V) \longrightarrow B\,\mathsf{Pic}^{\mathrm{gr}}(R).$$

There is a direct construction of this functor, which is based on the fact that $B\,\mathsf{Pic}^{\mathrm{gr}}(R)$ is equivalent to the 2-groupoid of $\mathsf{Pic}^{\mathrm{gr}}(R)$-torsors. Furthermore, we observe that $B\,\mathrm{Aut}(V)$ embeds into the groupoid of all elementary Tate R-modules (discard all non-isomorphisms). We denote this groupoid by $\mathsf{Tate}^{\mathrm{el}}(R)^{\times}$. It suffices therefore to associate an elementary Tate R-module a $\mathsf{Pic}^{\mathrm{gr}}(R)$-torsor, which is functorial in isomorphisms of Tate R-modules.

Definition 5.20 (Determinantal theories, Kapranov)**.** *Let* V *be a Tate vector space. A* determinantal theory *is a pair* $(\phi, (\alpha_{L,L'}))$*, where*

$$\phi\colon \{L \subset V | L \text{ is a lattice}\} \longrightarrow \mathsf{Pic}^{\mathrm{gr}}(R)$$

assigns to a lattice $L \subset V$ *a graded line, and for a pair* (L, L') *we have an isomorphism*

$$\alpha_{L,L'}\colon \phi(L) \simeq \phi(L') \otimes \mathsf{det}^{\mathrm{gr}}(L : L'),$$

such that the following properties hold

(a) $\alpha_{L,L} : \phi(L) \simeq \phi(L) \otimes \mathbf{1}$ *is the tautological isomorphism*

(b) for a triple of lattices L, L', L'' *we have a commutative diagram*

$$\begin{array}{ccc} \phi(L) & \longrightarrow & \phi(L') \otimes \mathsf{det}^{\mathrm{gr}}(L : L') \\ \downarrow & & \downarrow \\ \phi(L'') \otimes \mathsf{det}^{\mathrm{gr}}(L : L'') & \longrightarrow & \phi(L'') \otimes \mathsf{det}^{\mathrm{gr}}(L' : L'') \otimes \mathsf{det}^{\mathrm{gr}}(L : L'). \end{array}$$

Condition (a) and (b) imply that for a given lattice $L \subset (V)$, the map $(\phi, (\alpha_{L,L'})) \mapsto (\phi(L))$ is an equivalence. That is, a determinantal theory is uniquely determined (up to a unique isomorphism) by its value on a fixed lattice. This shows that the groupoid of determinantal theories is a $\mathsf{Pic}^{\mathrm{gr}}(k)$-torsor. We denote this torsor by $\mathcal{T}_V$.

Lemma-Definition 5.21. *There is a functor of* 2*-groupoids* $\mathsf{Tate}^{\mathrm{el}}(R)^{\times} \longrightarrow B\,\mathsf{Pic}^{\mathrm{gr}}(R)$ *which sends an elementary Tate vector space* V *to the* $\mathsf{Pic}^{\mathrm{gr}}(R)$*-torsor of determinantal theories* $\mathcal{T}_V$*. An isomorphism of elementary Tate vector spaces* $f\colon V \longrightarrow W$ *induces an isomorphism* $\mathcal{T}_V \simeq \mathcal{T}_W$ *of* $\mathsf{Pic}^{\mathrm{gr}}(R)$*-torsors, as the image of a lattice in* V *under the isomorphism* f *is a lattice in* W*.*

The torsor $\mathcal{T}_V$ induces a natural $\mathbb{Z}$-torsor for every elementary Tate space V. If $R = k$ is a finite field, this torsor is closely related to the $\mathbb{R}^\times_>$-torsor of non-zero Haar measures on the locally compact topological vector space V.

The graded central extension

$$\mathrm{Aut}(V) \longrightarrow \mathsf{Pic}^{\mathrm{gr}}(R)$$

discussed above, gives rise to a function $v\colon \mathrm{Aut}(V) \longrightarrow \mathbb{Z}$ and a graded commutator pairing $P(\mathrm{Aut}(V)) \longrightarrow R^\times$. In the case $V = k((t))$, both devices are closely related to classical constructions in number theory.

Lemma 5.22. *Let k be a field, and $f, g \in k((t))^\times$ units which we view as continuous automorphisms of the Tate vector space $V = k((t))$. We then have $v(f) = v_t(f)$ (that is, the map v agrees with the t-valuation), and $\langle f, g\rangle = (-1)^{v(f)v(g)} \frac{f^{v(f)}}{g^{v(f)}}|_{t=0}$ is the tame symbol.*

Proof. Since v is a group homomorphism $k((t))^\times \longrightarrow \mathbb{Z}$ it suffices to verify the claim for $f = t$, and $f \in k[[t]]^\times$ By definition, $v(t^{-1})$ equals the degree of the graded line $\mathsf{det}^{\mathrm{gr}}(t^{-1}k[[t]] : k[[t]])$. The latter is equal to $\mathsf{det}^{\mathrm{gr}}(t^{-1}k[[t]]/k[[t]]) = \mathsf{det}^{\mathrm{gr}}(k) = (k, 1)$. That is, $v(t) = 1$. If $f \in k[[t]]^\times$, then $\mathsf{det}^{\mathrm{gr}}(fk[[t]] : k[[t]]) = \mathsf{det}^{\mathrm{gr}}(k[[t]] : k[[t]]) = (k, 0)$. This concludes the proof of the first assertion.

For the second claim, we remark that $(f, g) \mapsto \langle f, g\rangle$ and $(f, g) \mapsto (-1)^{v(f)v(g)} \frac{f^{v(f)}}{g^{v(f)}}|_{t=0}$ are bimultiplicative and alternating. Thus it suffices to prove the assertion for $(f, g) = (t, t)$, $f = t$ and $g \in R[[t]]^\times$. In the first case, $\langle t, t\rangle = -1$, as well as $(-1)^1 \frac{t}{t} = 1$.

In the second case: assume $g \in R[[t]]^\times$, we then have

$$\langle f, g\rangle^{-1} = \det(g|_{t=0}\colon t^{-1}k[[t]]/k[[t]] \longrightarrow t^{-1}k[[t]]/k[[t]]) = g|_{t=0}$$

as we wanted. □

5.4 The Clifford extension

Remark 5.23. *In this subsection we assume for simplicity that $\mathbb{Q} \subset R$.*

There is a second viewpoint on the graded central extension $\mathrm{Aut}(V) \longrightarrow \mathsf{Pic}^{\mathrm{gr}}(R)$. In this approach one associates to a Tate R-module V its so-called Clifford algebra Cl_V (which is an infinite-dimensional topological R-algebra with a natural $\mathbb{Z}$-grading). A choice of a lattice $L \subset V$ gives rise to a Cl_V-module M_L, and we will see that M_L is a projective $\mathrm{Aut}(V)$-action (in fact the infinite-dimensional symplectic group $\mathrm{Sp}(V \oplus V^\vee)$ acts projectively on M_L). The existence of a central extension can then be inferred from the following well-known lemma.

Lemma 5.24. *Let M be a free R-module acted on projectively by a group G. Then there exists a central extension*

$$1 \longrightarrow R^\times \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1,$$

such that the projective G-action on M lifts to a $\widehat{G}$ action.

