Holomorphic sections of the Deligne–Hitchin moduli space

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joint work with F. Beck, I. Biswas, S. Heller

Motivation

- Σ compact Riemann surface, genus ≥ 2.
- $G = SL(2,\mathbb{C})$, $E \to \Sigma$ trivial rk 2 vector bundle
- $\overline{\partial}_E$ hol. structure, h hermitian metric, Chern connection ∇^h, Higgs field $\Phi \in \Omega^{1,0}(\mathfrak{sl}(E))$.
- Self-duality equations (Hitchin)

$$F^{\nabla^h} + [\Phi \wedge \Phi^{*_h}] = 0, \qquad \overline{\partial}_E \Phi = 0.$$

$\iff \nabla^\lambda = \nabla^h + \lambda^{-1}\Phi + \lambda\Phi^{*_h} \qquad$ flat $\forall \lambda \in \mathbb{C}^*$.

$\iff h : \tilde{\Sigma} \to H^3 = SL(2,\mathbb{C})/\,SU(2)$ equivariant harmonic

- Similarly, equivariant harmonic maps $\tilde{\Sigma} \to S^3 = SU(2)$ correspond to solutions of

$$F^{\nabla^h} - [\Phi \wedge \Phi^{*_h}] = 0, \qquad \overline{\partial}_E \Phi = 0.$$

$\iff \nabla^\lambda = \nabla^h + \lambda^{-1}\Phi - \lambda\Phi^{*_h}$ flat $\forall \lambda \in \mathbb{C}^*$.

- Equivariant harmonic map $\tilde{\Sigma} \to H^3, S^3$
  $\leftrightsquigarrow$ $\mathbb{C}^*$-family of flat G-connections ∇^λ + reality condition
  $\leftrightsquigarrow$ family of λ-connections + reality condition

Moduli space of solutions to the self-duality equations

- $\mathcal{M} = \{\text{solutions to self-duality equations}\}/gauge$
  irreducible solutions $\rightsquigarrow$ smooth manifold
- two complex structures, I_1, I_2:
- $(\mathcal{M}, I_1) = \mathcal{M}_{Dol} \cong \{\text{Higgs bundles } (\bar\partial_E, \Phi)\}/gauge$
- $(\mathcal{M}, I_2) = \mathcal{M}_{dR} = \{\text{flat } G\text{-connections } \nabla\}/gauge \cong \mathrm{Hom}(\pi_1(\Sigma), G)/G$
- $(g, I_1, I_2, I_3 = I_1 I_2)$ hyperkähler structure on $\mathcal{M}$
- $g = L^2$-metric induced by
  $\|(\alpha, \phi)\|^2 = 2i \int_\Sigma \mathrm{tr}(\alpha^* \wedge \alpha + \phi \wedge \phi^*)$
- I_1-holomorphic symplectic form $\omega_2 + i\omega_3$

$$(\omega_2 + i\omega_3)((\alpha, \phi), (\beta, \psi)) = \int_\Sigma \mathrm{tr}(\phi \wedge \beta - \psi \wedge \alpha).$$

- I_1-holomorphic S^1-action, $(\bar\partial_E, \Phi) \mapsto (\bar\partial_E, t\Phi)$, $t^*\Omega = t\Omega$
- moment map $\mu(\bar\partial_E, \Phi) = -\int_\Sigma \mathrm{tr}(\Phi \wedge \Phi^*)$.
- Fixed points: $\bar\partial_E$ stable, $\Phi = 0$, or
  $(E, \bar\partial_E) = L \oplus L^*, \Phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$

Twistor Space

Twistor Space of a HK manifold (Hitchin et. al. 1987)

- (M, g, I_1, I_2, I_3) hyperkähler
- S^2 of orthogonal complex structures:
  $(aI_1 + bI_2 + cI_3)^2 = -(a^2 + b^2 + c^2) = -1 \iff (a, b, c) \in S^2$
- Twistor space $Z = M \times S^2$ complex structure
  $\mathbb{I}|_{(m,\lambda)} = (I_\lambda, I_{\mathbb{C}P^1})$
- Holomorphic projection: $\pi : Z \to \mathbb{C}P^1, \quad \pi^{-1}(\lambda) = (M, I_\lambda)$.
- $\Omega_\lambda = \omega_2 + i\omega_3 + 2i\lambda\omega_1 + \lambda^2(\omega_2 - i\omega_3) \in H^0(\Lambda^2 T_F^*(2))$,
  twisted relative symplectic form
- $\tau : (m, \lambda) \mapsto (m, -\overline{\lambda}^{-1})$ anti-holomorphic involution.
- $m \in M \rightsquigarrow$ Twistor line $s_m : \mathbb{C}P^1 \to Z, s_m(\lambda) = (m, \lambda)$
  holomorphic, $\tau-$real: $\tau \circ s(\lambda) = s(-\overline{\lambda}^{-1})$. Normal bundle
  $\mathcal{O}(1)^{\oplus 2n}$.
- Given complex manifold Z with this data, get hyperkähler structure on moduli space of twistor lines.
- Question (Simpson): Let $\mathcal{S}$ be the space of holomorphic sections of $Z(M) \to \mathbb{C}P^1$. Is $M = \mathcal{S}^\tau$, i.e. is any τ-real holomorphic section $s : \mathbb{C}P^1 \to Z(M)$ a twistor line?

