The Multiplicative Hitchin System

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May 11, 2020

Abstract

Multiplicative Higgs bundles are an analogue of ordinary Higgs bundles where the Higgs field takes values in a Lie group instead of its Lie algebra. The moduli space of multiplicative Higgs bundles on $\mathbb{CP}^1$ with a framing at infinity has additional structure: that of an integrable system. In this talk I'll discuss a few different contexts where these integrable systems naturally appear: in physics (from supersymmetric gauge theory), in geometry (from the theory of periodic monopoles), and in representation theory (as symplectic leaves in a Poisson Lie group). This is based on joint work with Vasily Pestun.

1 Introduction

My goal in this talk is to introduce you to an algebro-geometric object – the *multiplicative Hitchin system* – and explain how it shows up in quantum field theory in a few different contexts. I'll also talk about a speculative version of the geometric Langlands conjecture for this object. This is joint work with Vasily Pestun [EP19].

I'll begin by defining the multiplicative Hitchin system on a curve C – which will be an algebraic integrable system, in particular a hyperkähler variety – then explain several contexts in which it arises:

1. Directly from geometry: as a moduli space of multiplicative Higgs bundles (has a nice derived picture).
2. As a symplectic leaf in a Poisson Lie group.
3. As a moduli space of monopoles on $C \times S^1$.
4. In physics: as the Seiberg-Witten integrable system associated to a quiver gauge theory.
5. Also in physics: from twisting a 5d $\mathcal{N} = 2$ supersymmetric gauge theory.

2 Multiplicative Higgs Bundles

Let's explain the main object of study: the moduli space of multiplicative Higgs bundles. The moduli space we'll consider has been discussed in various guises before: the earliest reference I'm aware of is due to Arutyunov, Frolov, and Medvedev [AFM97a, AFM97b]. We also refer to Frenkel-Ngô [FN11], Bouthier [Bou15a, Bou14, Bou15b], and particulatly Hurtubise-Markman [HM02] – it is sometimes referred to as the moduli space of "G-pairs". Throughout this talk I'll work over the complex numbers. I'll write C for a smooth complex curve and G for a complex reductive group.

Here's the idea. Recall that a *Higgs bundle* on C is a principal G-bundle P along with a section ϕ of the coadjoint bundle $\mathrm{ad}(P)^*$ twisted by the canonical bundle. The set of Higgs bundles can be promoted to the closed points of a stack: the moduli stack of Higgs bundles.

Remark 2.1. I'm going to ignore that tricky twist by the canonical bundle. For this talk I'll only be interested in the Calabi-Yau case, where C is either an elliptic curve or an object modelling $\mathbb{C}$ or $\mathbb{C}^\times$ with appropriate boundary conditions. While one can make sense of ordinary Higgs bundles on any curve this won't be true anymore for the multiplicative version I'm going to define in a moment, at least not along with all the structure that the moduli space of Higgs bundles usually includes.

Let's give a concise definition of the moduli space of Higgs bundles (without the canonical bundle twist).

Definition 2.2. The moduli space $\mathrm{Higgs}_G^{\mathcal{O}}(C)$ of $\mathcal{O}$-Higgs bundles on C is the moduli space $\underline{\mathrm{Map}}(C, \mathfrak{g}^*/G)$ of maps into the coadjoint quotient stack $\mathfrak{g}^*/G$.

The *multiplicative* version of this moduli space replaces the coadjoint bundle $\mathrm{ad}(P)^*$ by the *group* adjoint bundle $\mathrm{Ad}(P)$. So a closed point in the moduli space corresponds to a principal G-bundle P on C along with a section of $\mathrm{Ad}(P)$: i.e. an automorphism of P. Let's give the analogous concise definition as a mapping space.

Definition 2.3. The moduli space $\mathrm{mHiggs}_G(C)$ of multiplicative Higgs bundles on C is the moduli space $\underline{\mathrm{Map}}(C, G/G)$ of maps into the group adjoint quotient stack G/G.