Proof. We have a central extension of groups

$$1 \longrightarrow R^\times \longrightarrow \mathrm{Aut}(M) \longrightarrow \mathbb{P}\,\mathrm{Aut}(M) \longrightarrow 1.$$

A projective action corresponds to a group homomorphism $G \longrightarrow \mathbb{P}\operatorname{Aut}(M)$. We set $\widehat{G} = G \times_{\mathbb{P}\operatorname{Aut}(M)} \operatorname{Aut}(M)$ and observe that $\widehat{G}$ acts on M through the projection $\widehat{G} \longrightarrow \operatorname{Aut}(V)$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & R^\times & \longrightarrow & \widehat{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & R^\times & \longrightarrow & \operatorname{Aut}(M) & \longrightarrow & \mathbb{P}\operatorname{Aut}(M) & \longrightarrow & 1 \end{array}$$

By definition, $\widehat{G}$ is a central extension of G by $R^\times$. □

Definition 5.25 (Clifford Lie algebras)**.** *Let V be a Tate R-module. We consider the continuous bilinear map*

$$b\colon V \times V^\vee \longrightarrow R$$

given by $(v, \alpha) \mapsto \alpha(v)$. The topological R-module $V \oplus V^\vee$ is endowed with a continuous symmetric form

$$b((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_1(v_2) \oplus \alpha_2(v_1).$$

We consider the topological R-algebra generated by $V \oplus V^\vee \oplus R$ modulo the relation

$$xy + yx = b(x, y),$$

and $1 \in R \subset V \oplus V^\vee \oplus R$ is the unit in Cl_V.

We have an isomorphism $\operatorname{Cl}_V \simeq \operatorname{Cl}_{V^\vee}$. The topological R-module $V \oplus V^\vee$ is endowed with a non-degenerate alternating 2-form $\omega((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_1(v_2) - \alpha_2(v_1)$.

Let $L \subset V \oplus V^\vee$ be a lattice (that is, we assume it to be a compact Tate R-module with discrete quotient), such that $L^\perp = L$. We call such a subspace a maximally isotropic lattice.

Definition 5.26 (Vacuum module)**.** *We define M_L to be the topological Cl_V-module, such that*

$$\mathsf{Hom}_{\operatorname{Cl}_V}(M_L, N) = N^L.$$

One can define M_L in rather explicit terms. It is generated by a single element $|0\rangle$, called the *vacuum vector*, satisfying $L \cdot |0\rangle = 0$. That is, as a topological R-module it is isomorphic to $(V \oplus V^\vee)/L$.

Lemma 5.27 (Schur's lemma for vacuum modules)**.** *We have $\mathsf{End}_{\operatorname{Cl}_V}(M_L) = R$.*

Proof. By definition we have $\mathsf{End}_{\operatorname{Cl}_V}(M_L) = M_L^L$. As a topological R-module, M_L is isomorphic to $((V \oplus V^\vee)/L) \cdot |0\rangle$.

Let $y \in (V \oplus V^\vee)/L)$ be a non-zero element. We have to show that $y \cdot |0\rangle$ is not annihilated by L. Choose $x \in L$, such that $\omega(x, y) = 1$. Then we have

$$xy|0\rangle = yx|0\rangle + |0\rangle = |0\rangle \neq 0.$$

This proves $M_L^L = R \cdot |0\rangle$ and hence concludes the proof. □

Theorem 5.28. *Let $L, L' \subset V \oplus V^\vee$ be two maximally isotropic lattices. Then, Zariski-locally on $\operatorname{Spec} R$, there exist isomorphisms $M_L \simeq M_{L'}$. Furthermore, the set of such isomorphisms is a torsor under $R^\times$.*

The proof of this result will be given below. We start with a special case: V being a finite-dimensional vector space over a field k.

Lemma 5.29. *Let V be a finite-dimensional vector space over a field k. Then the algebra Cl_V is isomorphic to $\mathsf{End}_k(M_L)$ where $L \subset V \oplus V^\vee$ is an isotropic lattice.*

Proof. One sees that the morphism $\mathrm{Cl}_V \longrightarrow \mathsf{End}(M_L)$ is injective, and $\dim(\mathrm{Cl}_V) = 2^{2n} = (2^n)^2 = \dim \mathsf{End}(M_L)$. □

Corollary 5.30. *Let V be a finite projective R-module, then Cl_V is a split Azumaya algebra. Every vacuum module M_L for $L \subset V$ a maximal isotropic subspace, gives rise to a splitting.*

Now we are ready to prove the theorem above.

Proof of Theorem 5.28. The second assertion follows from the first and Lemma 5.27. So it remains to show that we have an isomorphism $M_L \simeq M_{L'}$. This is equivalent to showing that $M_L^{L'}$ is a free invertible R-module. We choose a lattice $K \subset L, L'$. It is easy to see that $K^\perp \subset K$, and the quotient $K^\perp/K$ is a projective R-module of finite rank, inheriting non-degenerate pairing. Furthermore we can write

$$M_L = \bigcup_{K \subset L, L'} (M_L)^K.$$

Each of the spaces $(M_L)^K$ is a finite-dimensional subset, as it's a quotient of $K^\perp/K$. As a consequence of the universal property of vacuum modules, $(M_L)^K$ itself is a vacuum module for the Clifford algebra of the quadratic space $K^\perp/K$. Hence, for every lattice $L'/K \subset K^\perp/K$, we have that $((M_L)^K)^{L'}$ is an invertible R-module. Taking the union over all K, we obtain that $(M_L)^{L'}$ is an invertible R-module. □

Corollary 5.31. *Let V be a Tate R-module. We denote by $\mathcal{V}_V$ the stack on $\operatorname{Spec} R$, given by Cl_V-modules which are étale-locally vacuum modules (discard all non-invertible morphisms). Then $\mathcal{V}_V$ is an $\mathcal{O}^\times$-gerbe on $\operatorname{Spec} R$.*

Fix a maximally isotropic lattice $L \subset V \oplus V^\vee$. An automorphism $g \in O(V \oplus V^\vee) \subset \mathrm{Aut}(V \oplus V^\vee)$ which preserves the symmetric bilinear form b, maps L to another maximally isotropic lattice gL. In particular, we see that M_L and the twisted Cl_V-representation $(M_L)^g = M_{gL}$ are non-canonically isomorphic (and these isomorphisms are unique up to an element of $R^\times$). Therefore, we obtain a projective representation of $O(V \oplus V^\times)$ on M_L. The corresponding central extension of $O(V \oplus V^\vee)$ will be denoted by $\widetilde{O(V \oplus V^\vee)}$.

Proposition 5.32. *Let $\mathrm{Aut}(V) \longrightarrow O(V \oplus V^\vee)$ be the canonical embedding. The pullback of the central extension of $O(V \oplus V^\vee)$ to $\mathrm{Aut}(V)$ agrees with the Tate central extension defined in the previous subsection.*

Proof. See [BBE02, Proposition 2.15]. □

The Clifford perspective implies the following property of the central extension of $\mathrm{Aut}(V)$ which we don't know how to explain purely in term of Tate modules.

