Complex line bundles and gerbes

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Abstract

Closely following Griffiths and Harris [4] and Hitchin [7], we review some of the theory of holomorphic and smooth complex line bundles especially on Riemann surfaces, including a summary of some necessary concepts and results from differential geometry and sheaf cohomology, assuming no prior knowledge of sheaves and only basic elements of the theory of smooth manifolds. Next, following Chatterjee [2], we consider the notion of a gerbe, which is a higher-order analogue of a line bundle, and discuss some related results especially on oriented 3-manifolds.

Contents

1 Complex and Riemannian manifolds **1**
- 1.1 Complex manifolds . . . 1
- 1.2 Vector bundles . . . 1
- 1.3 Complex differential forms and the Dolbeault operators . . . 4
- 1.4 Hermitian metrics, connections, and curvature . . . 6
- 1.5 Riemannian manifolds . . . 9
- 1.6 The Hodge star and the Laplacian . . . 10

2 Sheaves and cohomology **11**
- 2.1 Sheaves . . . 12
- 2.2 Morphisms and sheafification . . . 13
- 2.3 Cohomology of a sheaf . . . 15
- 2.4 Some calculations of sheaf cohomology . . . 16

3 Divisors and line bundles on Riemann surfaces **18**
- 3.1 Divisors . . . 18
- 3.2 From divisors to line bundles . . . 20
- 3.3 Holomorphic and meromorphic sections . . . 22
- 3.4 The first Chern class . . . 22

4 Currents and the Poincaré-Lelong equation **25**
- 4.1 Currents . . . 25
- 4.2 The Chern connection on a line bundle and the Poincaré-Lelong equation . . . 26

5 Gerbes **29**
- 5.1 Definition of a gerbe . . . 29
- 5.2 0-connections and 1-connections . . . 31
- 5.3 Construction of a 1-connection on $\mathcal{G}(R)$ . . . 32

6 Additional remarks **34**

1 Complex and Riemannian manifolds

Here we introduce (or review) various elements of complex geometry, vector bundles, and Riemannian geometry which will come into play in our discussion of line bundles and gerbes.

1.1 Complex manifolds

A *complex manifold* of *(complex) dimension* n is a smooth manifold X having an atlas $\{\varphi_\alpha : U_\alpha \to \mathbb{C}^n\}$ such that, for each pair $U_\alpha \cap U_\beta \neq \emptyset$, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$. Such an X then has dimension $2n$ as a real manifold. If $f : X \to Y$ is a smooth map between complex manifolds, we say that f is *holomorphic* if it is given locally, in charts, by holomorphic functions.

For example, the usual cover of the projective space $\mathbb{P}^n$ by $n+1$ affine charts has holomorphic transition maps, and so $\mathbb{P}^n$ is an n-dimensional complex manifold.

A 1-dimensional complex manifold is also known as a *Riemann surface*. The complex plane itself is a Riemann surface, and so is the Riemann sphere $\mathbb{P}^1 \cong S^2$.

Let X be an n-dimensional complex manifold. Considering X as a $2n$-dimensional real manifold, we denote by $T_{\mathbb{R},p}X$ the usual *real tangent space* to X at a point $p \in X$. A basis is given by $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\}$ where $z_j = x_j + iy_j$ for $1 \leq j \leq n$ are local complex coordinates. The *complexified tangent space* to X at p is the complex vector space $T_{\mathbb{C},p}X = T_{\mathbb{R},p}X \otimes \mathbb{C}$, which has $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\}$ as a basis over $\mathbb{C}$, but we often prefer the basis $\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\}$ where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$
$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Note that the dimension of $T_{\mathbb{C},p}X$ is $2n$ over $\mathbb{C}$. We denote by T'_pX the subspace of $T_{\mathbb{C},p}X$ spanned by the $\frac{\partial}{\partial z_j}$ and by T''_pX the subspace spanned by the $\frac{\partial}{\partial \bar{z}_j}$, so that $T_{\mathbb{C},p}X = T'_pX \oplus T''_pX$. These two subspaces are called the *holomorphic* and *antiholomorphic* tangent spaces, respectively. Identifying $T_{\mathbb{C},p}X$ with the space of derivations of smooth complex functions at p, the holomorphic tangent space consists of those derivations which vanish on antiholomorphic functions (f such that $\bar{f}$ is holomorphic) and the antiholomorphic tangent space consists of those derivations which vanish on holomorphic functions.

If $f : X \to Y$ is a smooth map, then f induces a linear map $f_* : T_{\mathbb{C},p}X \to T_{\mathbb{C},f(p)}Y$ in the usual way, but this f_* does not send T'_pX to $T'_{f(p)}Y$ in general; indeed, $f_*(T'_pX) \subset T'_{f(p)}Y$ if and only if f is holomorphic at p. The holomorphic maps $f : X \to Y$ are therefore precisely those smooth maps f whose differentials are well-defined as maps between the holomorphic tangent bundles of X and Y.

Note that composing the inclusion $T_{\mathbb{R},p}X \hookrightarrow T_{\mathbb{C},p}X = T'_pX \oplus T''_pX$ with the projection onto T'_pX gives an isomorphism $T_{\mathbb{R},p}X \to T'_pX$ of real vector spaces; $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}$ is sent to $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}$ is sent to $i\frac{\partial}{\partial z}$.

1.2 Vector bundles

Let X be a smooth manifold.

Definition 1. *A* (smooth complex) vector bundle *on X is a union $E = \bigsqcup_{x \in X} E_x$ of finite-dimensional complex vector spaces parametrised by the points of X such that E is equipped with a smooth structure satisfying the following properties:*

- *The projection $\pi : E \to X$, which takes $v \in E_x$ to x, is smooth.*
- *There exists an atlas $\{U_\alpha\}$ for X such that, for each α, there is a diffeomorphism*

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$$

which restricts to an isomorphism of vector spaces

$$\varphi_\alpha|_{E_x} : E_x \to \{x\} \times \mathbb{C}^k$$

for each $x \in U_\alpha$*. Such a chart* $(\pi^{-1}(U_\alpha), \varphi_\alpha)$ *is called a* local trivialization *of* E*.*

The second property above implies that the dimension of E_x is locally constant as x varies, and is therefore constant on each connected component of X. If $\dim E_x = k$ for all $x \in X$ then E is a vector bundle of *rank* k. The simplest example of a vector bundle of rank k on X is the *trivial bundle* $X \times \mathbb{C}^k$, where we take $\pi(x, v) = x$ and equip each fibre $\pi^{-1}(x) = \{x\} \times \mathbb{C}^k$ with the obvious vector space structure. A vector bundle of rank 1 is called a *line bundle*, and these objects will be a major focus of this discussion.

Let $E \xrightarrow{\pi} X$ be a vector bundle. For ease of notation, for a subset $S \subset X$ we write

$$E_S := \pi^{-1}(S) = \bigcup_{x \in S} E_x$$

and in general if $\{U_\alpha\}$ is a collection of open sets, we write

$$U_\alpha \cap U_\beta = U_{\alpha\beta}, U_\alpha \cap U_\beta \cap U_\gamma = U_{\alpha\beta\gamma},$$

etc. Now, given a complex vector bundle of rank k with local trivializations $\varphi_\alpha : E_{U_\alpha} \to U_\alpha \times \mathbb{C}^k$, the maps

$$\varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{C}^k \to U_{\alpha\beta} \times \mathbb{C}^k$$

restrict to vector space automorphisms of $\{x\} \times \mathbb{C}^k$, and so they are of the form $(x, v) \mapsto (x, g_{\alpha\beta}(x)v)$ where

$$g_{\alpha\beta} : U_{\alpha\beta} \to GL_k(\mathbb{C})$$

is a smooth map. These *transition functions* $g_{\alpha\beta}$ clearly satisfy the equations

$$\begin{aligned} g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} &= 1, \\ g_{\alpha\beta} \cdot g_{\beta\alpha} &= 1. \end{aligned}$$

Conversely, given an atlas $\{U_\alpha\}$ for X and a collection of smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to GL_k(\mathbb{C})$ satisfying these equations, we can construct a complex vector bundle over X with these maps as transition functions, by taking the disjoint union $\bigsqcup_\alpha U_\alpha \times \mathbb{C}^k$ and identifying $(x, v) \in U_\beta \times \mathbb{C}^k$ with $(x, g_{\alpha\beta}(x)v) \in U_\alpha \times \mathbb{C}^k$, for $x \in U_{\alpha\beta}$.

If U is an open subset of X, a *section* of E over U is a smooth map $s : U \to E$ such that $\pi \circ s = \mathrm{Id}_U$, *i.e.* $s(x) \in E_x$ for each $x \in U$. Such a map can be specified in terms of transition functions $g_{\alpha\beta}$ by a collection of smooth maps

$$s_\alpha : U_\alpha \to \mathbb{C}^k$$

satisfying

$$s_\alpha = g_{\alpha\beta} \cdot s_\beta$$

over $U_{\alpha\beta}$. The set of sections of E over U, which we denote by $\Gamma(U, E)$, is naturally a module over the space of smooth complex functions on U; in particular, it is a complex vector space. The space of *global* sections of E is $\Gamma(E) := \Gamma(X, E)$. Given two vector bundles $E \xrightarrow{\pi} X$ and $E' \xrightarrow{\pi'} X$ on the same manifold X, a *morphism* from E to E' is a smooth map $\varphi : E \to E'$ such that $\pi'\varphi = \pi$ and such that $\varphi|_{E_x} : E_x \to E'_x$ is a linear map for each $x \in X$. These maps are indeed the morphisms of a category of vector bundles over X. We say that a vector bundle of rank k is *trivial* if it is isomorphic to the trivial bundle $X \times \mathbb{C}^k$.

An equivalent way of thinking of the local trivializations of a vector bundle $E \to X$ is in terms of

frames. A *frame* for E over an open set $U \subset X$ is a collection e of sections $e_1, \ldots, e_k \in \Gamma(U, E)$ such that, for each $x \in U$, the vectors $e_1(x), \ldots, e_k(x)$ form a basis for E_x. Given such a frame, we obtain a local trivialization φ_U from the map

$$\begin{aligned} \varphi_U^{-1} \quad &: \quad U \times \mathbb{C}^k \to E_U \\ (x, \lambda_1, \ldots, \lambda_k) \quad &\mapsto \quad \sum_i \lambda_i e_i(x). \end{aligned}$$

Conversely, if we have a local trivialization φ_U, then by pulling back the standard basis for $\{x\} \times \mathbb{C}^k$ we obtain the values $e_i(x)$ of a frame over U. Such a frame exists, then, if and only if the vector bundle $E_U \to U$ is trivial.