2.1 Including Poles

So far this moduli space won't be very interesting, especially in the important example where $C = \mathbb{A}^1$. We get something more interesting by introducing simple poles for our multiplicative Higgs fields. Let $D \subseteq C$ be a finite set of points in C.

Definition 2.4. The moduli space $\mathrm{mHiggs}_G(C, D)$ of multiplicative Higgs bundles on C with poles at the subset D is the moduli space modelling a G-bundle P on C equipped with a section of $\mathrm{Ad}(P)|_{C \setminus D}$. Globally we define the moduli space as the fiber product

$$\mathrm{mHiggs}_G(C, D) := \mathrm{mHiggs}_G(C \setminus D) \times_{\mathrm{Bun}_G(C \setminus D)} \mathrm{Bun}_G(C).$$

We'd like to prescribe the behaviour of the multiplicative Higgs field near the punctures. In the neighbourhood of a point $z \in D$ the multiplicative Higgs field is described by an element of $\mathrm{Maps}(\mathbb{D}^\times, G) = G((z))$, where $\mathbb{D}^\times$ is the formal punctured disk. This element is only well-defined up to gauge transformations which extend across the puncture.

We'll actually only specify the local behaviour up to the action of $\mathrm{Maps}(\mathbb{D}, G)^2 = G[\![z]\!]^2$ on the left and right. The set of $G[\![z]\!]$ double cosets in $G((z))$ is in canonical bijection with the set of dominant coweights of the group G, so at each puncture $z \in D$ we fix a dominant coweight $\omega_z^\vee$. I'll write $\omega^\vee$ for short for the set $\{\omega_z^\vee : z \in D\}$.

Remark 2.5. Recall that a coweight is just a homomorphism $\omega \colon T \to \mathbb{G}_m$ from the maximal torus. Inside this lattice there is a "dominant cone" consisting of coweights that give a positive result when paired with any positive coroot.

To specify the moduli space, we can think of the space of double cosets as a stack

$$G[\![z]\!] \backslash G((z)) / G[\![z]\!] = G[\![z]\!] \backslash \mathrm{Gr}_G$$

where Gr_G is the affine Grassmannian. For each $z \in D$ fix a left $G[\![z]\!]$-orbit in Gr_G corresponding to a coweight $\omega_z^\vee$, and let $\Lambda_z \subseteq G[\![z]\!]$ be the corresponding stabilizer.

Remark 2.6. We can think of the moduli space of multiplicative Higgs bundles as the moduli space of G-bundles on the product $(\Sigma \times S^1) \setminus (D \times \{0\})$, which are algebraic in the Σ direction and locally constant in the S^1 direction. One nice way of making this precise is using derived algebraic geometry, and considering G bundles on the derived stack $(\Sigma \times S^1_B) \setminus (D \times \{0\})$, where S^1_B is the "Betti stack" of the circle, i.e. the stack whose functor of points is the constant functor with value S^1. We can think of fixing singularities as fixing the restriction to a "formal punctured neighbourhood" of D.

Definition 2.7. The moduli space $\mathrm{mHiggs}_G(C, D, \omega^\vee)$ of multiplicative Higgs bundles on C with poles at the subset $D = \{z_1, \ldots, z_k\}$ and fixed residue $\omega_z^\vee$ at each $z \in D$ is defined to be the fiber product

$$\mathrm{mHiggs}(C, D, \omega^\vee) = \mathrm{mHiggs}(C, D) \times_{(G[\![z]\!] \backslash \mathrm{Gr}_G)^k} (B\Lambda_{z_1} \times \cdots \times B\Lambda_{z_k}).$$

Remark 2.8. These moduli spaces are empty unless the coweights at each puncture are chosen appropriately. Specifically one needs to assume that the sum $\sum_{z \in D} \mathrm{ord}\langle \rho, \omega_z^\vee \rangle$ is equal to zero, where ρ is the Weyl vector, and ord denotes the order of the pole or zero of a representative element of $\mathbb{C}((z))$.