Corollary 5.33. *For $f \in R((t))^\times$ we denote by $\det(f)$ the fibre of the central extension $\widehat{\mathrm{Aut}}(V)$ over f. Then this $\mathbb{G}_{m,R}$-torsor has a reduction to $\mu_{2,R}$.*

Proof. We claim that in the group $O(V \oplus V^\vee)$ we can write $f = gh$, where $g^2 = 1$. Since

$$\det(f) \simeq \det(g) \otimes \det(h),$$

and $\det(g)^2 \simeq \det(h)^2 \simeq \mathbf{1}$ we obtain a μ_2-reduction of $\det(g)$, $\det(h)$ and hence $\det(g) \otimes \det(h)$.

The elements g and h are constructed as follows. At first we observe $R((t))^\vee \simeq R((t))dt$ (with respect to the residue pairing). A 1-form ν gives therefore rise to a morphism $R((t)) \longrightarrow R((t))dt = R((t))^\vee$. In particular we can view ν as an endomorphism $V \oplus V^\vee$. In matrix form these morphisms look like

$$\begin{pmatrix} 0 & \nu^{-1} \\ \nu & 0 \end{pmatrix}$$

We set $g = fdt$ and $h = dt$. These are orthogonal automorphisms, as the induced maps $R((t)) \longrightarrow R((t))dt$ are isomorphisms.

By definition we have $gh = f$, where f is viewed as the orthogonal automorphism in the image of $\mathrm{Aut}(R((t))) \hookrightarrow O(R((t)) \oplus R((t))^\vee)$. In matrix form:

$$\begin{pmatrix} 0 & \nu^{-1} \\ \nu & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & f^{-1}\nu^{-1} \\ f\nu & 0 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & f^{-1} \end{pmatrix}.$$

This concludes the proof. □

As a consequence of this μ_2-reduction we obtain that the $\mathbb{G}_m$-torsor $\det(f)$ carries the structure of a *crystal*. Recall the following definition going back to Grothendieck.

Definition 5.34. *Let Y be a k-scheme and $\mathcal{F}$ a G-torsor Y. The structure of a crystal on $\mathcal{F}$ is given by the following data: for every pair of morphisms $x, y\colon \operatorname{Spec} R \longrightarrow Y$, such that $x|_{R^{\mathrm{red}}} = y|_{R^{\mathrm{red}}}$ an isomorphism*

$$c_{x,y}\colon x^*\,\mathcal{F} \simeq y^*\,\mathcal{F},$$

such that $c_{x,y} \circ c_{y,z} = c_{x,z}$ for every triple of points $x, y, z \in Y(R)$, satisfying $x|_{R^{\mathrm{red}}} = y|_{R^{\mathrm{red}}} = z|_{R^{\mathrm{red}}}$.

Since $\mu_{2,R} \longrightarrow \operatorname{Spec} R$ is a finite étale group scheme, every μ_2-torsor carries a canonical crystal structure.

Corollary 5.35. *For every $f \in R((t))^\times$ the $\mathbb{G}_m$-torsor is endowed with the structure of a crystal.*

5.5 The epsilon connection

The definition below defines the de Rham epsilon line with respect to a family of 1-forms. We leave the connection $\mathcal{E}$ constant, in order to avoid introducing the notion of ϵ-nice families of flat connections.

Definition 5.36. *Let $\mathcal{E}$ be a flat connection on $U \subset X$, and $\nu \in \Omega^1_{U_R/R}(U_R)$ be a generating section. For $x \in X$ a closed point we define $\varepsilon_{\nu,x} = \mathsf{det}^{\mathrm{gr}}(M_R : \nu^{-1}(N \otimes \Omega^1(S)))_x$ where (M, N) is a good lattice pair for $\mathcal{E}$.*

The link of the central extension $\widehat{\mathrm{Aut}}(R((t)))$ with epsilon factors is provided by the following lemma.

Lemma 5.37. *For an invertible function f on U_R we have an isomorphism*

$$\varepsilon_{f\nu,x}(\mathcal{E}) \simeq \varepsilon_{\nu,x} \otimes \det_x(f)^{\operatorname{rk} E},$$

where we view f as an element in $R'((t_x))^\times$ where t_x is a choice of a uniformiser near x.

Proof. This follows from the transitivity property of relative graded determinants:

$$\varepsilon_{f\nu,x}(\mathcal{E}) = \mathsf{det}^{\mathrm{gr}}(M_R : f^{-1}\nu^{-1}(N \otimes \Omega^1(S)))_x \otimes \mathsf{det}^{\mathrm{gr}}(f^{-1}(\nu^{-1}(N \otimes \Omega^1(S))) : (\nu^{-1}(N \otimes \Omega^1(S)))).$$

The factor on the right hand side can be identified with the graded determinant line associated to the automorphism of the Tate R-module $V_x = R'((t_x)) \otimes N \otimes \Omega^1_X(S)$ given by multiplication with f. Since the rank of $N \otimes \Omega^1_X(S)$ is n, it isn't difficult to show that this line is isomorphic to $\det_x(f)^n$. □

We denote by $\not\leqslant_x$ the functor assigning to a commutative k-algebra R the set $R((t_x))^{\times dt_x}$ where t_x is a local uniformiser at x.

Corollary 5.38. *Let ν_1, ν_2 be two R-families of generating forms on U, such that $\nu_1|_{R^{\mathrm{red}}} = \nu_2|_{R^{\mathrm{red}}}$, then $\varepsilon_{\nu_1,x}(\mathcal{E}) \simeq \varepsilon_{\nu_2,x}(\mathcal{E})$, and these isomorphisms define the structure of a crystal on $\mathbf{\Omega}_x$ denoted $\varepsilon_x(M)$.*

Proof. By assumption, $\nu_2 = f\nu_1$ and $f|_{R^{\mathrm{red}}} = 1$. Hence we have an isomorphism $\det(f) \simeq \det(1) \simeq \mathbf{1}$. The right hand side being the trivial $\mathbb{G}_m$-torsor, we obtain an isomorphism

$$\varepsilon_{\nu_1,x}(\mathcal{E}) \simeq \varepsilon_{\nu_2,x}(\mathcal{E}).$$

The crystal property follows from the fact that the lines $\det(f)$ have a crystal structure. □

The product formula of Beilinson–Bloch–Esnault also holds on the crystalline level. In the following we denote by $\mathbf{\Omega}$ the functor assigning to a commutative k-algebra R the set $(\Omega^1_{X/k} \otimes_k R)^\times$.

Theorem 5.39 (BBE)**.** *Let X be a smooth proper curve over k and let $\mathcal{E}/U$ be a flat connection on an open subscheme $U \subset X$. For a generating section $\nu \in \Omega^1_{X_R/R}(U_R)$ we have*

$$\bigotimes_{x \in X} \varepsilon_x(M) \simeq (\mathsf{det}^{\mathrm{gr}}(R\Gamma_{dR}(M)))_{\mathbf{\Omega}},$$

where the right hand side denotes the constant crystal with value $\mathsf{det}^{\mathrm{gr}}(R\Gamma_{dR}(M))$.