Let $E \to X$ and $E' \to X$ be vector bundles. Each morphism $\phi : E \to E'$ defines a $\mathbb{C}$-linear map $\phi_* : \Gamma(E) \to \Gamma(E')$ by $\phi_*(s) = \phi \circ s$. If $f \in C^\infty(X)$, we have $\phi_*(fs) = f\phi_* s$ since ϕ is linear on each fibre E_x, so Φ is actually a map of $C^\infty(X)$-modules. Conversely, if

$$\Phi : \Gamma(E) \to \Gamma(E')$$

is a map of $C^\infty(X)$-modules, then $\Phi = \varphi_*$ for some morphism

$$\phi : E \to E' :$$

locally, with respect to a frame e, we define

$$\phi(\sum_i a_i e_i(x)) = \sum_i a_i(x)\Phi(e_i)(x).$$

It is easy to see that this defined independently of the choice of frame because Φ is $C^\infty(X)$-linear, and the transition functions between two frames are smooth. To summarize:

Proposition 1.1. *The vector bundle morphisms*

$$E \to E'$$

are naturally in one-to-one correspondence with the $C^\infty(X)$-linear maps

$$\Gamma(E) \to \Gamma(E').$$

It is a theorem due to Swan that the $C^\infty(X)$-modules $\Gamma(E)$ are always finitely-generated and projective, and that this correspondence actually gives an equivalence between the category of smooth vector bundles over X and the category of *all* finitely-generated projective $C^\infty(X)$-modules.

Let $E \to X$ be a vector bundle given by transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to GL_k(\mathbb{C})$. The *dual* bundle of E is the vector bundle $E^* \to X$ whose transition functions are $g^*_{\alpha\beta} : U_{\alpha\beta} \to GL_k(\mathbb{C}) : x \mapsto g^T_{\beta\alpha}$. If $E' \to X$ is another vector bundle with transition functions $g'_{\alpha\beta} : U_{\alpha\beta} \to GL_\ell(\mathbb{C})$, the *tensor product* of E and E' is the rank-$(k+\ell)$ vector bundle whose transition functions are

$$\begin{aligned} g_{\alpha\beta} \otimes g'_{\alpha\beta} \quad &: \quad U_{\alpha\beta} \to GL_{k+\ell}(\mathbb{C}) \\ x \quad &\mapsto \quad g'_{\alpha\beta}(x) \otimes g_{\alpha\beta}(x). \end{aligned}$$

We similarly define the *exterior powers* $\wedge^k E$ of E by taking exterior powers of its fibres.

The definition of a *holomorphic* vector bundle is obtained from the above by requiring that X and E be complex manifolds, and replacing all smooth maps above by holomorphic maps. A holomorphic vector bundle has holomorphic transition functions, in particular. In the particular case that X is a line bundle, since $GL_1(\mathbb{C}) \cong \mathbb{C}^*$ we can think of the transition functions simply as nowhere-vanishing

holomorphic functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}$ which satisfy the conditions

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1.$$

Since each complex n-manifold X is also a real $2n$-manifold, it comes equipped with a $2n$-dimensional real tangent bundle $T_{\mathbb{R}}X$ whose fibres are the tangent spaces $T_{\mathbb{R},x}X$ described earlier. However, the complexified tangent spaces $T_{\mathbb{C},x}X$ are more useful in this context, and these compose the fibres of a $2n$-dimensional *complex tangent bundle* $T_{\mathbb{C}}X$, which we henceforth denote by TX (or simply by T if the base is understood). Given a holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}$ for X, the bundle TX has transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to GL_n(\mathbb{C})$ given by

$$g_{\alpha\beta}(x) = \begin{pmatrix} \frac{\partial p^i}{\partial z_j}(x) & \frac{\partial q^i}{\partial z_j}(x) \\ \frac{\partial p^i}{\partial \bar{z}_j}(x) & \frac{\partial q^i}{\partial \bar{z}_j}(x) \end{pmatrix}$$

where

$$\varphi_\alpha \circ \varphi_\beta^{-1}(z) = \sum_j p^j z_j + \sum_j q^j \bar{z}_j.$$

However, since $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic, we have

$$g_{\alpha\beta}(x) = \begin{pmatrix} \frac{\partial p^i}{\partial z_j}(x) & 0 \\ 0 & \frac{\partial q^i}{\partial \bar{z}_j}(x) \end{pmatrix}.$$

We see from this that the tangent bundle TX splits naturally into a direct sum $T'X \oplus T''X$ of the *holomorphic tangent bundle* and the *antiholomorphic tangent bundle*, two n-dimensional complex vector bundles whose transition functions are given by the nonzero n-by-n blocks above, and whose fibres are the holomorphic and antiholomorphic tangent spaces respectively. In particular, the transition maps of $T'X$ are holomorphic, and so $T'X$ is a holomorphic vector bundle. As noted before, a smooth map $f : X \to Y$ is holomorphic if and only if $f_*(T'X) \subset T'Y$. Related vector bundles are the complex cotangent bundle $T^*X = (TX)^*$, the holomorphic and antiholomorphic cotangent bundles $T^{*\prime}X = (T'X)^*$ and $T^{*\prime\prime}X = (T''X)^*$, and tensor/exterior products of these. A particularly important line bundle is the *canonical line bundle*, denoted $K = \wedge^n T^*X$. (The canonical line bundle of a Riemann surface is simply its cotangent bundle.)

1.3 Complex differential forms and the Dolbeault operators

Let X be a complex manifold. For nonnegative integers k, p, q, the space of *(smooth) complex k-forms* is

$$A^k(X) = \Gamma(\wedge^k T^*X)$$

, and the space of *(smooth) complex (p, q)-forms* is

$$A^{p,q}(X) = \Gamma(\wedge^p T^{*\prime}X \otimes \wedge^q T^{*\prime\prime}X).$$

From the decomposition $TX = T'X \oplus T''X$, we have

$$\wedge^k T^*X = \bigoplus_{p+q=k} \wedge^p T^{*\prime}X \otimes \wedge^q T^{*\prime\prime}X$$

and so

$$A^k(X) = \bigoplus_{p+q=k} A^{p,q}(X).$$

The usual de Rham derivative induces a linear map $d : A^k(X) \to A^{k+1}(X)$. If ρ is a (p, q)-form, in

local coordinates z_j we can write

$$\rho = \sum_{\#I=p,\#J=q} \rho_{I,J} \cdot dz_I \wedge d\bar{z}_J$$

where $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ for $I = \{i_1 < \cdots < i_p\}$, and similarly for $d\bar{z}_J$. Locally,

$$d\rho = \sum_{\#I=p,\#J=q} d\rho_{I,J} \wedge dz_I \wedge d\bar{z}_J.$$

Since $d\rho_{I,J} \in A^1(X) = A^{1,0}(X) \oplus A^{0,1}(X)$, we see that d takes $A^{p,q}(X)$ into $A^{p+1,q}(X) \oplus A^{p,q+1}(X)$. Where $\pi^{(r,s)}$ denotes the natural projection $A^{r+s}(X) \to A^{r,s}(X)$, we then have

$$d = \partial + \bar{\partial}$$

where

$$\partial = \pi^{(p+1,q)} \circ d : A^{p,q}(X) \to A^{p+1,q}(X),$$
$$\bar{\partial} = \pi^{(p,q+1)} \circ d : A^{p,q}(X) \to A^{p,q+1}(X)$$

are the *Dolbeault operators*. We have the local expressions

$$\partial\rho = \sum_{\#I=p,\#J=q,i} \frac{\partial \rho_{I,J}}{\partial z_i} dz_i \wedge dz_I \wedge d\bar{z}_J,$$

$$\bar{\partial}\rho = \sum_{\#I=p,\#J=q,i} \frac{\partial \rho_{I,J}}{\partial \bar{z}_i} d\bar{z}_i \wedge dz_I \wedge d\bar{z}_J.$$

We also define the operator $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$. Note that

$$\begin{aligned} dd^c &= \frac{1}{4\pi i}(\partial + \bar{\partial})(\partial - \bar{\partial}) \\ &= \frac{-1}{2\pi i}\partial\bar{\partial}. \end{aligned}$$

A complex form ρ is holomorphic if and only if the locally defined coefficient functions $\rho_{I,J}$ are holomorphic, and this is clearly equivalent to the condition $\bar{\partial}\rho = 0$. We denote the space of holomorphic k-forms on X by $\Omega^k(X)$.

We can extend the definition of forms to have values in arbitrary vector bundles. Given a complex vector bundle $E \to X$, the space of E-valued k-forms is $A^k(E) = \Gamma(\wedge^k T^*X \otimes E)$, and the space of E-valued (p,q)-forms is $A^{p,q}(E) = \Gamma(\wedge^p T^{*\prime}X \otimes \wedge^q T^{*\prime\prime}X \otimes E)$. In particular, $A^0(E) = \Gamma(E)$. These coincide with the previous definitions of k-forms and (p,q)-forms when E is the trivial line bundle $X \times \mathbb{C}$. As above, we have the decomposition

$$A^k(E) = \bigoplus_{p+q=k} A^{p,q}(E).$$

Using local coordinates z_j for X and a local frame $e_1, \ldots, e_k$ for E, a form $\varphi \in A^{p,q}(E)$ can be written

$$\varphi = \sum_{\#I=p,\#J=q,i} \varphi_{I,J,i} \cdot dz_I \wedge d\bar{z}_J \otimes e_i.$$

If E is a holomorphic vector bundle, then there is a naturally defined linear operator $\bar{\partial} : A^{p,q}(E) \to A^{p,q+1}(E)$ which generalizes the Dolbeault operator $\bar{\partial}$ above. Given φ expressed locally as above, in terms of a *holomorphic* frame e, we define $\bar{\partial}\varphi$ locally by

$$\bar{\partial}\varphi = \sum_{I,J,i} \bar{\partial}\varphi_{I,J,i} \wedge dz_I \wedge d\bar{z}_J \otimes e_i.$$

To see that this is well-defined, suppose e' is another holomorphic frame. Then $e_i = \sum_j h_{ij} e'_j$ for some holomorphic functions h_{ij}, and the local representation of φ with respect to this frame is

$$\varphi = \sum_{I,J,i,j} \varphi_{I,J,i} h_{ij} \cdot dz_I \wedge d\bar{z}_J \otimes e'_j.$$

Since the functions h_{ij} are holomorphic we have $\bar{\partial} h_{ij} = 0$, so $\bar{\partial}(\varphi_{I,J,i} \cdot h_{ij}) = (\bar{\partial}\varphi_{I,J,i}) \cdot h_{ij}$ using the product rule, and

$$\begin{aligned} \sum_{I,J,i,j} \bar{\partial}(\varphi_{I,J,i} \cdot h_{ij}) \wedge dz_I \wedge d\bar{z}_J \otimes e'_j &= \sum_{I,J,i,j} \bar{\partial}(\varphi_{I,J,i}) \cdot h_{ij} \wedge dz_I \wedge d\bar{z}_J \otimes e'_j \\ &= \sum_{I,J,i} \bar{\partial}\varphi_{I,J,i} \wedge dz_I \wedge d\bar{z}_J \otimes e_i, \end{aligned}$$

agreeing with our expression above. As before, an E-valued complex form ρ is holomorphic if and only if $\bar{\partial}\rho = 0$.