Examples 2.9.

1. Elliptic: Now let $C = E$, an elliptic curve. In this case we can just use the above definition and consider the moduli space $\mathrm{mHiggs}_G(E, D, \omega^\vee)$. This is now a stack with a smooth map down to the stack $\mathrm{Bun}_G(E)$ of principal G-bundles on the elliptic curve. In particular in this case one can consider the non-trivial moduli space where $D = \varnothing$: $\mathrm{mHiggs}_G(E) = \underline{\mathrm{Map}}(E, G/G)$. This example was studied in great depth by Hurtubise and Markman.
2. Trigonometric: Let $C = \mathbb{CP}^1$, but now we fix the following data. Choose a pair B_+, B_- of opposite Borel subgroups of G with unipotent radicals N_+ and N_-. We consider G-bundles on $\mathbb{CP}^1$ with a meromorphic Higgs field $g(z)$ with fixed poles, and with $g(0) \in B_+$ and $g(\infty) \in N_-$. With fixed residues this again defines a finite-dimensional smooth variety.
3. Rational: We still $C = \mathbb{CP}^1$ and consider the moduli space of multiplicative Higgs bundles with a fixed framing at infinity. In other words, consider the fiber product $\mathrm{mHiggs}(\mathbb{CP}^1, D, \omega^\vee) \times_{G/G} g_\infty$, where we view G/G as $\underline{\mathrm{Map}}(\{\infty\}, G/G)$. This is a finite-dimensional smooth variety whose points are G-valued rational functions with fixed simple poles and zeroes in $\mathbb{C}$ and asymptotic to $g_\infty z^d$ near $z = \infty$.

Remark 2.10. There are actually two rational/trigonometric/elliptic trichotomies that appear in this story. The one we just discussed, and trichotomy: the ordinary (additive) spaces of Higgs bundles, the space of multiplicative Higgs bundles, and a space of "elliptic" Higgs bundles. This latter moduli space is defined as the moduli space $\mathrm{Bun}_G(C \times E_q)$, where C is one of the curves above, and E_q is a fixed elliptic curve with parameter q. Alternatively you can think of them as a space of "q-twisted" multiplicative Higgs bundles for the loop group LG. Note that the multiplicative and additive cases arise when E_q is degenerated to a nodal or a cuspidal curve respectively.

3 The Poisson Lie Group

Let's discuss the connection between the classical moduli spaces and Poisson Lie groups.

Definition 3.1. Write $G_1[[z^{-1}]]$ for the group of G-valued power series in the parameter z^{-1} with constant leading term. This group has a natural Poisson structure, coming – for instance – from the Manin triple $(G(\!(z^{-1})\!), G_1[[z^{-1}]], G[z])$. We'll sometimes refer to it as the *rational Poisson Lie group*.

Remark 3.2. Alternatively, we can describe the Poisson structure as a formula in terms of the classical r-matrix $r = \frac{\Omega}{z-w}$, where Ω is the quadratic Casimir element. This element lives in $\mathfrak{g}^{\otimes 2}[w][\![z^{-1}]\!]$, and can be paired with two tangent vectors to the Poisson Lie group using the residue pairing. We define the Poisson bracket of f_1 and f_2, evaluated at g, by pairing r with the difference $\nabla_L(f_1)(g) \otimes \nabla_L(f_2)(g) - \nabla_R(f_1)(g) \otimes \nabla_R(f_2)(g)$.

Theorem 3.3. The algebraic map $r_\infty \colon \mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee) \to G_1[[z^{-1}]]$ defined by restricting to a formal neighbourhood of ∞ is a Poisson map: the inclusion of a symplectic leaf.

Dually, in terms of algebras of functions, there is a map of Poisson algebras $\mathcal{O}(G_1[[z^{-1}]]) \to \mathcal{O}(\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee))$. It turns out that we can quantize this structure, using some pre-existing structures from the literature. First of all, the quantization of the rational Poisson Lie group is well-known: it's the so-called *Yangian* quantum group $Y(\mathfrak{g})$.