6 Patel's epsilon lines via algebraic K-theory

The graded determinant, that is, the functor

$$\mathsf{det}^{\mathrm{gr}} \colon \mathsf{Vect}_k \longrightarrow \mathsf{Pic}^{\mathrm{gr}}(k),$$

plays a particularly important role in these notes. Recall that $\mathsf{Pic}^{\mathrm{gr}}(k)$ is a so-called *Picard groupoid*; a groupoid endowed with a group-like symmetric monoidal structure $\otimes$. Furthermore, the functor

$\mathsf{det}^{\mathrm{gr}}$ is symmetric monoidal with respect to the symmetric monoidal structure $\oplus$ on Vect_k. For every short exact sequence of k-vector spaces

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

we have an isomorphism

$$\mathsf{det}^{\mathrm{gr}}(V) \simeq \mathsf{det}^{\mathrm{gr}}(V') \otimes \mathsf{det}^{\mathrm{gr}}(V'').$$

The theory of higher algebraic K-theory provides a far-reaching generalisation of this picture. The category of k-vector spaces can be replaced by an arbitrary category endowed with a notion of *extensions*[9], and the target $\mathsf{Pic}^{\mathrm{gr}}(k)$ is replaced by a topological space with a product structure well-defined and commutative up to homotopy. The target's true nature is that of a *spectrum*. We begin by giving a biased account of the theory of spectra.

6.1 Spectra as generalisations of Picard groupoids

We begin our spectral tour with a panoramic overview of stable homotopy theory, emphasising the similarity with derived categories. We follow Thomason–Trobaugh's unforgettable credo

> *"ignore any pointless examples involving baroque curiosities like "MU," "MSO," "MSpin," or the "Steenrod algebra""*[10]

and thereby completely overlook the historical development of this branch of algebraic topology.

Spectra give rise to a category $\mathsf{Ho}(\mathrm{Sp})$, the homotopy category of spectra, which possesses many similar traits to the derived category of abelian groups $D(\mathbb{Z})$:

(a) The homotopy category $\mathsf{Ho}(\mathrm{Sp})$ is a triangulated category.

(b) It is endowed with a t-structure whose heart $\mathsf{Ho}(\mathrm{Sp})^{\heartsuit}$ is equivalent to the category of abelian groups. The induced homology functors are denoted by $(\pi_i)_{i\in\mathbb{Z}}\colon \mathsf{Ho}(\mathrm{Sp}) \longrightarrow \mathsf{Mod}(\mathbb{Z})$.

(c) There exists an exact functor of triangulated categories $D(\mathbb{Z}) \longrightarrow \mathsf{Ho}(\mathrm{Sp})$ which is compatible with the standard t-structure.

These similarities continue when focusing attention on the subcategory of *connective spectra*, that is, spectra whose negative homotopy group vanish ($\pi_i(X) = 0$ for $i < 0$).

Theorem 6.1 (Deligne)**.** *There exists an equivalence between the homotopy category of* strict Picard groupoids*, and the full subcategory $D_{[0,1]}(\mathbb{Z})$ of the derived category $D(\mathbb{Z})$, consisting of chain complexes supported in degrees $[0, 1]$.*[11]

A Picard groupoid is called *strict*, if it is equivalent to a strict commutative group object in the *category* of groupoids. Dropping strictness we arrive at a similar theorem (a proof of which can be found in [Pat12, 5.1]) linking Picard groupoids to spectra.

Theorem 6.2. *There exists an equivalence between the homotopy category of Picard groupoids and the full subcategory $\mathsf{Ho}(\mathrm{Sp})_{[0,1]} \subset \mathsf{Ho}(\mathrm{Sp})$ consisting of spectra $\mathbb{E}$ whose homotopy groups $\pi_i(\mathbb{E})$ vanish whenever $i \neq 0, 1$.*

[9] That is, an exact category, or Waldhausen category, a stable ∞-category, ...

[10] [TT90, p. 249]

[11] In this subsection we use homological grading conventions when referring to chain complexes. That is, differentials lower degrees.

This theorem in turn is just a special case of description of connective spectra $\mathsf{Ho}(\mathrm{Sp})_{\geq 0}$ in terms of symmetric monoidal structures on ∞-groupoids. Tracing this through the by now classical equivalence of the homotopy category of ∞-groupoids and topological spaces (with respect to weak homotopy equivalence), this is a theorem of Segal.

Theorem 6.3. *The homotopy category of* connective spectra $\mathsf{Ho}(\mathrm{Sp})_{\geq 0}$ *is equivalent the homotopy category of* Picard ∞-groupoids*, that is, ∞-groupoids endowed with a group-like symmetric monoidal structure.*

In particular we see that there is a functor $\Omega^\infty \colon \mathsf{Ho}(\mathsf{Sp}) \longrightarrow \mathsf{Ho}(\mathsf{Spaces})$, which forgets the symmetric monoidal structure.

The theorem above should be compared to the *Dold-Kan* correspondence.

Theorem 6.4 (Dold–Kan)**.** *The category of simplicial abelian groups is equivalent to the category of connective chain complexes $D(\mathbb{Z})_{\geq 0}$ (that is, chain complexes $C^\bullet$ satisfying $H^i(C^\bullet) = 0$ for $i < 0$).*

In the light of the above a simplicial abelian group is a *model* for a *strict Picard ∞-groupoid*. The algebraic K-theory discussed in the next subsection is a connective spectrum which is hardly every strict. We can therefore only use the analogy between spectra and chain complexes as a rough guiding principle when thinking about algebraic K-theory.

6.2 Algebraic K-theory

Algebraic K-theory (as developed by Quillen and Waldhausen) assigns to a category C with a notion of extensions (this is usually an additional structure) a connective spectrum $K(\mathsf{C})$. Taking the point of view of Theorem 6.3 we may say that algebraic K-theory assigns a Picard ∞-groupoid to C. Furthermore there exists a functor of ∞-groupoids

$$\mathsf{C}^\times \longrightarrow K(\mathsf{C}),$$

where $\mathsf{C}^\times$ denotes the groupoid obtained from C by discarding non-invertible morphisms. For every extension in C,

$$X \hookrightarrow Y \twoheadrightarrow Z$$

one obtains a homotopy

$$[Y] \simeq [X] \otimes [Z]$$

in the ∞-groupoid $K(\mathsf{C})$.

Furthermore, denoting $\pi_i K(\mathsf{C})$ by $K_i(\mathsf{C})$, one has that $K_0(\mathsf{C})$ is isomorphic to the Grothendieck group of the category C.

The right class of input categories C for K varies from purpose to purpose. In the context of algebraic geometry it is most convenient to work with your favourite enhancement of triangulated categories, be it dg-categories or stable ∞-categories. A triangulated category by itself doesn't contain enough information to recover the K-theory spectrum. However, it gets close to doing so: if $F \colon \mathsf{C} \longrightarrow \mathsf{D}$ is an exact functor of enhancement of triangulated categories (e.g. dg or stable ∞-categories), such that F induces an equivalence of triangulated categories, then F induces an equivalence $K(\mathsf{C}) \simeq K(\mathsf{D})$.