1.4 Hermitian metrics, connections, and curvature

Let $E \to X$ be a complex vector bundle.

Definition 2. *A hermitian metric on E is a smooth section h of the vector bundle $(E \otimes \bar{E})^*$ (where $\overline{E_x}$ is E_x equipped with the scalar multiplication $\lambda \cdot v = \overline{\lambda} v$) such that for each $x \in X$, the bilinear form $h(,)$ on E_x defined by $h(u,v) = h(x)(u \otimes v)$ is a hermitian inner product. A* hermitian vector bundle *is a holomorphic vector bundle equipped with a hermitian metric.*

This can be thought of as a smooth choice of hermitian inner products for the fibres of E, a collection of hermitian inner products $(,)_x$ on the fibres E_x such that, if $e = (e_1, \ldots, e_k)$ is a frame for E over U, then the functions $h_{ij} : U \to \mathbb{C} : x \mapsto (e_i(x), e_j(x))_x$ are all smooth. We note that every complex vector bundle E can be equipped with a hermitian metric: choosing trivializations $\varphi_\alpha : E_{U_\alpha} \to U_\alpha \times \mathbb{C}^k$ where $\{U_\alpha\}$ is a locally finite cover of X, we obtain hermitian metrics h_α on the trivial bundles $E_{U_\alpha} \to U_\alpha$ from the usual hermitian inner product on $\mathbb{C}^k$. Choose a smooth real partition of unity $\{\rho_\alpha\}$ subordinate to the cover $\{U_\alpha\}$, extend $\rho_\alpha h_\alpha$ by zero to a global section of $(E \otimes \bar{E})^*$, and the sum $\sum_\alpha \rho_\alpha h_\alpha$ is then a hermitian metric on E.

A *connection* on a vector bundle can be thought of as a way of differentiating the sections of that vector bundle.

Definition 3. *A* connection *is a $\mathbb{C}$-linear map $\nabla : A^0(E) \to A^1(E)$ satisfying the Leibniz rule*

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$$

for all smooth functions f and sections s of E.

Given such a connection and a vector field V on X, by composing ∇ with the contraction

$$i_V : A^1(E) \to A^0(E)$$

we obtain a map $\nabla_V : A^0(E) \to A^0(E)$ which can be thought of as giving the derivative of a section of E in the direction of V at each point, and conversely these derivatives ∇_V determine the connection. Given a frame e for E, we have $\nabla e_i = \sum_j \theta_{ij} e_j$ for some 1-forms θ_{ij}. The matrix $\theta = (\theta_{ij})$ is called the *connection matrix* of ∇ with respect to e. The frame e and matrix θ determine ∇ locally, and θ in turn depends on the choice of frame e.

In general there is no distinguished connection on a vector bundle, but if E is a hermitian vector bundle then there are two natural compatibility conditions that can be imposed on a connection, and there exists a unique connection satisfying these conditions, as we now describe.

Suppose ∇ is a connection on a complex vector bundle E. From the decomposition

$$A^1(E) = A^{1,0}(E) \oplus A^{0,1}(E)$$

we can write

$$\nabla = \nabla' + \nabla''$$

for uniquely determined maps

$$\begin{aligned} \nabla' &: A^0(E) \to A^{1,0}(E), \\ \nabla'' &: A^0(E) \to A^{0,1}(E). \end{aligned}$$

Since E is a holomorphic vector bundle, we also have the Dolbeault operator $\bar{\partial} : A^0(E) \to A^{0,1}(E)$ described above. We say that ∇ is *compatible with the complex structure* of E if $\nabla'' = \bar{\partial}$.

Now assume further that E is equipped with a hermitian metric h. If s and t are sections of E, then $h(s,t)$ defines a smooth function on X, so $d(h(s,t))$ is a 1-form. We say that ∇ is *unitary*, or *compatible with the metric* on E if for all $s, t \in \Gamma(E)$, we have the equation of 1-forms

$$d(h(s,t)) = h(\nabla s, t) + h(s, \nabla t)$$

, where the first term on the right-hand side are given by $h(\sum_i \rho_i \otimes s_i, t) = \sum_i h(s_i, t)\rho_i$ for 1-forms ρ, and similarly for the second term. This can be equivalently expressed by saying that, for each vector field V, we have the equation of functions $V(h(s,t)) = h(\nabla_V s, t) + h(s, \nabla_V t)$. We show that there is a unique connection ∇ on a hermitian vector bundle E which is compatible with both the metric and the complex structure.

First, suppose that ∇ is such a connection. Let e be a holomorphic frame for E, and define the smooth functions $h_{ij} = (e_i, e_j)$ on X. Then since ∇ is compatible with the metric,

$$\begin{aligned} dh_{ij} &= (\nabla e_i, e_j) + (e_i, \nabla e_j) \\ &= \left(\sum_k \theta_{ik} e_k, e_j\right) + \left(e_i, \sum_k \theta_{jk} e_k\right) \\ &= \sum_k \theta_{ik} h_{kj} + \sum_k \bar{\theta}_{jk} h_{ik}. \end{aligned}$$

Since ∇ is compatible with the complex structure and e is holomorphic, $\nabla e_i = \nabla' e_i + \bar{\partial} e_i = \nabla' e_i$. It follows that the 1-forms θ_{ij} contain only terms of the form $f dz_\ell$, and the 1-forms $\bar{\theta}_{ij}$ contain only terms of the form $f d\bar{z}_\ell$. The two sums above are therefore of respective types $(1,0)$ and $(0,1)$. Then since $dh_{ij} = \partial h_{ij} + \bar{\partial} h_{ij}$, we have

$$\begin{aligned} \partial h_{ij} &= \sum_k \theta_{ik} h_{kj}, \\ \bar{\partial} h_{ij} &= \sum_k \bar{\theta}_{jk} h_{ik}. \end{aligned}$$

Writing $h = (h_{ij})$ as a matrix of functions, we can express these equations as $\partial h = \theta h, \bar{\partial} h = h \bar{\theta}^t$. (Here, ∂ and $\bar{\partial}$ act on the entries of the matrix.) As $\{e_i\}$ is a basis for E at each point, $h = ((e_i, e_j))$ is invertible and hermitian at each point, and so there is a unique solution $\theta = (\partial h) h^{-1}$ for θ, and hence a unique connection ∇.

Definition 4. *The* metric connection *or* Chern connection *on a hermitian vector bundle is the unique connection that is compatible with both the metric structure and the complex structure.*

Just as the action $f \mapsto df$ on smooth functions extends to give the (de Rham) exterior derivative d on differential forms, given a connection $\nabla : A^0(E) \to A^1(E)$ there is a unique extension $\nabla : A^p(E) \to$

$A^{p+1}(E)$ obtained by forcing the Leibniz rule

$$\nabla(\psi \wedge s) = d\psi \otimes s + (-1)^p \psi \wedge \nabla s$$

for p-forms ψ and sections s of E. In particular, we have the map $\nabla^2 : \Gamma(E) \to \Gamma(\wedge^2 T^* \otimes E)$. This operator is in fact a map of $C^\infty(X)$-modules. To see this, it suffices to show that $\nabla^2(f \cdot s) = f \cdot \nabla^2 s$ for all smooth functions $f : X \to \mathbb{C}$ and sections s of E:

$$\begin{aligned} \nabla^2(f \cdot s) &= \nabla(df \otimes s + f \cdot \nabla s) \\ &= (d^2 f \otimes s - df \otimes \nabla s) + (df \otimes \nabla s + f \cdot \nabla^2 s) \\ &= f \cdot \nabla^2 s. \end{aligned}$$

It follows that ∇^2 is induced by a bundle morphism $F : E \to \wedge^2 T^* \otimes E$ in the sense that $\nabla^2 s = F \circ s$. We can then think of this morphism as a section F of $E^* \otimes \wedge^2 T^* \otimes E \cong \wedge^2 T^* \otimes (E^* \otimes E)$. We call this the *curvature* of the connection ∇, denoted $F(\nabla)$. Given a frame e for E, we have $\nabla^2 e_i = \sum_j F_{ij} \otimes e_j$ for some 2-forms F_{ij}, and indeed (F_{ij}), the *curvature matrix* with respect to e, is the matrix representation of the section F in terms of the frame $e_i^* \otimes e_j$ for $E^* \otimes E$. In the important case that E is a line bundle, the canonical isomorphism $E_x^* \otimes E_x = \mathbb{C}$ implies that $E^* \otimes E$ is the trivial bundle, and hence F is a section of $\wedge^2 T^*$, *i.e.* a global 2-form.

One important thing to point out about connections on a vector bundle is that one always exists. For example, we could use a partition of unity to construct a metric (as previously shown) and then take the metric connection thereby induced. Notice that the zero map $\Gamma(E) \to \Gamma(T^* \otimes E)$ is *not* a connection, and so a linear combination of connections will generally fail to be a connection itself. There are other ways to produce new connections from old ones, of course. For example, if ∇ is a connection on E and ζ is a global 1-form, we obtain a new connection $\nabla + \zeta$ on E by $(\nabla + \zeta)s = \nabla s + \zeta \otimes s$. It is easy to see that this satisfies the Leibniz rule. Conversely, it is easy to see from the Leibniz rule that *the difference of two connections is a global 1-form.*

Definition 5. *Let E, E' be vector bundles on X, equipped with connections ∇, ∇' respectively. The induced connection $\nabla + \nabla'$ is the connection on $E \otimes E'$ given by*

$$(\nabla + \nabla')_V (s \otimes s') = \nabla_V s \otimes s' + s \otimes \nabla'_V s'$$

for all vector fields V and all sections s, s' of E and E' respectively.