Circle action on Z, the hyperholomorphic line bundle

- If M comes with I_1-holomorphic S^1-action, moment map μ, $t^*(\omega_2 + i\omega_3) = t(\omega_2 + i\omega_3) \rightsquigarrow S^1$-action on Z. Vector field Y.
- Holomorphic involution $N = (-1) \in S^1$.
- $\rightsquigarrow$ second real structure $\rho = \tau \circ N$, covers $\lambda \mapsto \bar{\lambda}^{-1}$.
- Haydys, Hitchin: If $\omega_1 \in H^2(M, 2\pi\mathbb{Z})$. $\rightsquigarrow$ holomorphic line bundle $\mathcal{L} \to Z$.
- $\mathcal{L}$ trivial on twistor lines.
- Meromorphic connection $\nabla^{\mathcal{L}}$, poles at $\lambda = 0, \infty$.
- Curvature $\mathcal{F}$, closed meromorphic two-form on Z, s.t. for $\lambda \in \mathbb{C}^*$
  $\mathcal{F}|_{\pi^{-1}(\lambda)} = \frac{1}{2i\lambda}\Omega_\lambda = \frac{1}{2i}(\omega_2 + i\omega_3)\lambda^{-1} + \omega_1 + \frac{1}{2i}(\omega_2 - i\omega_3)\lambda$
- Holomorphic function $\mathrm{res}_0 : \mathcal{S} \to \mathbb{C}, \quad s \mapsto \mathrm{res}_{\lambda=0}(s^*\nabla^{\mathcal{L}})$
  $\mathrm{res}_0(s) = -\frac{1}{2}\, i_Y(\omega_2 + i\omega_3)\,(\dot{s}(0) - \dot{s}_m(0)) + \mu(m)$
  (Beck-Heller-R.)

Deligne–Hitchin Twistor Space

Twistor space of $\mathcal{M}$, λ-connections

- Deligne-Simpson: A holomorphic λ-connection is a pair $(\bar{\partial}_E, \mathcal{D})$ such that
  - $\bar{\partial}_E$ holomorphic structure,
  - $\mathcal{D} : \Omega^0(E) \to \Omega^{1,0}(E)$, $\quad \mathcal{D}(fs) = \lambda \partial f \otimes s + f\mathcal{D}s$.
  - holomorphic: $\bar{\partial} \circ \mathcal{D} + \mathcal{D} \circ \bar{\partial} = 0$.

- $\lambda = 1$: Holomorphic connection, $\nabla = \bar{\partial}_E + \mathcal{D}$ flat connection
- $\lambda = 0$: $\mathcal{D}(fs) = f\mathcal{D}s \implies \mathcal{D} = \Phi$, Higgs field, $(\bar{\partial}_E, \Phi)$ Higgs bundle.
- $\lambda \in \mathbb{C}^*$: $\nabla = \bar{\partial}_E + \lambda^{-1}\mathcal{D}$ flat connection

Twistor space of $\mathcal{M}$, λ-connections

- $\pi^{-1}(\mathbb{C}) \subset Z(\mathcal{M}) =$ "Hodge moduli space"

$$\mathcal{M}_{Hod}(\Sigma) = \{(\bar{\partial}_E, \mathcal{D}, \lambda) \colon \lambda \in \mathbb{C}, (\bar{\partial}_E, \mathcal{D}) \text{ hol. } \lambda\text{-conn.}\}/\sim .$$

- Similarly $\pi^{-1}(\lambda \neq 0) = \mathcal{M}_{Hod}(\overline{\Sigma})$.
- $\mathcal{M}_{DH} = Z(\mathcal{M}) = (\mathcal{M}_{Hod}(\Sigma) \cup \mathcal{M}_{Hod}(\overline{\Sigma}))/\sim$, $(\bar{\partial}, \mathcal{D}, \lambda) \sim \left(\frac{\mathcal{D}}{\lambda}, \frac{\bar{\partial}}{\lambda}, \frac{1}{\lambda}\right)$ Deligne-Hitchin twistor space.
- $\mathbb{C}^*$-action on Higgs bundle moduli space induces $\mathbb{C}^*$-action on $\mathcal{M}_{DH}$ via

$$(\bar{\partial}_E, \mathcal{D}, \lambda) \mapsto (\bar{\partial}_E, t\mathcal{D}, t\lambda).$$

