Multiplicative Higgs Bundles

Joint work with Vasily Petun (arXiv:1812:05516)

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Introduction

In this talk I'd like to introduce a moduli space that appears in several guises.

- A multiplicative version of the **Hitchin system**, on a flat curve C, with singularities (Arutyunov–Frolov–Medvedev, Hurtubise–Markman, Frenkel-Ngô).
- A moduli space of **monopoles** on $C \times S^1$, again with singularities (Cherkis–Kapustin, Charbonneau–Hurtubise).
- **Symplectic leaves** in a Poisson Lie group (Shapiro).
- **Coulomb branches** in a 4d $\mathcal{N} = 2$ ADE quiver gauge theory (Nekrasov–Pestun).
- Solutions to the **equations of motion** in a twist of 5d $\mathcal{N} = 2$ super Yang–Mills theory.

I'll say something today about the first three points.

Multiplicative Higgs Bundles

Slogan: a multiplicative G-Higgs bundle on C is a Higgs bundle where the Higgs field takes values in the group G instead of its Lie algebra. Let's make this precise.

Let me start by defining Higgs bundles in a suggestive way. Let C be a curve with trivial canonical bundle.

Definition

The moduli stack of **G-Higgs bundles** on C is the mapping stack $\mathrm{Map}(C, \mathfrak{g}^*/G)$ into the coadjoint quotient stack $\mathfrak{g}^*/G$. So a closed point consists of a principal G-bundle P and a section of the associated coadjoint bundle $\mathrm{ad}(P)^*$.

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We modify this definition as follows:

Definition

The moduli stack of **multiplicative G-Higgs bundles** on C is the mapping stack $\mathrm{Map}(C, G/G)$ into the group adjoint quotient stack G/G. So a closed point consists of a principal G-bundle P and a section of the associated group valued adjoint bundle $\mathrm{Ad}(P)$. Equivalently, a G-bundle with an automorphism.

Unlike ordinary Higgs bundles, there isn't a natural extension of this definition to the setting where C is not flat.

Introducing Singularities

So far this definition isn't very interesting, purely because there aren't very many flat curves. We get interesting examples when we allow our multiplicative Higgs fields to become singular. To get a finite-dimensional moduli space we need to specify local data at each singular point. On the formal neighbourhood of a singular point, a singular multiplicative Higgs field gives an element of the loop group $G((z))$. This element is well-defined up to the action of $G[\![t]\!]$ on the left and the right, so induces a well-defined element of

$$G[\![t]\!] \setminus G((t))/G[\![t]\!].$$

Such double cosets are in canonical bijection with **dominant coweights** of G.

Write

$$\mathrm{mHiggs}_G(C, D, \omega^\vee)$$

to denote the moduli stack of multiplicative G-Higgs bundles on C, with singularities given by a dominant coweight coloured divisor $(D, \omega^\vee)$.

Examples

One can consider three classes of examples:

- **Elliptic:** The case where C is an elliptic curve. Studied by Hurtubise and Markman (2002).
- **Trigonometric:** Now $C = \mathbb{CP}^1$, but we fix boundary conditions at 0 and ∞: reductions to opposite Borels $B_\pm$, so that the resulting H-reductions coincide.
- **Rational:** Again $C = \mathbb{CP}^1$, but we now fix a **framing** at ∞. We'll focus on this example for the rest of the talk.

All three examples will produce finite-dimensional smooth varieties with symplectic structures.

Occurence as Symplectic Leaves

The **rational Poisson Lie group** is the algebraic group $G_1[\![z^{-1}]\!]$, the subscript 1 indicating that the constant term is 1. There is a Poisson structure coming from the **rational r-matrix** $r = \frac{\Omega}{z-w}$, where Ω is the quadratic Casimir element in $\mathfrak{g}^{\otimes 2}$. The Poisson bracket is given by the formula

$$\{f_1, f_2\}(g) = (r, \nabla_L(f_1)(g) \otimes \nabla_L(f_2)(g) - \nabla_R(f_1)(g) \otimes \nabla_R(f_2)(g)).$$

There is a map

$$\rho_\infty \colon \mathrm{mHiggs}_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee) \to G_1[\![z^{-1}]\!]$$

given by restricting to a formal neighbourhood of ∞.

Theorem (E-Pestun)

For every coloured divisor, the map ρ_∞ is the inclusion of a symplectic leaf.

Periodic Monopoles

Monopoles on a 3-manifold M are solutions (A, Φ) to Bogomolny's equation

$$F_A = *\mathrm{d}_A \Phi.$$

A **Dirac singularity** for a monopole is a simple type of spherically symmetric singularity where Φ has a simple pole. Such singularities are described by their **charge**, which is a choice of dominant coweight.

There is an analytic map H from the moduli space of monopoles on $C \times S^1_t$ with fixed Dirac singularities, and multiplicative Higgs bundles on C with corresponding singularities, given by taking the holonomy of $A + i\Phi \mathrm{d}t$ in the S^1_t direction.

Theorem (Charbonneau–Hurtubise, Smith)

The map H is an isomorphism.

If $C = \mathbb{CP}^1$, and we fix a framing at infinity, the moduli space of monopoles on $C \times S^1$ can be realized as a symplectic quotient (or even more: a hyperkähler quotient).

Theorem (E-Pestun)

In this setting, the map H is a symplectomorphism.

Twistor Rotation

The equivalence H promotes the symplectic structure on the moduli space of multiplicative Higgs bundles to a hyperkähler structure. We can give a nice description of what our moduli space looks like in other complex structures on the twistor sphere.

Theorem

In the complex structure J_q, our moduli space becomes algebraically isomorphic to the space of ***q-connections*** *$q\text{-}Conn_G^{\mathrm{fr}}(\mathbb{CP}^1, D, \omega^\vee)$. That is, the space of pairs (P, g) where P is a principal G-bundle and g is a meromorphic bundle map $P \to q^*P$ with prescribed singularities.*

Thanks for listening!