We will be most interested in line bundles, and so we examine connections and curvature a little more closely in this situation. Suppose $L \to X$ is a line bundle given by transition functions g_{ij} and local trivializations e_i over U_i, so that $e_i = g_{ij} \cdot e_j$ over U_{ij}. If ∇ is a connection on L, then $\nabla e_i = A_i \otimes e_i$ for some 1-forms A_i over U_i. From the Leibniz rule, we calculate

$$\begin{aligned} A_i \otimes e_i &= \nabla(g_{ij} \cdot e_j) \\ &= dg_{ij} \otimes e_j + g_{ij} \nabla e_j \\ &= (dg_{ij} + g_{ij} A_j) \otimes e_j \end{aligned}$$

whereas $A_i \otimes e_i = g_{ij} A_i \otimes e_j$, and we conclude

$$dg_{ij} + g_{ij}(A_j - A_i) = 0,$$

or

$$d \log g_{ij} - (A_i - A_j) = 0.$$

As shorthand, we write this as $d \log g - \delta A = 0$. We can then consider a connection on L either as a map $\Gamma(L) \to \Gamma(T^* \otimes L)$, or as a collection of local 1-forms A_i with $d \log g - \delta A = 0$.

Now if F is the curvature of such a connection, we represent it by local 2-forms F_i, and we have

$$\begin{aligned} F_i \otimes e_i &= \nabla(A_i \otimes e_i) \\ &= dA_i \otimes e_i - A_i \wedge A_i \otimes e_i \\ &= dA_i \otimes e_i \end{aligned}$$

since $\omega \wedge \omega = 0$ for any 1-form ω, and so we write $F = dA$. In particular, *the curvature of a line bundle is a closed 2-form* (but usually not exact, since A is not a global 1-form in general).

Suppose that ∇, ∇' are two connections on a line bundle L, given by local 1-forms A_i, A_i' as above. From the condition $d \log g - \delta A = 0$ we have $\delta(A - A') = 0$, whence $A_i - A_i' = A_j - A_j'$ on each U_{ij}, so the difference $A - A'$ is a *global* 1-form.

Next, suppose L and L' are line bundles on X, equipped with connections ∇, ∇' respectively. As above, we write $\nabla e_i = A_i \otimes e_i$ and $\nabla' e_i' = A_i' \otimes e_i'$ locally. The induced connection $\nabla + \nabla'$ then takes a particularly simple form: for a vector field V, we have

$$\begin{aligned} (\nabla + \nabla')_V (e_i \otimes e_i') &= (A_i(V)e_i) \otimes e_i' + e_i \otimes (A_i'(V)e_i') \\ &= (A_i + A_i')(V)(e_i \otimes e_i'), \end{aligned}$$

which is to say that $\nabla + \nabla'$ is represented, with respect to the local trivializations $e_i \otimes e_i'$, by the 1-forms $A_i + A_i'$. In particular, since $F(\nabla) = dA_i$ locally, we have

$$F(\nabla + \nabla') = F(\nabla) + F(\nabla').$$

1.5 Riemannian manifolds

Let X be a smooth real manifold.

Definition 6. *A (Riemannian) metric on X is a smooth section g of $(T \otimes T)^*$ such that on each tangent space T_pX, the bilinear form given by $g(x, y) = g(p)(x \otimes y)$ is a real inner product. (That is, it is symmetric, nondegenerate and positive-definite.) A Riemannian manifold is a manifold equipped with a choice of Riemannian metric.*

This is the real analogue of a hermitian metric on the tangent bundle of a complex manifold. The tangent bundle of a Riemannian manifold can thus be thought of as a bundle of *Hilbert spaces*. Given two vector fields V, W on a Riemannian manifold, their inner product $g(V, W)$ is a smooth function. In particular, each vector field has a length given by $\sqrt{g(V, V)}$ which varies continuously over X. Every manifold admits a (non-canonical) Riemannian metric, as we can construct one using a partition of unity much as was previously described for a hermitian metric.

An inner product on a finite-dimensional vector space can be thought of as an isomorphism to the dual, by taking a vector v to $(v, \cdot)$. A Riemannian metric g, being a "bundle of inner products", thus gives rise to a vector bundle isomorphism $T \xrightarrow{\cong} T^*$: on vector fields, the isomorphism is given by $V \mapsto g(V, \cdot)$. The cotangent bundle thereby inherits a metric itself, which we also denote by g.

For each integer k, $\wedge^k T^*$ inherits a metric too. Let $\{e_i\}$ be an orthonormal frame (which is easily seen to exist by using the Gram-Schmidt process locally) for T^*. A metric is then defined on $\wedge^k T^*$ by declaring $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ to be an orthonormal frame. Explicitly, we have

$$g(\rho_{i_1} \wedge \cdots \rho_{i_k}, \rho_{j_1} \wedge \cdots \rho_{j_k}) = \det[g(\rho_{i_r}, \rho_{j_s})]_{r,s}$$

for all 1-forms ρ_{i_j}.

If X is an oriented manifold, then a Riemannian metric g induces a *canonical volume form* on X. Since $\wedge^n T^*$ is trivial, it follows that it has exactly two sections of constant length 1 with respect to the metric induced by g, and there is a unique positive choice of such a section. This is the canonical volume

form, which we henceforth denote by ω. If $\{e_i\}, \{e_i'\}$ are orthonormal frames for T^*, then they differ by an orthogonal transformation which has determinant ± 1, and so

$$e_1 \wedge \cdots \wedge e_n = \pm e_1' \wedge \cdots \wedge e_n'.$$

Indeed, $\omega = e_1 \wedge \cdots \wedge e_n$ for any positively oriented orthonormal frame $\{e_i\}$.

Proposition 1.2. *Suppose that, in (positively-oriented) local coordinates for X,*

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j.$$

Then

$$\omega = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

Proof. Choose an orthonormal frame $\{e_i\}$ for T^*, and write $dx^r = \sum_k \alpha_{rk} e_k$. We intend to show that $\det(\alpha_{ij}) = \sqrt{\det(g_{ij})}$ since

$$\begin{aligned} \omega(dx^1 \wedge \cdots \wedge dx^n) &= \det(\alpha_{ij})\omega(e_1 \wedge \cdots \wedge e_n) \\ &= \det(\alpha_{ij}). \end{aligned}$$

But

$$\begin{aligned} g_{ij} &= g(dx^i, dx^j) \\ &= g(\sum_k \alpha_{ik} e_k, \sum_l \alpha_{jl} e_l) \\ &= \sum_k \alpha_{ik}\alpha_{jk} \end{aligned}$$

whence $\det(g_{ij}) = \det(\alpha_{ij})^2$, and the claim follows. □

1.6 The Hodge star and the Laplacian

Here we introduce some important operators on differential forms on a Riemannian manifold and state, without proof, some important results on them. For details, see [4] or [3].

Let (X, g) be a Riemannian manifold with canonical volume form ω. Recall the operation of contraction along a vector field: if V is a vector field, then $i_V : A^k(X) \mapsto A^{k-1}(X)$ is the unique graded derivation on the algebra $A^\bullet(X)$ of differential forms such that $i_V f = 0$ and $i_V df = df(V)$ for all smooth functions f. The volume form ω induces a vector bundle isomorphism $b_\omega : \wedge^k T \xrightarrow{\cong} \wedge^{n-k} T^*$, given on sections by

$$V_1 \wedge \cdots \wedge V_k \mapsto i_{V_1} \cdots i_{V_k} \omega.$$

At the same time, the metric induces a vector bundle isomorphism $g : T \to T^*$, hence $\wedge^k g : \wedge^k T \xrightarrow{\cong} \wedge^k T^*$.

Definition 7. *The Hodge star is the linear isomorphism $* : A^k(X) \to A^{n-k}(X)$ induced by the composition $b_\omega \circ (\wedge^k g)^{-1} : \wedge^k T^* \xrightarrow{\cong} \wedge^{n-k} T^*$ as above.*

In general, we call $*\alpha$ the *Hodge dual* of α. The operator $*$ is characterized by the following property: if $\{e_i\}$ is a positively oriented orthonormal frame for T^*, then

$$*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n.$$

(In particular, $*(1) = \omega$.) Note that this property implies

$$** \alpha = (-1)^{k(n-k)} \alpha$$

for k-forms α. Another useful identity is

$$\alpha \wedge *\alpha = \|\alpha\|^2 \omega.$$

Definition 8. *The codifferential $d^* : A^k(X) \to A^{k-1}(X)$ is given by*

$$\begin{aligned} d^*\alpha &= *^{-1} d * \alpha \\ &= (-1)^{n(k+1)+1} * d * . \end{aligned}$$

Notice that $(d^*)^2 = 0$, immediately from the definition $d^* = *^{-1} d *$. We say that a form φ is *coclosed* if $d^*\varphi = 0$, and *coexact* if $d^*\alpha = \varphi$ for some form α.

Definition 9. *The Laplacian is the linear operator $\Delta : A^k(X) \to A^k(X)$ given by*

$$\Delta = (d + d^*)^2 = dd^* + d^*d.$$

This generalizes the usual Laplacian operator on functions given by $\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$. By analogy, we say that a differential form φ is *harmonic* if $\Delta\varphi = 0$. Clearly any form which is both closed and coclosed is harmonic.

For each $k \geq 0$, the Hodge star induces an inner product on $A_c^k(X)$, the space of compact k-forms, given by

$$(\alpha, \beta) = \int_X \alpha \wedge *\beta;$$

this is indeed positive-definite since

$$(\alpha, \alpha) = \int_X \|\alpha\|^2 \omega.$$

With respect to this inner product, d^* is adjoint to d in the sense that

$$(d\alpha, \beta) = (\alpha, d^*\beta)$$

for α a k-form and β a $k+1$-form, both of compact support, and Δ is self-adjoint as

$$(\Delta\alpha, \beta) = (d\alpha, d\beta) + (d^*\alpha, d^*\beta) = (\alpha, \Delta\beta)$$

for k-forms α, β of compact support. If X is compact and $\Delta\alpha = 0$, then $(d\alpha, d\alpha) + (d^*\alpha, d^*\alpha) = 0$ and since $(,)$ is positive definite this implies $d\alpha = d^*\alpha = 0$. Hence on a compact manifold, we have that a form is harmonic if and only if it is closed and coclosed.