Definition 6.5. *Let X be a variety, we define $K(X)$ to be the K-theory spectrum of the dg or stable ∞-category $\mathrm{Perf}(X)$ of perfect complexes on X. For a closed subvariety $Z \subset X$ we define $K(X, Z)$ to be the K-theory of the full subcategory $\mathrm{Perf}(X, Z) \subset \mathrm{Perf}(X)$ (that is, perfect complexes on X which are acyclic when restricted to $X \setminus Z$).*

With respect to these definition we have the following celebrated result by Thomason–Trobaugh:

Theorem 6.6 (Thomason–Trobaugh, proto-localisation)**.** *We have a fibre diagram*

$$\begin{array}{ccc} K(X, Z) & \longrightarrow & K(X) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & K(X \setminus Z). \end{array}$$

of connective spectra*.*

In concrete terms this means that we obtain a long exact sequence

$$K_0(X, Z) \longrightarrow K_0(X) \longrightarrow K(X \setminus Z) \longrightarrow \cdot \longrightarrow K_{i-1}(X \setminus Z) \longrightarrow K_i(X, Z) \longrightarrow K_i(X) \longrightarrow K_i(X \setminus Z) \longrightarrow \cdots,$$

however we don't have an injective map $K_0(X, Z) \longrightarrow K_0(X)$ in general. It is tempting to believe that this lack of injectivity can be explained through K-groups in negative degrees. This is indeed the case.

Theorem 6.7 (Thomason–Trobaugh, localisation)**.** *There exist spectra $\mathbb{K}(X, Z)$, $\mathbb{K}(X)$ and $\mathbb{K}(X \setminus Z)$, such that we have a bicartesian diagram*

$$\begin{array}{ccc} \mathbb{K}(X, Z) & \longrightarrow & \mathbb{K}(X) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{K}(X \setminus Z). \end{array}$$

In particular there is a long exact sequence

$$\cdots \longrightarrow \mathbb{K}_{i-1}(X \setminus Z) \longrightarrow \mathbb{K}_i(X, Z) \longrightarrow \mathbb{K}_i(X) \longrightarrow \mathbb{K}_i(X \setminus Z) \longrightarrow \cdots$$

for all integers $i \in \mathbb{Z}$. Furthermore one has $\mathbb{K}_i = K_i$ for $i \geq 0$.[12]

For a regular variety X one has $\mathbb{K}(X) = K(X)$. Related to this is the following intriguing property of algebraic K-theory.

Theorem 6.8 ($\mathbb{A}^1$-invariance)**.** *Let X be a regular variety and let $\pi \colon V \longrightarrow X$ be a fibration into affine spaces which is Zariski-locally trivial. Then the induced functor $L\pi^* \colon K(X) \longrightarrow K(V)$ is an equivalence of spectra.*

[12]The spectra $\mathbb{K}$ can be defined for arbitrary enhancements of triangulated categories or exact categories. We have $\mathbb{K}_0(\mathsf{C}) = K_0(\mathsf{C})$ if C is idempotent complete.

6.3 Patel's epsilon factor

In [Pat12] Patel introduced a formalism of de Rham epsilon factors for higher-dimensional schemes. This subsection is devoted to reviewing Patel's construction. We denote by k a field of characteristic 0.

Definition 6.9. *Let X be a scheme of finite presentation and $D \subset X$ a closed subset. We say that a finite presentation morphism of schemes $f\colon Y \longrightarrow X$ is* an isomorphism infinitely near D, *if the induced morphism of formal schemes $\widehat{Y}_D \longrightarrow \widehat{X}_D$ is an isomorphism.*

Situation 6.10. *Let X be a smooth separated k-scheme, $D \subset X$ a closed subset, and $\nu \in \Omega^1_X(X \setminus D)$ a nowhere vanishing 1-form. We denote by $D_{\mathsf{hol}}(X)$ the bounded derived category of holonomic D-modules on X, and by $D^{\nu}_{\mathsf{hol}}(X)$ the full subcategory of objects $M \in D_{\mathsf{hol}}(X)$ whose singular support S does not intersect the graph of ν.*

Theorem 6.11 (Patel)**.** *There exists a morphism of spectra $\mathcal{E}^P_\nu\colon \mathbb{K}(D_{\mathsf{hol}}(X)) \longrightarrow \mathbb{K}(X, D)$, satisfying the following properties.*

(a) "Excision": For a morphism of smooth varieties $f\colon Y \longrightarrow X$ which is an isomorphism infinitely near $D \subset X$ we have a commutative diagram of spectra

$$\begin{array}{ccc} \mathbb{K}(D^{\nu}_{\mathsf{hol}}(X)) & \longrightarrow & \mathbb{K}(D^{f^*\nu}_{\mathsf{hol}}(Y)) \\ \Big\downarrow{\scriptstyle \mathcal{E}^P_\nu} & & \Big\downarrow{\scriptstyle \mathcal{E}^P_{f^*\nu}} \\ \mathbb{K}(X, D) & \longrightarrow & \mathbb{K}(Y, f^{-1}(D)). \end{array}$$

(b) "Product formula": If X is proper we have a commutative diagram

$$\begin{array}{ccc} \mathbb{K}(D^{\nu}_{\mathsf{hol}}(X)) & \xrightarrow{\mathcal{E}^P_\nu} & \mathbb{K}(X, D) \\ & {\scriptstyle R\Gamma_{dR}} \searrow & \Big\downarrow{\scriptstyle R\Gamma} \\ & & \mathbb{K}(k) \end{array}$$

relating the morphisms $R\Gamma$ and $R\Gamma_{dR}$ induced by the exact functors $R\Gamma = R\Gamma(X, -)\colon D_{\mathrm{perf}}(X, D) \longrightarrow D_{\mathrm{perf}}(k)$, respectively $R\Gamma_{dR} = R\Gamma_{dR}(X, -)\colon D_{\mathsf{hol}}(X) \longrightarrow D_{\mathrm{perf}}(k)$.

Remark 6.12. *According to Thomason–Trobaugh [TT90, Theorem 2.6.3(d)], every $f\colon Y \longrightarrow X$ as in Definition 6.9 induces an equivalence K-theory spectra $Lf^*\colon \mathbb{K}(X, D) \xrightarrow{\simeq} \mathbb{K}(Y, D)$. Thomason–Trobaugh's definition of* isomorphisms infinitely near D *in [TT90, Definition 2.6.2.1] is different from our Definition 6.9. Yet, these two definitions are equivalent as is shown in [TT90, Lemma-Definition 2.6.2.2]. In the light of Thomason–Trobaugh's result, the excision property (a) in Patel's Theorem 6.11 can therefore be accurately described as stating that the epsilon factor $\mathcal{E}^P_\nu(M)$ of a holonomic D-modules depends only on the geometry of X and M near D.*

If D is proper, we may consider the composition $R\Gamma \circ \mathcal{E}^P_\nu\, \mathbb{K}(D^{\nu}_{\mathsf{hol}}(X)) \longrightarrow \mathbb{K}(k)$. Post-composing this with the graded determinant map $\mathbb{K}(k) \longrightarrow \mathsf{Pic}^{\mathbb{Z}}$ we can define a de Rham epsilon line $\varepsilon^P_\nu(M)$. Furthermore, by virtue of the definition of algebraic K-theory we have the following.