Theorem 3.4 (Following from work of Gerasimov-Kharchev-Lebedev-Oblezin)**.** The algebra $\mathcal{O}(\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee))$ admits a deformation quantization $\mathcal{O}_\hbar(\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee))$ with the canonical structure of a representation for the Yangian $Y(\mathfrak{g})$.

The additional observation that we need to make in order to see this is fairly simple. Gerasimov, Kharchev, Lebedev and Oblezin [GKLO05] constructed $Y(\mathfrak{g})$-modules whose classical limits sweep out the symplectic leaves of $G_1[[z^{-1}]]$ of each "type", meaning each possible sum of residues. Our moduli spaces are symplectic leaves of this form, therefore quantize to representations within the GKLO classification.

4 Monopoles and Integrable System Structures

From now on, we'll focus our attention on the rational case from example 2.9. The elliptic version of the story we're about to tell is a theorem of Hurtubise and Markman [HM02]. The trigonometric version has not yet, to my knowledge, been studied.

Let's talk about the upshot of our discussion so far. The fact that we've identified the moduli space of multiplicative Higgs bundles with a Coulomb branch tells us that it should have the structure of an *algebraic integrable system*: that is, it admits a holomorphic symplectic structure, and can be written as a fibration over an affine base space whose fibers are, generically Lagrangian algebraic tori. Our work gives a natural algebraic description for this integrable system structure, which we can explicitly compare with previously known constructions. Let me elaborate on this.

A *monopole* on a Riemannian 3-manifold M for the group G is a G-bundle P on M with connection A along with a section Φ of $\mathrm{ad}(P)$ that satisfy the Bogomolny equation

$$*F_A - \mathrm{d}_A\Phi = 0.$$

I won't explain exactly what a Dirac singularity is, except to note that they are well-behaved local singularities indexed by coweights of G. By a framing we mean a fixed limit for the holonomy of $A + i\Phi$ around S^1 at ∞.

Now we can explain what this has to do with the moduli space of multiplicative Higgs bundles that I introduced earlier!

Theorem 4.1 (Charbonneau-Hurtubise [CH10] (for $\mathrm{GL}(n)$), Smith [Smi16] (for general G))**.** Let $D \subseteq C \times S^1$ be a finite subset, write π for the projection $C \times S^1 \to C$ and assume that D contains at most one point in each fiber of π. Fix a dominant coweight $\omega_z^\vee$ at each point $z \in D$. There is an analytic equivalence

$$\mathrm{Mon}_G(C \times S^1, D, \omega^\vee) \xrightarrow{\sim} \mathrm{mHiggs}_G(C, \pi(D), \omega^\vee)$$

between the moduli space of monopoles on $C \times S^1$ and the moduli space of (poly-stable) multiplicative Higgs bundles on C with compatible singularities, given by taking the holonomy of $A + i\Phi$ around the circle S^1.

This equivalence commutes with natural maps down to the base $\mathcal{B}$ on the two sides. Here one can identify

$$\mathcal{B} = \Gamma\left(C; \bigoplus_{i=1}^{r} \mathcal{O}_C\left(\omega_D^\vee(\lambda_i) \cdot D\right)\right)$$

where $\omega_D^\vee(\lambda_i) \cdot D$ denotes the divisor $\{\omega_z^\vee(\lambda_i) \cdot z : z \in D\}$. This space of global sections is the same as the space of maps $C \setminus D \to T/W$ with simple poles with prescribed residues at the divisor D. The space of multiplicative Higgs bundles maps down to $\mathcal{B}$ by composing with the characteristic polynomial (Chevalley) map $\chi \colon G/G \to T/W$. On the other hand the projection down to $\mathcal{B}$ in the integrable system as calculated by Nekrasov and Pestun is indeed given by the map $\chi(\oint_{S^1} A + i\Phi)$.