Proofs of the following results can be found in [3].

Proposition 1.3. *Given $\beta \in A^k(X)$ and a point $x \in X$, there is a neighbourhood U of x and $\mu \in A^k(U)$ such that $\Delta\mu = \beta$ on U.*

Theorem 1.1. *(Theorem 22 in [3]) Let X be a compact Riemannian manifold. For each k, the space of harmonic k-forms is finite-dimensional, and the equation $\Delta\mu = \beta$ of k-forms has a global solution μ if and only if β is orthogonal to all harmonic k-forms.*

2 Sheaves and cohomology

Divisors, line bundles, and gerbes are intimately related to certain sheaves and their cohomologies, which we define and discuss in some detail here. As a motivating example, we consider the sheaf $\mathcal{C}$ of continuous complex-valued functions on a space. Given a topological space X, for each open $U \subset X$ there is a ring $\mathcal{C}(U)$ of continuous complex functions on U, and for any inclusion $V \subset U$ of open sets, the restriction map

$$\mathcal{C}(U) \to \mathcal{C}(V) : f \mapsto f|_V$$

is a homomorphism of rings. Obviously the composition of two restriction maps is itself a restriction map. We also have the property that if $\{U_\alpha\}$ is a cover of X by open sets, and we are given functions $f_\alpha \in \mathcal{C}(U_\alpha)$ that agree on pairwise intersections ($f_\alpha|_{U_{\alpha\beta}} = f_\beta|_{U_{\alpha\beta}}$), then there is a unique $f \in \mathcal{C}(X)$ which satisfies $f|_{U_\alpha} = f_\alpha$ for all α. The general definition of a sheaf captures these essential properties of the rings $\mathcal{C}(U)$.

2.1 Sheaves

We begin with the notion of a presheaf, which assigns an object to each open subset of a given topological space, together with abstract restriction morphisms.[1]

Definition 10. *Let X be a topological space. Let $\boldsymbol{X}$ denote the subcategory of **Set** whose objects are the open subsets of X, and whose morphisms are precisely the inclusions $U \hookrightarrow V$. Let $\boldsymbol{C}$ be the category of commutative unital rings or the category of modules over some fixed ring. A presheaf on X with values in $\boldsymbol{C}$ is then a contravariant functor $\mathcal{F}$ from $\boldsymbol{X}$ to $\boldsymbol{C}$. For each open subset U, the elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.*

In other words, we assign to each open U an object $\mathcal{F}(U)$, and to each inclusion $U \subset V$ a unique *restriction* morphism $\rho_{UV} : \mathcal{F}(V) \to \mathcal{F}(U)$ such that two restrictions compose to give another restriction, with $\rho_{UU} = \mathrm{Id}_U$ and $\mathcal{F}(\emptyset) = 0$. To reinforce the idea that these morphisms are abstract restrictions, we may write $s|_U := \rho_{UV}(s)$ for each $s \in \mathcal{F}(V)$, when this is convenient notation.

The presheaf of rings $\mathcal{C}$ described above has some additional properties: if you have compatible continuous functions defined on the elements of an open cover of X, there is a unique global continuous function on X whose restrictions to elements of the cover are those functions. This leads us to the definition of a sheaf:

Definition 11. *Let $\mathcal{F}$ be a presheaf on a topological space X. We say that $\mathcal{F}$ is a sheaf if it satisfies the following additional properties:*

- $\mathcal{F}(\emptyset) = 0$
- *Let U be an open subset of X and let $\{U_\alpha\}$ be an open cover of U. Suppose we are given a collection of $s_\alpha \in \mathcal{F}(U_\alpha)$ such that $s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}}$ for all α and β. Then there exists a section $s \in \mathcal{F}(U)$, called the gluing of $\{s_\alpha\}$, such that $s|_{U_\alpha} = s_\alpha$ for all α.*
- *Let U be an open subset of X and let $\{U_\alpha\}$ be an open cover of U. If $s, t \in \mathcal{F}(U)$ such that $s|_{U_\alpha} = t|_{U_\alpha}$ for all α, then $s = t$.*

The first of these properties can be thought of as a sort of normalization axiom. The third property amounts to saying that sections are determined by their local properties, and it implies that the gluing given in the second property is uniquely determined.

Not every presheaf is a sheaf. A simple example is given by the presheaf $\mathcal{B}_\mathbb{R}$ of bounded continuous functions on $\mathbb{R}$. Taking the cover $\{U_\alpha = (-\alpha, \alpha) : \alpha > 0\}$ of $\mathbb{R}$, the function $f_\alpha(x) = x$ is in $\mathcal{B}_\mathbb{R}(U_\alpha)$ for each α, but clearly there is no $f \in \mathcal{B}_\mathbb{R}(\mathbb{R})$ which restricts to these, as the identity function $f(x) = x$ is not bounded on $\mathbb{R}$.

The local behaviour of a presheaf at a point is described by the stalk over that point. Let $\mathcal{F}$ be a presheaf on a topological space X, and fix a point $x \in X$. The collection of homomorphisms $\{\rho_{UV} : x \in U \subset V\}$ is a direct system, whose direct limit is the *stalk of $\mathcal{F}$ over x*, denoted $\mathcal{F}_x$. An element of $\mathcal{F}_x$ can be explicitly described by a neighbourhood U of x and a section $s \in \mathcal{F}(U)$, where (U, s) is identified with (V, t) if $s|_W = t|_W$ for some open neighbourhood $W \subset U \cap V$ of x. For each neighbourhood U of x and a section $s \in \mathcal{F}(U)$, the image of s under the canonical morphism $\mathcal{F}(U) \to \mathcal{F}_x$ is the *germ* of s at x, often denoted s_x. Note that if $\mathcal{F}$ is a sheaf of functions (for example, $\mathcal{A}$ or $\mathcal{O}$ as defined below) then this generalizes the usual definition of the germ of a function at a point.

[1] Here we only define a sheaf of rings or of modules over a ring, but sheaves can take values in other categories (e.g. **Set**).

Sections of a sheaf, due to uniqueness of gluing, are determined completely by their germs: if $\mathcal{F}$ is a sheaf and $s, t \in \mathcal{F}(U)$ such that $s_x = t_x$ for all $x \in U$, then there exist neighbourhoods $U_x \subset U$ such that $s|_{U_x} = t|_{U_x}$, and so $s = t$.

The following table describes some important sheaves defined on any complex manifold. E denotes a complex vector bundle.

Some important sheaves of functions		
Symbol	**Type**	**Definition**
$\mathcal{A}$	rings or additive groups	smooth complex functions
$\mathcal{A}^*$	multiplicative groups	nowhere-vanishing smooth complex functions
$\mathcal{A}^k$	vector spaces	k-forms
$\mathcal{A}^{r,s}$	vector spaces	(r, s)-forms
$\mathcal{Z}^k$	vector spaces	closed k-forms
$\mathcal{O}$	rings or additive groups	holomorphic functions
$\mathcal{O}^*$	multiplicative groups	nowhere-vanishing holomorphic functions
$\mathcal{A}^k(E)$	vector spaces	E-valued k-forms
$\mathcal{A}^{r,s}(E)$	vector spaces	E-valued (r, s)-forms
$\mathcal{O}(E)$	vector spaces	holomorphic sections of E
Ω^k	vector spaces	holomorphic k-forms
$\mathcal{A}^k$	vector spaces	k-forms
$\mathcal{M}$	rings or additive groups	meromorphic functions
$\mathcal{M}^*$	multiplicative groups	nonzero meromorphic functions
$\underline{A}$	abelian groups	locally constant A-valued functions, A an abelian group

This last class of sheaves of the form $\underline{A}$ is defined on any topological space, and several of the other sheaves are also defined on real manifolds.

2.2 Morphisms and sheafification

Definition 12. *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on a topological space X. A morphism from $\mathcal{F}$ to $\mathcal{G}$ is a collection of homomorphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for each inclusion $U \subset V$, $\rho_{UV} \circ \varphi_V = \varphi_U \circ \rho_{UV}$ (these two restrictions maps being for $\mathcal{G}$ and $\mathcal{F}$ respectively).*

Clearly, this is the same as a natural transformation $\varphi : \mathcal{F} \to \mathcal{G}$, considering these presheaves again as contravariant functors on the category $\mathbf{X}$. Just as a homomorphism of rings or modules has a kernel, image, and cokernel, so does a morphism of sheaves with values in one of these categories. Defining the image and cokernel requires some work, but the kernel is defined in an obvious way. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of, say, abelian groups. For each U we have a group $\ker \varphi_U \subset \mathcal{F}(U)$, and from the definition of a morphism of sheaves we see that the restriction maps ρ_{UV} take $\ker \varphi_V$ to $\ker \varphi_U$. We thereby obtain a new presheaf $\ker \varphi$, defined by $\ker \varphi(U) = \ker \varphi_U$. Indeed, $\ker \varphi$ is a sheaf: if $\{U_\alpha\}$ is an open cover of U and we are given $s_\alpha \in \ker \varphi_{U_\alpha}$ with $s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}}$, there is a unique $s \in \mathcal{F}(U)$ with $s|_{U_\alpha} = s_\alpha$ since $\mathcal{F}$ is a sheaf. Moreover, we have $\varphi(s)|_{U_\alpha} = \varphi_{U_\alpha}(s_\alpha) = 0 \in \mathcal{G}(U_\alpha)$, so by uniqueness of gluing in $\mathcal{G}$ it follows that $\varphi(s) = 0 \in \mathcal{G}(U_\alpha)$, whence $s \in \ker \varphi_U$.

We say that φ is *injective* if $\ker \varphi$ is the zero sheaf, which is to say that $\ker \varphi_U = 0$ for all open sets U. In this situation, $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$: by the injections $\varphi_U : \mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$ we identify $\mathcal{F}(U)$ with its image in $\mathcal{G}(U)$, and since $\rho_{UV} \circ \varphi_V = \varphi_U \circ \rho_{UV}$ for each $U \subset V$, we see that ρ_{UV} takes the image of $\mathcal{F}(V)$ to that of $\mathcal{F}(U)$. Conversely, if we have a collection of subobjects $\mathcal{S}(U) \subset \mathcal{G}(U)$ satisfying $\rho_{UV}\mathcal{S}(V) \subset \mathcal{S}(U)$, this defines a sheaf with restriction maps inherited from $\mathcal{G}$, and the inclusions $\mathcal{S}(U) \hookrightarrow \mathcal{G}(U)$ give a morphism of sheaves $\mathcal{S} \to \mathcal{G}$ with trivial kernel.