- Get trivialisation

$$\pi^{-1}(\mathbb{C}^*) \cong \mathcal{M}_{dR} \times \mathbb{C}^*, \qquad (\bar{\partial}_E, \mathcal{D}, \lambda) \mapsto (\nabla^\lambda = \bar{\partial}_E + \lambda^{-1}\mathcal{D}, \lambda).$$

- real structure τ given by $\nabla^\lambda \mapsto \overline{\left(\nabla^{-\bar{\lambda}^{-1}}\right)}^*$.

The Energy Functional

Sections of $\mathcal{M}_{DH} \to \mathbb{C}P^1$

- $\mathcal{S}$ space of holomorphic sections of $\mathcal{M}_{DH}$. Real structures ρ, τ.
- If $(\nabla, \Phi) \in \mathcal{M}$ solves self-duality equations, get twistor line in $\mathcal{S}^\tau$

$$\lambda \mapsto (\bar{\partial}^\nabla + \lambda\Phi^*, \lambda\partial^\nabla + \Phi, \lambda).$$

- Similarly, harmonic map $\tilde{\Sigma} \to S^3$ gives section in $\mathcal{S}^\rho$

$$\lambda \mapsto (\bar{\partial}^\nabla - \lambda\Phi^*, \lambda\partial^\nabla + \Phi, \lambda).$$

- $\mathbb{C}^*$-action on $\mathcal{S}$
- fixed points $\mathcal{S}^{\mathbb{C}^*}$ = closures of $\mathbb{C}^*$-orbits in $\mathcal{M}_{DH}$.
  Explicit description in terms of BB-stratification of $\mathcal{M}_{Hod}$ (Collier–Wentworth)

The Energy Functional (Beck-Heller-R.)

- Let $s : \mathbb{C}P^1 \to \mathcal{M}_{DH}$ be a holomorphic section.
- May write for $\lambda \in \mathbb{C}$

$$s(\lambda) = (\bar{\partial}_E + \lambda \Psi + \ldots, \Phi + \lambda \partial_E + \ldots, \lambda).$$

- Energy

$$\mathcal{E} : \mathcal{S} \to \mathbb{C}, \qquad \mathcal{E}(s) = \frac{1}{2\pi i} \int_\Sigma \mathrm{tr}(\Phi \wedge \Psi).$$

- Energy of twistor line: $\mathcal{E}(s) = \frac{-1}{2\pi} \|\Phi\|^2_{L^2} \leq 0$,
- Essentially the energy of the harmonic metric $h : \tilde{\Sigma} \to H^3$.
- Energy of harmonic map $\tilde{\Sigma} \to S^3$.

Results

- Biswas-Heller-R.
  - $\mathcal{S}^\tau \neq \mathcal{M}$, i.e there exist τ-real holomorphic sections which are not obtained from solutions to the self-duality equations.
  - Characterization of $\mathcal{M} \subset \mathcal{S}$.
- Beck-Heller-R.
  - $\mathcal{E}(s) = \mathrm{res}_0(s^* \nabla^{\mathcal{L}})$. Note that RHS makes sense for arbitrary hyperkähler manifolds M.
  - Energy $\mathcal{E}$ can be used to distinguish between different components of $\mathcal{S}^\tau$.
  - L.Heller and S.Heller have constructed other types of τ-real sections s from certain Willmore surfaces in S^3. In this case $\mathcal{E}(s)$ is essentially the Willmore energy of the Willmore surface.

Work in progress (Beck-Biswas-Heller-R.)

- Critical set of $\mathcal{E}$ given by $\mathcal{S}^{\mathbb{C}^*}$. Explicit formula for second variation at $s \in \mathcal{S}^{\mathbb{C}^*}$.
- Existence of sections $s \in \mathcal{S}$ such that $\deg(s^*\mathcal{L}) \neq 0$. In particular, $\mathcal{S}$ is not connected.
- Holomorphic symplectic structure on $\mathcal{S}$, extending the Kähler form ω_1 on $\mathcal{M} \subset \mathcal{S}^\tau$.
- $\mathcal{E}$ essentially momentum map for $\mathbb{C}^*$-action on $\mathcal{S}$.
- Determine normal bundle of $s \in \mathcal{S}^{\mathbb{C}^*}$, generalizing results of S. Heller for "grafting sections"

References

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