Theorem 4.2. In the rational case, the multiplicative Hitchin system with fixed residues $\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee)$ naturally has the structure of an algebraic completely integrable system with base $\mathcal{B}$. With respect to this symplectic structure the Charbonneau–Hurtubise–Smith isomorphism is a symplectomorphism.

Remark 4.3. We can write the symplectic structure down very explicitly. Intuitively we think of it as coming from the formalism of shifted symplectic structures and shifted Poisson structures (as developed in [PTVV13, CPT⁺17, MS16, MS17]). We don't have time to talk about this construction today, but the idea is to exhibit a 1-shifted Lagrangian structure on the restriction map to the 1-shifted symplectic stack $G[\![z]\!]\backslash G((z))/G[\![z]\!]$ representing G-bundles on the formal bubble $\mathbb{B} = \mathbb{D} \sqcup_{\mathbb{D}^\times} \mathbb{D}$. While this is an attractive intuitive story, there are technical obstructions that make it hard to make this rigorous using current technology.

In particular, this theorem implies that the moduli spaces $\mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee)$ are hyperkähler (although not necessarily canonically: fixing a canonical structure requires fixing a polarization, i.e. a positive integral 1, 1-form on each generic fiber of the integrable system). We can actually go further and explain what happens when we vary the complex structure in the twistor sphere. The deformed moduli spaces have a natural description in terms of *q-difference connections*.

Definition 4.4. Let q be an automorphism of a curve C – as usual we'll think of the three examples where $C = \mathbb{C}, \mathbb{C}^\times$ or E, in which case we can think of C as its own group of automorphisms. A *q-difference connection* on C is a G-bundle P along with an isomorphism $A \colon P \to q^*P$ of G-bundles. One can consider q-difference connections with poles at a finite subset $D \subseteq C$: just as for multiplicative Higgs fields we can fix the behaviour of a q-difference connection near a pole by fixing a closed point in $G[\![z]\!] \setminus \mathrm{Gr}_G$, or equivalently a dominant coweight.

The *moduli space* of q-difference connections on C with singularities at D and residues $\{\omega^\vee\}$ is defined to be

$$q\text{-}\mathrm{Conn}_G(C, D, \omega^\vee) := \underline{\mathrm{Map}}(C \times_q S^1_B \setminus D, BG) \times_{(G[\![z]\!] \setminus \mathrm{Gr}_G)^k} (B\Lambda_1 \times \cdots \times B\Lambda_k)$$

where by $C \times_q S^1_B$ we mean the mapping torus of the automorphism q, viewed as a derived stack, and by $C \times_q S^1_B \setminus D$ we mean the complement of the subset D of the fiber over $1 \in S^1$. When $q \to 0$ this recovers the moduli space of multiplicative Higgs fields.

Let's discuss what Theorem 4.2 tells us about the twistor deformation of our moduli space.

If we choose a radius r, the moduli space of periodic G-monopoles (so monopoles on $\mathbb{R}^2 \times S^1_r$) is not just holomorphic symplectic but hyperkähler – it's defined as a hyperkähler quotient. Our theorem then gives a canonically associated hyperkähler structure to the multiplicative Hitchin system (depending on r).

In the limit $r \to \infty$ one can explicitly describe the twistor family of holomorphic symplectic spaces on the monopole side. Consider the space $\mathrm{Mon}_G(C \times S^1_r, D, \omega^\vee)_{J_\zeta}$, i.e. the space considered in the complex structure at ζ in the twistor sphere. Take the limit $r \to \infty$ and $\zeta \to 0$, keeping the product $r\zeta = q$ fixed. In this limit we can identify $\mathrm{Mon}_G(C \times S^1_r, D, \omega^\vee)_{J_\zeta}$ with monopoles on the twisted product $\mathrm{Mon}_G(C \times_q S^1_1, D, \omega^\vee)_{J_0}$ in the untwisted complex structure. In fact the argument of Charbonneau and Hurtubise works equally well for this twisted product, providing an equivalence

$$\mathrm{Mon}_G(C \times_q S^1_1, D, \omega^\vee) \to q\text{-}\mathrm{Conn}_G(C, D, \omega^\vee).$$

Therefore our theorem implies the following.