One may be tempted to define the cokernel of morphism by setting $\mathrm{coker}\varphi(U) = \mathrm{coker}\varphi_U = \mathcal{G}(U)/\varphi_U\mathcal{F}(U)$. The restriction maps $\mathcal{G}(V) \to \mathcal{G}(U)$ do descend to homomorphisms $\mathrm{coker}\varphi_V \to \mathrm{coker}\varphi_U$, as is easily seen

from φ being a morphism, and so this does define a presheaf. But it does not define a sheaf in general. For example, consider the sheaves $\mathcal{O}, \mathcal{O}^*$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and the sheaf morphism $\exp : \mathcal{O} \to \mathcal{O}^*$ which takes $f \in \mathcal{O}(U)$ to $e^{2\pi i f} \in \mathcal{O}^*(U)$. There is no global logarithm of the identity function $f(z) = z$, so this f is nonzero in $\mathcal{O}^*(\mathbb{C}^*)/\exp \mathcal{O}(\mathbb{C}^*)$, but taking $\{U_\alpha\}$ to be an cover of $\mathbb{C}^*$ by simply connected open subsets, the restrictions $f|_{U_\alpha}$ all have logarithms, and are therefore zero in $\mathcal{O}^*(U_\alpha)/\exp \mathcal{O}(U_\alpha)$. This contradicts the "uniqueness of gluing" property given before.

Nevertheless, there is a natural way to correct this lack of ability to glue sections together. More generally, there is a canonical "best" way of constructing a sheaf from any presheaf, called *sheafification*. Let $\mathcal{F}$ be a presheaf on X. For U an open subset of X, let $\mathcal{F}^+(U)$ be the set of all functions $s : U \to \bigsqcup_{x \in U} \mathcal{F}_x$ such that $s(x) \in \mathcal{F}_x$ for each x, and such that there is an open cover $\{U_\alpha\}$ of U and sections $s_\alpha \in \mathcal{F}(U_\alpha)$ satisfying $(s_\alpha)_x = s(x)$ for all $x \in U_\alpha$. An element of $\mathcal{F}^+(U)$ is then represented by such a collection $\{(U_\alpha, s_\alpha)\}$, where we identify this collection with $\{(V_\beta, t_\beta)\}$ if each $x \in U_\alpha \cap V_\beta$ has an open neighbourhood $W \subset U_\alpha \cap V_\beta$ with $s_\alpha|_W = t_\beta|_W$. Since sections of $\mathcal{F}^+(U)$ are defined as functions $U \to \bigsqcup_x \mathcal{F}_x$ with $s(x) \in \mathcal{F}_x$, it is clear that for $U \subset V$ we can restrict sections over V to U and thereby obtain another section. It is also clear that compatible local sections over $U_\alpha \subset U$ glue together uniquely to give a section over U, and so $\mathcal{F}^+$ is indeed a sheaf.

There is a natural morphism $\gamma : \mathcal{F} \to \mathcal{F}^+$ defined on U by taking s to the function $x \mapsto s_x$, and this satisfies a universal property:

Proposition 2.1. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves and $\mathcal{G}$ is a sheaf, then there is a unique morphism $\tilde{\varphi} : \mathcal{F}^+ \to \mathcal{G}$ with $\varphi = \tilde{\varphi}\gamma$.*

Proof. For $s \in \mathcal{F}^+(U)$, choose an open cover $\{U_\alpha\}$ of U with $s_\alpha \in \mathcal{F}(U_\alpha)$ such that $s(x) = (s_\alpha)_x$ for all $x \in U_\alpha$. Let $t_\alpha = \varphi_{U_\alpha}(s_\alpha) \in \mathcal{G}(U_\alpha)$. For $x \in U_{\alpha\beta}$, we have $(s_\alpha)_x = (s_\beta)_x = s(x)$ and so $s_\alpha|_{U_{\alpha\beta x}} = s_\beta|_{U_{\alpha\beta x}}$ for some neighbourhood $U_{\alpha\beta x} \subset U_{\alpha\beta}$ of x. Since $U_{\alpha\beta} = \bigcup_x U_{\alpha\beta x}$, we conclude by uniqueness of gluing in $\mathcal{G}$ that $s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{\alpha\beta}$, hence $t_\alpha|_{U_{\alpha\beta}} = t_\beta|_{\alpha\beta}$ as φ commutes with restrictions. Then since $\mathcal{G}$ is a sheaf, there is a unique $t \in \mathcal{G}(U)$ with $t|_{U_\alpha} = t_\alpha$. For each $x \in U_\alpha$, let φ_x denote the homomorphism $\mathcal{F}_x \to \mathcal{G}_x$ induced by φ. We have $t_x = (t_\alpha)_x = \varphi_x((s_\alpha)_x) = \varphi_x(s(x))$, and so t_x is independent of the choice of U_α and s_α. Since t is determined by its germs, the map $s \mapsto t$ is therefore well-defined, and we take $\tilde{\varphi}_U(s) = t$. $\square$

Given a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, we then define the *cokernel* of φ to be the sheafification of the presheaf $U \mapsto \mathrm{coker}\varphi_U$, and we denote this sheaf by $\mathrm{coker}\varphi$. Explicitly, an element of $\mathrm{coker}\varphi(U)$ is represented by an open cover $\{U_\alpha\}$ of X together with a collection of sections $s_\alpha \in \mathcal{G}(U_\alpha)$, where $\{(U_\alpha, s_\alpha)\}$ is identified with $\{(V_\beta, t_\beta)\}$ if each $x \in U_\alpha \cap V_\beta$ has an open neighbourhood $W \subset U_\alpha \cap V_\beta$ such that $s_\alpha|_W - t_\beta|_W \in \varphi_W \mathcal{F}(W)$. As an important special case, if $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ then we define the *quotient sheaf* $\mathcal{G}/\mathcal{F}$ to be the cokernel of the inclusion $\mathcal{F} \hookrightarrow \mathcal{G}$.

If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then we define the *image* of φ to be the sheaf $\mathrm{im}\varphi$ associated to the presheaf given by $U \mapsto \mathrm{im}\varphi_U$. By the universal property of sheafification, there is an induced morphism $\mathrm{im}\varphi \to \mathcal{G}$, and this can be seen to be injective and so $\mathrm{im}\varphi$ is naturally identified with a subsheaf of $\mathcal{G}$. We say that φ is *surjective* if $\mathrm{im}\varphi = \mathcal{G}$. (This does not imply that each of the induced homomorphisms φ_U is surjective.)

Given the existence of kernels, cokernels, and images of sheaves, we can define exact sequences just as we do in other categories. To us, the most important type is a *short exact sequence* of sheaves, which is a sequence of sheaf morphisms

$$0 \to \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \to 0$$

such that φ is injective, ψ is surjective, and $\ker \psi = \mathrm{im}\varphi$.

2.3 Cohomology of a sheaf

Let $\mathcal{F}$ be a sheaf on X, and let $\underline{U} = \{U_\alpha\}$ be an open cover of X. For each integer $p \geq 0$ we define the group of *p-cochains*

$$C^p(\underline{U}, \mathcal{F}) = \prod_{\alpha_0 \neq \cdots \neq \alpha_p} \mathcal{F}(U_{\alpha_0 \cdots \alpha_p})$$

where $U_{\alpha_0 \cdots \alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$. We define a *coboundary map* $\delta : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$ by the formula

$$(\delta\sigma)_{\alpha_0, \ldots, \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \sigma_{\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_{p+1}} |_{U_{\alpha_0 \cdots \alpha_{p+1}}}.$$

For example, if $\sigma \in C^0(\underline{U}, \mathcal{F})$, we have $(\delta\sigma)_{U,V} = \sigma_U - \sigma_V$, each of the terms on the right implicitly restricted to $U \cap V$. We denote the kernel of $\delta : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$ by $Z^p(\underline{U}, \mathcal{F})$, the group of *p-cocycles*. If σ is a p-cocycle, it must satisfy the alternating condition $\sigma_{\alpha_0, \ldots, \alpha_p} = \sigma_{\alpha_{\pi(0)}, \ldots, \alpha_{\pi(p)}}$ for each permutation π of $\{0, \ldots, p\}$. It is easy to see that $\delta^2 = 0$, *i.e.* $\delta C^{p-1}(\underline{U}, \mathcal{F}) \subset Z^p(\underline{U}, \mathcal{F})$, and we can therefore define the groups

$$H^p(\underline{U}, \mathcal{F}) = Z^p(\underline{U}, \mathcal{F}) / \delta C^{p-1}(\underline{U}, \mathcal{F}).$$

These are not the cohomology groups that we want to associate to $\mathcal{F}$, for they may depend on the choice of open cover $\underline{U}$. The solution to this is to take a direct limit of these groups over *all* open covers of X, ordered by refinement. Suppose that $\underline{U}' = \{U'_\beta\}_{\beta \in I'}$ is a refinement of $\underline{U} = \{U_\alpha\}_{\alpha \in I}$, which is to say that there exists a function $\phi : I' \to I$ such that $U'_\beta \subset U_{\phi\beta}$. We then define a homomorphism $\rho_\phi : C^p(\underline{U}, \mathcal{F}) \to C^p(\underline{U}', \mathcal{F})$ by the formula

$$(\rho_\phi \sigma)_{\beta_0 \cdots \beta_p} = \sigma_{\phi\beta_0 \cdots \phi\beta_p} |_{U_{\beta_0 \cdots \beta_p}}.$$

This map satisfies $\rho_\phi \circ \delta = \delta \circ \rho_\phi$:

$$\begin{aligned}
(\delta\rho_\phi\sigma)_{\beta_0 \cdots \beta_p} &= \sum_{i=0}^{p+1} (-1)^i (\rho_\phi \sigma)_{\beta_0 \cdots \widehat{\beta_i} \cdots \beta_{p+1}} |_{U_{\beta_0 \cdots \beta_{p+1}}} \\
&= \sum_{i=0}^{p+1} (-1)^i \left(\sigma_{\phi\beta_0 \cdots \widehat{\phi\beta_i} \cdots \phi\beta_{p+1}} |_{U_{\beta_0 \cdots \hat{\beta}_i \cdots \beta_{p+1}}} \right) |_{U_{\beta_0 \cdots \beta_{p+1}}} \\
&= \sum_{i=0}^{p+1} (-1)^i \sigma_{\phi\beta_0 \cdots \widehat{\phi\beta_i} \cdots \phi\beta_{p+1}} |_{U_{\beta_0 \cdots \beta_{p+1}}} \\
&= (\delta\sigma)_{\phi\beta_0 \cdots \phi\beta_{p+1}} |_{U_{\beta_0 \cdots \beta_{p+1}}} \\
&= (\rho_\phi \delta\sigma)_{\beta_0 \cdots \beta_{p+1}}
\end{aligned}$$

and so it descends to a homomorphism $\rho : H^p(\underline{U}, \mathcal{F}) \to H^p(\underline{U}', \mathcal{F})$. Moreover, if $\psi : I' \to I$ is another function satisfying $U'_b t \subset U_{\psi\beta}$, then it can be shown that ρ_ϕ and ρ_ψ are chain-homotopic, hence the homomorphism ρ does not depend on the choice of ϕ. Ordering the set of open covers of X by refinement, these maps ρ give a direct system of groups $H^p(\underline{U}, \mathcal{F})$ whose direct limit is the pth *(Čech) cohomology group* of $\mathcal{F}$ over X, denoted $H^p(X, \mathcal{F})$ or $\check{H}^p(X, \mathcal{F})$.