Lemma-Definition 6.13. *For every short exact sequence of holonomic D-modules*

$$\xi\colon M' \hookrightarrow M \twoheadrightarrow M''$$

we have an isomorphism of graded lines

$$\beta_\xi^P\colon \varepsilon_\nu^P(M) \simeq \varepsilon_\nu^P(M') \otimes \varepsilon_\nu^P(M''),$$

such that for every diagram

$$\begin{array}{ccccc} M & \hookrightarrow & N & \hookrightarrow & P \\ & & \downarrow & & \downarrow \\ & & N/M & \hookrightarrow & P/M \\ & & & & \downarrow \\ & & & & P/N \end{array}$$

we have a commutative diagram of isomorphisms of graded lines

$$\begin{array}{ccc} \varepsilon_\nu^P(P) & \longrightarrow & \varepsilon_\nu^P(N) \otimes \varepsilon_\nu^P(P/N) \\ \downarrow & & \downarrow \\ \varepsilon_\nu^P(M) \otimes \varepsilon_\nu^P(P/M) & \longrightarrow & \varepsilon_\nu^P(M) \otimes \varepsilon_\nu^P(N/M) \otimes \varepsilon_\nu^P(N/M)^{-1} \otimes \varepsilon_\nu^P(P/M). \end{array}$$

If $\dim X = 1$, $D \subset X$ a divisor, we will show in Theorem 6.18 that Patel's epsilon line satisfies

$$\varepsilon_\nu^P(M) \simeq \varepsilon_\nu^{BBE}(M),$$

and that with respect to this equivalence, β_ξ^P corresponds to β_ξ^{BBE}.

In the remainder of this subsection we will sketch the main steps of Patel's construction. The starting point is a theorem of Quillen about the K-theory of D-modules on a smooth k-scheme X. We denote by $\mathsf{FMod}_{\mathsf{coh}}(D_X)$ the quasi-abelian category of D_X-modules with a good filtration. The associated graded defines an exact functor

$$\mathsf{gr}\colon \mathsf{FMod}_{\mathsf{coh}}(D_X) \longrightarrow \mathrm{Coh}(T^*X).$$

Proposition 6.14 (Quillen)**.** *There exists a morphism of spectra $Q\colon \mathbb{K}(D_X) \longrightarrow \mathbb{K}(T^*X)$, such that the diagram*

$$\begin{array}{ccc} \mathbb{K}(D_X) & \longleftarrow & \mathbb{K}(\mathsf{FMod}_{\mathsf{coh}}(D_X)) \\ & \searrow & \downarrow \mathsf{gr} \\ & & \mathbb{K}(T^*X) \end{array}$$

commutes.

The key result underlying Patel's epsilon factors is a refinement of Quillen's result on filtered rings. Let $S \subset T^*X$ be a closed subset. We denote by $D_{\mathsf{hol}}(X, S)$ the derived ∞-category of holonomic D-modules on X with singular support contained in S.

Proposition 6.15 (Patel)**.** *There exists a morphism of spectra* $Q_S\colon \mathbb{K}(D_{\mathsf{hol}}(X,S)) \longrightarrow \mathbb{K}(T^*X,S)$ *fitting into a commutative square*

$$\begin{array}{ccc} \mathbb{K}(D_{\mathsf{hol}}(X,S)) & \longrightarrow & \mathbb{K}(T^*X,S) \\ \downarrow & & \downarrow \\ \mathbb{K}(D_{\mathsf{hol}}(X)) & \longrightarrow & \mathbb{K}(T^*X). \end{array}$$

We denote by $U = X \setminus D$. The 1-form ν defines a section $\nu\colon U \longrightarrow T^*X$ which does not intersect S (by assumption). Let $\pi\colon T^*X \longrightarrow X$ be the canonical projection.

The identity $\pi \circ \nu = \mathrm{id}_U$ gives rise to a commutative diagram of spectra

$$\begin{array}{ccc} \mathbb{K}(X) & \xrightarrow{L\pi^*} & \mathbb{K}(T^*X) \\ & \searrow & \downarrow{\scriptstyle L\nu^*} \\ & & \mathbb{K}(U). \end{array}$$

Since X is regular and $\pi\colon T^*X \longrightarrow X$ is Zariski-locally a fibration in to affine spaces, the induced morphism of spectra $\pi^*\colon \mathbb{K}(X) \xrightarrow{\simeq} \mathbb{K}(T^*X)$ is an equivalence. Therefore we also have a commutative diagram

$$\begin{array}{ccc} \mathbb{K}(X) & \xleftarrow{(\pi^*)^{-1}} & \mathbb{K}(T^*X) \\ & \searrow & \downarrow{\scriptstyle \nu^*} \\ & & \mathbb{K}(U). \end{array}$$

In particular, we get a commutative square

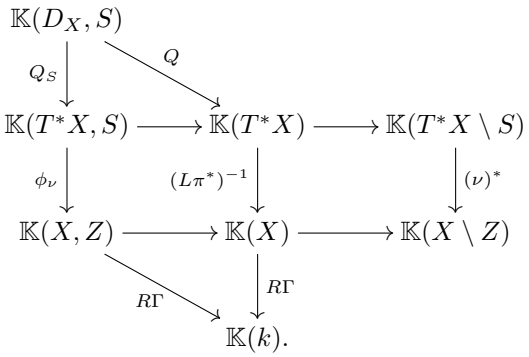
$$\begin{array}{ccc} \mathbb{K}(T^*X) & \longrightarrow & \mathbb{K}(T^*U) \\ {\scriptstyle (L\pi^*)^{-1}}\downarrow & & \downarrow{\scriptstyle L\nu^*} \\ \mathbb{K}(X) & \longrightarrow & \mathbb{K}(U) \end{array}$$

This induces a morphism between the fibres (of the rows) $\phi_\nu\colon \mathbb{K}(T^*X,S) \longrightarrow \mathbb{K}(X,D)$.

Definition 6.16 (Patel)**.** *The morphism* $\mathcal{E}^P_\nu$ *is defined to be* $\phi_\nu \circ Q_S\colon \mathbb{K}(D^\nu_{\mathsf{hol}}(X)) \longrightarrow \mathbb{K}(X,D)$.

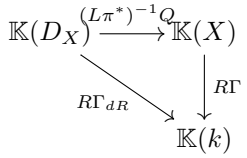
Let's prove the product formula for Patel's epsilon factor.

Proof of the product formula. We have a commutative diagram

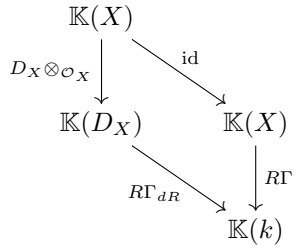


Therefore we obtain a homotopy $\mathcal{E}_\nu \simeq R\Gamma \circ \phi_\nu \circ Q_S \simeq R\Gamma \circ (L\pi^*)^{-1} \circ Q$. It remains to show that the right hand side is homotopic to morphism $R\Gamma_{dR}$. This follows from the following assertion.

Claim 6.17. *There is a commutative diagram*



The claim can be reduced to checking commutativity of



since $K(X) \longrightarrow K(D_X)$ is an equivalence of spectra (this is analogue to the $\mathbb{A}^1$-invariance property for the induced map in K-theory of the projection $T^*X \longrightarrow X$). This square commutes since $R\Gamma_{dR}(D_X \otimes_{\mathcal{O}_X} \mathcal{F}) \simeq R\Gamma(\mathcal{F})$ (see Proposition 3.7). □

6.4 Comparison

This subsection is devoted to confirming that Patel's epsilon line for curves agrees with Beilinson–Bloch–Esnault's epsilon line.