Corollary 4.5. If $\mathrm{mHiggs}_G(C, D, \omega^\vee)$ is equipped with the hyperkähler metric in the $r \to \infty$ limit, in the complex structure at q in a neighbourhood of 0 in the twistor sphere it becomes algebraically isomorphic to $q\text{-}\mathrm{Conn}_G(C, D, \omega^\vee)$.

5 Origins via Gauge Theory

Let me briefly describe two contexts in which the moduli space of multiplicative Higgs bundles naturally appears: via Seiberg-Witten theory for $\mathcal{N} = 2$ 4d quiver gauge theories, and via the "twisting" construction applied to $\mathcal{N} = 2$ 5d gauge theories.

5.1 Quiver Gauge Theory

The first appearance that we'll discuss arises from work of Nekrasov, Pestun and Shatashvili [NP12, NPS18]. They studied the theory of $\mathcal{N} = 2$ quiver gauge theories in dimension 4, in particular analysing, and later quantizing, their Coulomb branches.

To be a little more specific, to specify an $\mathcal{N} = 2$ gauge theory one need to fix some data:

1. A compact semisimple group G_{gauge}: the *gauge group* (I'm saving the notation G for something else shortly).
2. A representation V of G_{gauge}: the *matter representation*.
3. A complex number τ_i for each simple factor of G_{gauge}: the *coupling constants*.
4. A complex number m_j for each irreducible summand of V: the *masses*.

Most of these theories are pretty badly behaved when you try to quantize them, but there's an especially nice family of "quiver gauge theories". One chooses the group G_{gauge} to be a product of $\mathrm{SU}(n_i)$, where we think of the factors as associated to the vertices of an ADE quiver. One then chooses the representation V to have a summand V_{ij} for each edge of the quiver isomorphic to the bifundamental representation of $\mathrm{SU}(n_i) \times \mathrm{SU}(n_j)$, plus a summand for each vertex that looks like k_i copies of the fundamental representation of $\mathrm{SU}(n_i)$. The masses associated to the bifundamental representations are fixed, but the masses of the fundamental representations are free: we label them as $m_{i,f}$.

These theories are "superconformal" when $k_i = \sum_j C_{ij} n_j$, where C_{ij} is the Cartan matrix of the ADE quiver. This is why we used an ADE quiver specifically: you can build a theory like the above out of any quiver but these are almost the only superconformal examples

Remark 5.1. There's another family of superconformal quiver gauge theories: you can also use an affine ADE quiver with $k_i = 0$. These will be related to the "elliptic" Higgs moduli spaces we briefly mentioned at the end of the previous section.

In the example of an ADE quiver gauge theory, Nekrasov and Pestun [NP12] calculated its Coulomb branch of vacua – an algebraic integrable system – generalizing work of Cherkis and Kapustin [CK98].

Theorem 5.2 (Nekrasov-Pestun)**.** The Seiberg-Witten integrable system for the $\mathcal{N} = 2$ quiver gauge theory associated to a complex simple group G of ADE type is isomorphic, as a complex manifold, to the moduli space of multiplicative Higgs bundles on $\mathbb{CP}^1$ for the group G, with singularities at the points $(m_{i,f}, 1)$ with charge given by the fundamental coweight $\lambda_i^\vee$, and with a fixed framing at ∞.