In particular, the 0th cohomology group $H^0(X, \mathcal{F})$ is the direct limit of the kernels of $\delta : C^0(\underline{U}, \mathcal{F}) \to C^1(\underline{U}, \mathcal{F})$, and so an element of this group is represented by an open cover $\underline{U}$ together with a collection of sections $\{\sigma_U\}_{U \in \underline{U}}$ satisfying $\sigma_U|_{U \cap V} = \sigma_V|_{U \cap V}$. But by the gluing axiom of sheaves, each such collection is specified uniquely by a global section of $\mathcal{F}$, and this clearly gives an isomorphism $H^0(X, \mathcal{F}) \cong \mathcal{F}(X)$.

Cohomology classes can be often thought of as measuring the degree to which "locally trivial" objects fail to be "globally trivial". For example, consider the de Rham cohomology of a manifold. A closed k-form ω may fail to be exact; that is, there may not exist a global $(k-1)$-form α with $d\alpha = \omega$. But near any given point in the manifold, we may restrict ω to an open contractible neighbourhood, which has trivial cohomology, so there is a solution to $d\alpha = \omega$ over this neighbourhood. The cohomology class $[\omega]$

can be thought of as measuring the obstruction to gluing these local solutions together to form a global solution to $d\alpha = \omega$.

Similar remarks apply to sheaf cohomology. Let $\mathcal{F}$ be a sheaf on X, and represent an element $[\sigma]$ of $H^p(X, \mathcal{F})$ by a cocycle $\sigma \in Z^p(\underline{U}, \mathcal{F})$ for some open cover $\underline{U} = \{U_\alpha\}$, so that

$$(\delta\sigma)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \sigma_{\alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_{p+1}} = 0$$

over $U_{\alpha_0 \cdots \alpha_{p+1}}$. Fix $U_\beta \in \underline{U}$, and consider the open cover $\underline{U}^\beta = \{U_\beta \cap U_\alpha\}$ of U_β. For each tuple $(\alpha_0, \ldots, \alpha_{p-1})$ of distinct indices, set

$$\tau^\beta_{\alpha_0 \cdots \alpha_{p-1}} = \begin{cases} \sigma_{\beta\alpha_0 \cdots \alpha_{p-1}} & \text{if } \alpha_i \neq \beta \text{ for all } i \\ 0 & \text{otherwise} \end{cases} .$$

$$\rho_{UV}(f) = \begin{cases} 0 & \text{if } U \text{ is a singleton and } U \neq V \\ \mathrm{f}\text{—}_U & \text{otherwise} \end{cases} .$$

We thereby obtain an element τ^β of $C^{p-1}(\underline{U}^\beta, \mathcal{F})$. Denote by σ^β the cocycle in $Z^p(\underline{U}^\beta, \mathcal{F})$ obtained by restricting the components of σ. From the equation $\delta\sigma = 0$, it is easily checked that $\delta\tau^\beta = \sigma^\beta$, and so there exist "local solutions" to the equation $\delta\tau = \sigma$. The cohomology class $[\sigma]$ measures the obstruction to the existence of a global solution to $\delta\tau = \sigma$.

Any short exact sequence of sheaves

$$0 \to \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \to 0$$

gives rise to a long exact sequence in cohomology,

$$0 \to H^0(X, \mathcal{E}) \to \cdots \to H^n(X, \mathcal{E}) \xrightarrow{\varphi_*} H^n(X, \mathcal{F}) \xrightarrow{\psi_*} H^n(X, \mathcal{G}) \xrightarrow{\delta_*} H^{n+1}(X, \mathcal{E}) \xrightarrow{\varphi_*} H^{n+1}(X, \mathcal{F}) \to \cdots .$$

Given $[g] \in H^p(X, \mathcal{G})$, by refining sufficiently we choose a representative cocycle $\{g_{\alpha_0 \cdots \alpha_p}\} \in Z^p(\underline{U}, \mathcal{G})$ such that there is $\{f_{\alpha_0 \cdots \alpha_p}\} \in C^p(\underline{U}, \mathcal{F})$ with $\psi f_{\alpha_0 \cdots \alpha_p}) = g_{\alpha_0 \cdots \alpha_p}$; then $\psi\delta f = \delta g = 0$, so there is a cochain $e \in C^{p+1}(X, \mathcal{E})$ with $\varphi e = \delta f$, and we set $\delta_*[g] = [e]$.

2.4 Some calculations of sheaf cohomology

Here we calculate the cohomologies of certain sheaves on manifolds, which will be useful in our investigation of line bundles and gerbes.

As discussed above, the cohomology of a sheaf measures the obstruction to finding global solutions to equations of the form $\delta\tau = \sigma$ for τ, given that local solutions exist. Partitions of unity, on the other hand, are objects that are most useful for "gluing local objects together". We can then use these to prove that certain sheaves have trivial cohomology:

Proposition 2.2. *Let X be a complex manifold. For all $r, s \geq 0$ and $p > 0$, we have $H^p(X, \mathcal{A}^{r,s}) = 0$. In particular, $H^p(X, \mathcal{A}) = 0$ for $p > 0$.*

Proof. Let $\underline{U} = \{U_\alpha\}$ be a locally finite open cover of X, and let σ be a p-cocycle of the pair $(\underline{U}, \mathcal{A}^{r,s})$. Choose a smooth partition of unity $\{\rho_\alpha\}$ subordinate to $\underline{U}$. Define $\tau \in C^{p-1}(\underline{U}, \mathcal{A}^{r,s})$ by

$$\tau_{\alpha_0 \cdots \alpha_{p-1}} = \sum_\beta \rho_\beta \cdot \sigma_{\beta\alpha_0 \cdots \alpha_{p-1}},$$

implicitly extending $\rho_\flat t \cdot \sigma_{\beta\alpha_0\cdots\alpha_{p-1}}$ to be identically zero on $U_{\alpha_0\cdots\alpha_p} \setminus U_\beta$. We then have

$$\begin{aligned}
(\delta\tau)_{\alpha_0\cdots\alpha_p} &= \sum_{i=0}^{p}(-1)^i \sum_\beta \rho_\beta \sigma_{\beta\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_p} \\
&= \sum_\beta \rho_\beta \sum_{i=0}^{p}(-1)^i \sigma_{\beta\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_p} \\
&= \sum_\beta \rho_\beta(\sigma_{\alpha_0\cdots\alpha_p} - \left(\sigma_{\alpha_0\cdots\alpha_p} - \sum_{i=0}^{p}(-1)^i \sigma_{\beta\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_p}\right) \\
&= \sum_\beta \rho_\beta(\sigma_{\alpha_0\cdots\alpha_p} - (\delta\sigma)_{\beta\alpha_0\cdots\alpha_p}) \\
&= \sigma_{\alpha_0\cdots\alpha_p},
\end{aligned}$$

this last equation following from $\sum_\beta \rho_\beta = 1$ and $\delta\sigma = 0$. Then $\delta\tau = \sigma$, hence $[\sigma] = 0$ in cohomology, and the claim follows. □

Fix an abelian group A and a topological space X. On one hand, we have the sheaf cohomology groups $\check{H}^p(X, \underline{A})$, and on the other hand we have the singular cohomology groups $H^p(X, A)$ with coefficients in A. These groups do not always coincide, but they do for sufficiently nice spaces. In particular, these groups coincide for simplicial complexes:

Proposition 2.3. *Let X be a topological space equipped with the structure of a simplicial complex K, and let $H^p(K, A)$ be its simplicial cohomology groups with coefficients in A. Then $H^p(X, \underline{A})$ is isomorphic to $H^p(K, A)$, hence to the singular cohomology $H^p(X, A)$.*

Proof. Let $\{v_\alpha\}$ denote the set of vertices (0-simplices) of K. For each α, let $St(v_\alpha)$ be the interior of the union of all simplices having v_α as a vertex. Then $\underline{U} = \{U_\alpha = St(v_\alpha)\}$ is an open cover of X. Denote the simplicial cochain groups of K with coefficients in A by $C^p(K, A)$. Given a $(p+1)$-tuple $\alpha_0 \cdots \alpha_p$, either the set $U_{\alpha_0\cdots\alpha_p}$ is nonempty and connected, in the event that some p-simplex has $v_{\alpha_0}, \ldots, v_{\alpha_p}$ as its vertices, or $U_{\alpha_0\cdots\alpha_p}$ is empty. Hence $\underline{A}(U_{\alpha_0\cdots\alpha_p}) = A$ in the former case and $\underline{A}(U_{\alpha_0\cdots\alpha_p}) = 0$ in the latter case. We then have a group isomorphism $C^p(\underline{U}, \underline{A}) \cong C^p(K, A)$: a p-cochain $\sigma = \{\sigma_{\alpha_0\cdots\alpha_p}\} \in C^p(\underline{U}, \underline{A})$ corresponds to the p-cochain $\sigma' \in C^p(K, A)$ given by $\sigma'[v_{\alpha_0}, \ldots, v_{\alpha_p}] = \sigma_{\alpha_0\cdots\alpha_p} \in A$. This isomorphism gives rise to an isomorphism of cochain complexes since

$$\begin{aligned}
(\delta\sigma)'[v_{\alpha_0}, \ldots, v_{\alpha_{p+1}}] &= \sum_{i=0}^{p+1}(-1)^i \sigma_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{p+1}}|_{U_{\alpha_0\cdots\alpha_{p+1}}} \\
&= \delta(\sigma')[v_{\alpha_0}, \ldots, v_{\alpha_{p+1}}]
\end{aligned}$$

and so this descends to an isomorphism $H^p(\underline{U}, \underline{A}) \cong H^p(K, A)$ on cohomology. By subdividing K to make the cover $\underline{U}$ arbitrarily fine without changing the cohomology $H^p(\underline{U}, \underline{A})$, we obtain $H^p(X, \underline{A}) \cong H^p(K, A)$ as claimed. □

In particular, if X is a manifold, then these cohomology groups coincide. We shall then use the notation $H^p(X, A)$ to refer to either the cohomology of the constant sheaf $\underline{A}$ or the singular cohomology with coefficients in A, without ambiguity.