Theorem 6.18. *For a holonomic D-module M on X we have an isomorphism of graded lines*

$$\alpha_M \colon \varepsilon_\nu^P(M) \xrightarrow{\simeq} \varepsilon_\nu^{BBE}(M).$$

The proof of this result will be given below. At first we link Patel's Q_S and Quillen's Q with Deligne's good pairs. This is the content of Lemma 6.22 below.

We denote by $\pi_{\leq 1} \colon \infty - \mathsf{Gpd} \longrightarrow \mathsf{Grpd}$ the functor sending a space X to the Poincaré groupoid consisting of points of X and homotopy classes of paths between points, and by $\pi_{\leq}$ the Poincaré 2-groupoid of a space.

Lemma-Definition 6.19. *(a) Let X, U, and $\mathcal{E} = (E, \nabla)$ be as in Situation* **??**. *There is a morphism*

$$\gamma_0 \colon \pi_{\leq 2} K(\mathsf{Loc}(U)) \longrightarrow \pi_{\leq 2} K(X),$$

such that for a good lattice pair (M, N) for $\mathcal{E}$ we have

$$\gamma_0(\mathcal{E}) \simeq [M \xrightarrow{0} N \otimes \Omega^1_X(D)].$$

(b) For $\nu \in \Omega^{1,\times}_X(U)$ we have a morphism

$$\gamma_\nu \colon \pi_{\leq 1} K(\mathsf{Loc}(U)) \longrightarrow \pi_{\leq 1} K(X, D),$$

such that the square

$$\begin{array}{ccc} \pi_{\leq 1} K(\mathsf{Loc}(U)) & \xrightarrow{\gamma_\nu} & \pi_{\leq 1} K(X, D) \\ \downarrow & & \downarrow \\ \pi_{\leq 1} K(\mathsf{Loc}(U)) & \xrightarrow{\gamma_0} & \pi_{\leq 1} K(X) \end{array}$$

commutes, and for a good lattice pair (M, N) for $\mathcal{E}$ we have

$$\gamma_\nu(\mathcal{E}) \simeq [M \xrightarrow{\nu} N \otimes \Omega^1_X(D)].$$

Proof. (a): We check first that there's a well-defined map $\mathsf{Loc}(U)^\times \longrightarrow \pi_{\leq 2} K(X)$ sending $\mathcal{E} \in \mathsf{Loc}(U)$ to $[M \xrightarrow{0} N \otimes \Omega^1_X(D)]$.

Claim 6.20. *Let (M_1, N_2), and (M_2, N_3) be good lattices for $\mathcal{E}$. There exists a homotopy q_{12} in $\pi_{\leq 2} K(X)$ between $[M_1 \xrightarrow{0} N_1 \otimes \Omega^1_X(D)]$ and $[M_2 \xrightarrow{0} N_2 \otimes \Omega^1_X(D)]$. Furthermore, given a third good lattice pair (M_3, N_3) we have*

$$q_{12} \cdot q_{23} \simeq q_{13}.$$

This construction is compatible with quadruples of good lattice pairs.

Proof. This proof is a facsimile of the proof of the existence of good epsilon lines (with the additional dimension of taking the 2-categorical nature of $\pi_{\leq 2}$ into account). Without loss of generality (since the poset of good lattice pairs is filtered) we may assume that $(M_1, N_2) \subset (M_2, N_2) \subset (M_3, N_3)$. Using the H-group structure on $\pi_{\leq 2} K(X)$ we see that it suffices to construct a homotopy between $0 \in \pi_{\leq 2} K(X, Z)$ and $[M_{i+1}/M_i \xrightarrow{0} N_{i+1}/N_i \otimes \Omega^1_X(D)]$.

By virtue of the definition of good lattice pairs, the complexes

$$[M_{i+1}/M_i \xrightarrow{\nabla} N_{i+1}/N_i \otimes \Omega^1_X(D)]$$

are acyclic for $i = 1, 2$. The map ∇ is not $\mathcal{O}_X$-linear. However, we observe that $K(X, Z)$ is equivalent to the $K(\mathsf{Sh}^f_{X,Z}(k))$, where $\mathsf{Sh}^f_{X,Z}$ denotes the abelian category of sheaves of k-vector spaces on X with support Z. In $\pi_{\leq 2}K(\mathsf{Sh}^f_{X,Z}(k))$ we have a homotopy

$$[M_{i+1}/M_i \xrightarrow{\nabla} N_{i+1}/N_i \otimes \Omega^1_X(D)] \simeq [M_{i+1}/M_i] \ominus [N_{i+1}/N_i \otimes \Omega^1_X(D)] \simeq [M_{i+1}/M_i \xrightarrow{0} N_{i+1}/N_i \otimes \Omega^1_X(D)],$$

where $\ominus$ denotes subtraction with respect to the H-group structure on $\pi_{\leq 2}K(X)$. The left hand side is represented by an acylic complex, and therefore homotopic to 0. This concludes the proof of the claim. □

It remains to check that there exists a map $K(\mathsf{Loc}(U)) \longrightarrow K(X)$, such that the diagram

$$\begin{array}{ccc} \mathsf{Loc}(U)^\times & & \\ \downarrow & \searrow & \\ \pi_{\leq 1}(\mathsf{Loc}(U)) & \longrightarrow & K(X) \end{array}$$

commutes. This follows at once from the behaviour of lattice pairs with respect to short exact sequences.

Claim 6.21. *A short exact sequence of flat connections $\mathcal{E}' \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E}''$ on U can be lifted to a short exact sequence of good lattice pairs:*

$$\begin{array}{ccccc} M' & \hookrightarrow & M & \twoheadrightarrow & M'' \\ \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle\nabla} \\ N' \otimes \Omega^1_X(D) & \hookrightarrow & N \otimes \Omega^1_X(D) & \twoheadrightarrow & N'' \otimes \Omega^1_X(D). \end{array}$$

Proof. By virtue of formal descent we may assume that X is a trait. Without loss of generality we assume that $X = \operatorname{Spec} k[[t]]$ and $U = \operatorname{Spec} F$ where $F = k((t))$. For split short exact sequences the assertion is obvious. It suffices therefore to prove the claim when $\mathcal{E}'$ and $\mathcal{E}''$ are indecomposable and the short exact sequence is non-split. It then follows from Levelt–Turritin that there exists a finite étale morphism $q\colon \operatorname{Spec} F' \longrightarrow \operatorname{Spec} F$ (of generic points of traits), such that the short exact sequence is given by the push-forward q_* applied to the short exact sequence

$$\mathcal{L} \otimes (\mathcal{E}_{(n-1)} \hookrightarrow \mathcal{E}_{(n)} \longrightarrow \mathcal{E}_{(i)})$$

where $\mathcal{E}_{(n)}$ is the flat connection $(\mathcal{O}_{F'}, d + \frac{J_{(n)}}{z} dz)$ (where $J_{(n)}$ is an $(n \times n)$-Jordan block), and $\mathcal{L}$ is a rank 1 flat connection on $\operatorname{Spec} F'$.