5.2 Twisted 5d Gauge Theory

There's another, quite different, origin of the moduli space of multiplicative Higgs bundles from supersymmetric gauge theory. One can compute a *partially topological twist twist* of $\mathcal{N} = 2$ supersymmetric 5d gauge theory. It makes sense to consider this twisted theory on manifolds of the form $\mathbb{D} \times C \times S^1$ where C is a Calabi-Yau curve (so one of our three examples) and $\mathbb{D}$ is a formal disk. If one computes the space of solutions to the equations of motion in this twisted theory one recovers the space $\underline{\mathrm{Map}}(\mathbb{D}, \mathrm{mHiggs}_G(C))$ of maps with target multiplicative Higgs bundles [1]. One can also define the twisted theory in the presence of 't Hooft type surface defects at a finite set D of points in $C \times S^1$. This twisted theory introduces Dirac singularities at D, and so yields the space of maps into the more interesting moduli space $\mathrm{mHiggs}_G(C, D, \omega^\vee)$.

[1] More precisely, the result is the 1-shifted cotangent space with this base.

Remark 5.3. A key piece of motivation for our study of these multiplicative Higgs moduli spaces arises from a generalization of the geometric Langlands program, as a 5d analogue of Kapustin and Witten's work on geometric Langlands and 4d $\mathcal{N} = 4$ gauge theory. When we take the limit where the radius of the circle becomes very small, we end up with a holomorphic twist of 4d $\mathcal{N} = 4$ gauge theory studied by Kapustin. It has two further twists to an A-model and a B-model. Kapustin and Witten argued that S-duality interchanges these two twists, while replacing the gauge group G by its Langlands dual. By studying boundary conditions in these two dual twisted theories they recovered the geometric Langlands conjecture.

We can, therefore, try to tell the same story but keeping the circle large. One of the two further twists corresponds to a quantization of the moduli space of multiplicative Higgs bundles, and the other to a deformation (rotation in the twistor sphere, which we'll discuss in a moment). There are some special cases where we can actually formulate, and try to prove, a careful mathematical conjecture.

6 Consequences and Extensions

Let me conclude with some future directions in which we hope to extend this work.

1. There is a gap in our trichotomy: we still haven't analysed the trigonometric case, where we have opposite Borel reductions at two points in $\mathbb{CP}^1$. We believe that the trigonometric version of our multiplicative Hitchin system will still have the natural structure of an algebraic integrable system, and quantizations of the moduli space with prescribed singularities should give modules for the quantum affine algebra.
2. There's an analogue of our moduli space corresponding to the vacua of an asymptotically free, rather than conformal, ADE quiver gauge theory. This corresponds to allowing the multiplicative Higgs field to have a singularity at infinity. These moduli spaces should correspond to symplectic leaves in the Poisson variety $\mathcal{W}^\mu$ from [KWWY14], and quantize to modules for the shifted Yangian generalizing the modules appearing in that work. Some analysis has been done for the group SO(3) [Fos13, Moc17], but there are significant new analytic subtleties corresponding to the fact that our symplectic structure is no longer defined in terms of spaces of sections of vector bundles over $\mathbb{CP}^1$, but one has to use Sobolev space techniques involving appropriate singularity conditions at infinity.
3. In our paper we described a multiplicative analogue of the brane of opers, using the Steinberg section of the adjoint quotient map $G \to H/W$. That is, the brane of opers arises by considering a multiplicative analogue of the Hitchin section, and viewing it in complex structure J_q. We conjecture, and provide computational evidence, that by triangulizing the multiplicative Hitchin section in as an element of $G[[z^{-1}]]$ we obtain an inverse for the Yangian q-character map first described by Knight [Kni95]. This is a quantization of the fact that the Steinberg section is a section, and therefore inverse to the character map.
4. We previously discussed the elliptic Hitchin system corresponding to the moduli space of vacua of an affine ADE quiver gauge theory. By quantizing these moduli spaces we expect to obtain representations of doubly affine quantum groups, such as affine Yangians (in complex structure I) or quantum toroidal algebras (in complex structure J). This would generalize known results in the case $G = \mathrm{GL}(1)$, c.f. results of Oblomkov, Schiffman and Vasserot, and work in progress of Gukov, Koroteev, Nawata, Saberi.

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