We can then construct the short exact sequence of good lattice pairs by pushing-forward the pairs $(L^{\oplus \ell}, L(\mathsf{irr}(\mathcal{L}))^{\oplus \ell})$ for $\ell = n - i, n, i$, where $L \subset \mathcal{L}$ is a Deligne lattice and $\mathsf{irr}(\mathcal{L})$ denotes the irregularity of $\mathcal{L}$. □

This concludes the construction of the map γ_0. We now briefly turn to γ_ν: as in the proof of Claim 6.20 one verifies that there is a well-defined map $\mathsf{Loc}_n(U)^\times \longrightarrow \pi_{\leq 1}K(X, Z)$ given by $\mathcal{E} \mapsto [M_1 \xrightarrow{\nu} N_1 \otimes \Omega^1_X(D)]$. At first we remark that the complex $[M_1 \xrightarrow{\nu} N_1 \otimes \Omega^1_X(D)]$ is acyclic

when restricted to U, since $\nu(U) \cap S = \emptyset$. Therefore it defines indeed a point in $\pi_{\leq 1}K(X, Z)$. Furthermore, we may assume that $(M_1, N_2) \subset (M_2, N_2) \subset (M_2, N_3)$. As in the proof of Claim 6.20 we see that in $\pi_{\leq 1}K(\mathsf{Sh}^f_{X,Z}(k))$

$$[M_2/M_1 \xrightarrow{\nu} N_2/N_1 \otimes \Omega^1_X(D)] \simeq [M_2/M_1] \ominus [N_2/N_1 \otimes \Omega^1_X(D)] \simeq [M_2/M_1 \xrightarrow{\nabla} N_2/N_1 \otimes \Omega^1_X(D)] \simeq 0.$$

The compatibility of good lattice pairs with short exact sequences (Claim 6.21) implies the assertion. □

Lemma 6.22. *There is a commutative diagram of Picard groupoids*

$$\begin{array}{ccc} \pi_{\leq 1}K(\mathsf{Loc}_n(U)) & \xrightarrow{\mathcal{E}_\nu} & \pi_{\leq 1}K(X, D) \\ & \searrow^{\gamma_\nu} & \| \\ & & \pi_{\leq 1}K(X, D). \end{array}$$

Proof. We will verify these statements after formal completion at D (that is, after replacing X by the disjoint union of traits $\widehat{X}_D$). This is justified by the following well-known property of algebraic K-theory.

Claim 6.23 (Formal descent for algebraic K-theory)**.** *The commutative square of spectra*

$$\begin{array}{ccc} K(X) & \xrightarrow{j^*} & K(U) \\ \downarrow & & \downarrow \\ K(\widehat{X}_D) & \xrightarrow{\widehat{j}^*} & K(\widehat{X}_D \times_X U) \end{array}$$

is cartesian.

Proof. It suffices to show that the induced map of fibres $\mathsf{fib}(j^*) \longrightarrow \mathsf{fib}(\widehat{j}^*)$ is an equivalence. Thomason–Trobaugh's localisation theorem implies that this map is homotopic to the natural morphism

$$K(X, D) \longrightarrow K(\widehat{X}_D, D).$$

According to [TT90, Theorem 2.6.3(d)] this is an equivalence. □

In order to conclude the assertion, we have to show that the following commutative diagrams of 2-Picard groupoids are equivalent:

$$\begin{array}{ccc} \pi_{\leq 2}K(\mathsf{Loc}(U)) & \xrightarrow{\gamma_0} & \pi_{\leq 2}K(U) \\ {\scriptstyle\gamma_0}\downarrow & \searrow & \downarrow{\scriptstyle f^*} \\ \pi_{\leq 2}K(\widehat{X}_D) & \xrightarrow{\widehat{j}^*} & \pi_{\leq 2}K(\widehat{X}_D \times_X U) \end{array} \qquad \begin{array}{ccc} \pi_{\leq 2}K\,\mathsf{Loc}(U) & \xrightarrow{(\pi^*)^{-1}} & \pi_{\leq 2}K(U) \\ {\scriptstyle(\pi^*)^{-1}}\downarrow & \searrow & \downarrow{\scriptstyle f^*} \\ \pi_{\leq 2}K(\widehat{X}_D) & \xrightarrow{\widehat{j}^*} & \pi_{\leq 2}K(\widehat{X}_D \times_X U). \end{array}$$

In order to construct such an equivalence we "cut" the commutative squares into two halves along the dashed arrow, as indicated in the diagram above. We will then compare these halves individually. We claim that we have a two equivalent commutative triangles of Picard 2-groupoids

$$\begin{array}{ccc} \pi_{\leq 2}K(\mathsf{Loc}(U)) & \xrightarrow{\gamma_0} & \pi_{\leq 2}K(U) \\ & \searrow^{F} & \downarrow{\scriptstyle f^*} \\ & & \pi_{\leq 2}K(\widehat{X}_D \times_X U) \end{array} \qquad \begin{array}{ccc} \pi_{\leq 2}K\,\mathsf{Loc}(U) & \xrightarrow{(\pi^*)^{-1}} & \pi_{\leq 2}K(U) \\ & \searrow^{F} & \downarrow{\scriptstyle f^*} \\ & & \pi_{\leq 2}K(\widehat{X}_D \times_X U), \end{array}$$

where $F\colon D^b(\mathsf{Loc}(U)) \longrightarrow \mathrm{Perf}(\widehat{X}_D \times_X U)$ is the exact functor sending (E, ∇) to $[E \xrightarrow{0} E \otimes \Omega^1_X] \otimes_{\mathcal{O}_U} \widehat{\mathcal{O}}_{X,D}$. For the left hand side this follows from the definition of γ_0, for the right hand side this is a consequence of $(\pi^*)^{-1} \simeq i_0^*$, where $i_0\colon X \longrightarrow T^*X$ denotes the zero section. The same argument provides a homotopy between γ_0 and $(\pi^*)^{-1}$ which extends to a comparison of the two commutative diagrams. Mutatis mutandis we compare the remaining two commutating triangles. □

Proof of Theorem 6.18. For a holonomic D-module M on X, there exists an open subset $j\colon U \hookrightarrow X$, such that $M|_U \simeq \mathcal{E}$ is a vector bundle with a flat connection. The fibre of the map $M \longrightarrow j_*\mathcal{E}$ is a complex of D-modules with support on $D = X \setminus U$. Thus it suffices to prove the theorem for the D-module $j_*\mathcal{E}$.

Applying the determinant of cohomology, Lemma 6.22 yields a commutative diagram of Picard groupoids.

$$\begin{array}{ccc} \pi_{\leq 1}K(\mathsf{Loc}_n(U)) & \xrightarrow{\varepsilon^P_\nu} & \mathsf{Pic}^{\mathrm{gr}}(k) \\ & \searrow^{\det^{\mathrm{gr}}(R\Gamma(\gamma_\nu))} & \| \\ & & \mathsf{Pic}^{\mathrm{gr}}(k). \end{array}$$

It remains to show that $\det^{\mathrm{gr}}(R\Gamma(X, \gamma_\nu))$ is isomorphic to $\tilde{\varepsilon}^{BBE}(X, \mathcal{E})$. This is a consequence of Corollary **??**. □

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