There are a few other results which we will invoke later, whose proofs we either sketch or omit entirely. The reader is referred to [5] for proofs of the following two results.

Proposition 2.4. *Let X be a Riemann surface, and let $L \to X$ be a holomorphic line bundle. Then for all $p > 1$, $H^p(X, \mathcal{O}(L)) = 0$.*

Proposition 2.5. ***(Serre duality)*** *Let X be a compact Riemann surface and let $L \to X$ be a holomorphic line bundle. Then $H^1(X, \mathcal{O}(L)) \cong H^0(X, \mathcal{O}(K \otimes L^*))^*$ where $K = T^*X$ is the canonical line bundle.*

Proofs of the next two results can be found in [4].

Proposition 2.6. ***(Hodge decomposition)*** *Let X be a Kähler manifold (*e.g. *any Riemann surface equipped with a metric). Let $H^{p,q}(X)$ denote the qth cohomology of the differential complex $(A^{p,\bullet}(X), \bar{\partial})$ (that is, the $(p,q)^{th}$* Dolbeault cohomology *of X). Then $H^{p,q}(X) = \overline{H^{q,p}(X)}$, and*

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

Proposition 2.7. ***(Dolbeault's theorem)*** *The $(p,q)^{th}$ Dolbeault cohomology is isomorphic to the qth cohomology of the sheaf of holomorphic p-forms, $H^{p,q}(X) \cong H^q(X, \Omega^p)$.*

Using these last two results, we make an important observation about compact Riemann surfaces. Suppose X is a connected compact Riemann surface of genus g. Combining the Hodge decomposition with Dolbeault's theorem, we then have

$$\begin{aligned} \mathbb{C}^{2g} &\cong H^{1,0}(X) \oplus \overline{H^{1,0}(X)} \\ &\cong H^0(X, \mathcal{O}(K)) \oplus \overline{H^0(X, \mathcal{O}(K))} \end{aligned}$$

and we conclude that *the complex dimension of the space of holomorphic* 1*-forms on X is the genus of X.*

3 Divisors and line bundles on Riemann surfaces

Throughout this section, X denotes a Riemann surface.

3.1 Divisors

Let f be a meromorphic function on X. To each point $p \in X$ we assign the integer weight a_p, the order of f at that point. The set S of points which have nonzero weights, *i.e.* the set of zeroes and poles of f, is *locally finite*: each point of X has a neighbourhood which contains only finitely many points from S. Indeed, this is equivalent to asserting that each point $p \in X$ has a neighbourhood chart U such that either $U \cap S = \emptyset$ or $U \cap S = \{p\}$, by sufficiently shrinking the given neighbourhood.

Definition 13. *A divisor on X is a $\mathbb{Z}$-linear combination*

$$\sum_{p \in X} a_p \cdot p$$

which is locally finite, by which it is meant that the set $\{p : a_p \neq 0\}$ is locally finite.

Definition 14. *The divisor associated to f is the formal linear combination*

$$(f) = \sum_p a_p \cdot p.$$

Any divisor of this form is called a principal divisor.

Not every divisor is principal, in general.

Definition 15. *A divisor D is effective if $D \geq 0$; that is, $D = \sum_p a_p \cdot p$ for $a_p \geq 0$.*

For a meromorphic function f, evidently (f) is effective if and only if f is holomorphic. It is clear that the negative of a divisor is also a divisor, and so is the sum of two divisors, and therefore the set of divisors on X forms an additive group which we denote by $\mathrm{Div}(X)$.

Definition 16. *A divisor $\sum_p a_p \cdot p$ is finite if $a_p \neq 0$ for only finitely many p.*

If X is compact, then every divisor on X is finite since every discrete subset of X is finite. In general, the subgroup of finite divisors is precisely the group $C_0(X, \mathbb{Z})$ of integral cycles on X, which is therefore identified with $\mathrm{Div}(X)$ if X is compact.

It should be mentioned that on a complex manifold of arbitrary dimension, the appropriate definition of divisor is as a locally finite linear combination of complex codimension-1 irreducible analytic hypersurfaces, and most of what follows generalizes to this situation. (See [4] for details.) For simplicity, however, we restrict our attention to Riemann surfaces.

Since there is a natural inclusion of sheaves $\mathcal{O}^* \hookrightarrow \mathcal{M}^*$, we have a quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$. Informally, a section of this sheaf can be thought of as locally given by compatible meromorphic functions, but where we have thrown away all information about them except the locations and orders of their zeroes and poles. On the other hand, we can think of a divisor in a similar way. Indeed, we have the

Proposition 3.1. *There is a canonical isomorphism* $\Phi : H^0(X, \mathcal{M}^*/\mathcal{O}^*) \to \mathrm{Div}(X)$ *of abelian groups.*

Proof. Let f be a global section of $\mathcal{M}^*/\mathcal{O}^*$, represented by an open cover $\{U_\alpha\}$ of X together with $f_\alpha \in \mathcal{M}^*(U_\alpha)$ such that $f_\alpha/f_\beta \in \mathcal{O}^*(U_{\alpha\beta})$ for all α, β. At any point $p \in U_{\alpha\beta}$, we have

$$\mathrm{ord}_p(\frac{f_\alpha}{f_\beta}) = 0,$$

whence

$$\mathrm{ord}_p(f_\alpha) = \mathrm{ord}_p(f_\beta).$$

We can then define

$$\mathrm{ord}_p(f) := \mathrm{ord}_p(f_\alpha)$$

for any $U_\alpha \ni p$. This is independent of the choice of representative $\{(U_\alpha, f_\alpha)\}$: if $\{(U'_{\alpha\prime}, f'_{\alpha\prime})\}$ is another choice and $p \in U_\alpha \cap U'_{\alpha\prime}$, then $f_\alpha/f'_{\alpha\prime}$ is nonzero and holomorphic on a neighbourhood of p, hence $\mathrm{ord}_p(f_\alpha) = \mathrm{ord}_p(f'_{\alpha\prime})$.

Now, by refining the cover $\{U_\alpha\}$, we may assume without loss of generality that it is locally finite, and it follows that the linear combination

$$\Phi(f) := \sum_p \mathrm{ord}_p(f) \cdot p$$

is locally finite. From the identity $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$, it follows immediately that

$$\Phi : H^0(X, \mathcal{M}^*/\mathcal{O}^*) \to \mathrm{Div}(X)$$

is a group homomorphism. It is easy to see that Φ is injective: if $f \sim \{(U_\alpha, f_\alpha)\}$ is in its kernel, then $\mathrm{ord}_p(f_\alpha) = 0$ for all $p \in U_\alpha$, which is to say that $f_\alpha \in \mathcal{O}^*(U_\alpha)$, and so $f \sim \{(U_\alpha, 1)\}$ is the identity element of $H^0(X, \mathcal{M}^*/\mathcal{O}^*)$. Finally, we prove surjectivity. Let $D = \sum_p a_p \cdot p$ be a divisor, and let $S = \{p : a_p \neq 0\}$. From remarks given previously, each $p \in S$ has a neighbourhood chart U_p with $U_p \cap S = \{p\}$, and each $p \notin S$ has a neighbourhood U_p with $U_p \cap S = \emptyset$. For $p \in S$, let

$$z_p : U_p \to \mathbb{C}$$

be a choice of coordinate with $z_p(p) = 0$, and define

$$f_p = z_p^{a_p} \in \mathcal{M}^*(U_p).$$

For $p \notin S$, let $f_p = 1 \in \mathcal{M}^*(U_p)$. For all $p, q \in X$, f_p/f_q is nonvanishing and holomorphic on U_{pq}, since both f_p and f_q are so on this set. The collection $\{(U_p, f_p)\}$ then defines a global section f of $\mathcal{M}^*/\mathcal{O}^*$, and $\Phi(f) = D$ since $\mathrm{ord}_p(f_p) = a_p$ for each p. Therefore Φ is an isomorphism as claimed. $\square$

We shall henceforth implicitly use Φ to identify divisors with global sections of $\mathcal{M}^*/\mathcal{O}^*$.

3.2 From divisors to line bundles

If L is a holomorphic line bundle on X, the isomorphisms $L_x \otimes L_x^* = \mathbb{C}$ give rise to a bundle isomorphism $L \otimes L^* \cong X \times \mathbb{C}$, the trivial bundle, and $L \otimes (X \times \mathbb{C}) = L$ canonically. As the tensor product is associative and commutative up to isomorphism, this implies that it induces an abelian group law on the set of isomorphism classes of holomorphic line bundles on X, where the trivial line bundle is the identity and the inverse of an element is given by its dual.

Definition 17. *The Picard group of X is the abelian group* $\mathrm{Pic}(X)$ *of isomorphism classes of holomorphic line bundles on X, under the group operation given by the tensor product.*

Let $L \to X$ be a holomorphic line bundle, given by an open cover $\{U_\alpha\}$ of X and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$, with trivializations $\varphi_\alpha : L_{U_\alpha} \to U_\alpha \times \mathbb{C}$. (So $\varphi_\alpha \circ \varphi_\beta^{-1}(z, v) = (z, g_{\alpha\beta}(z)v)$.) Suppose we took a collection of $f_\alpha \in \mathcal{O}^*(U_\alpha)$ and defined new trivializations $\varphi'_\alpha = f_\alpha \cdot \varphi_\alpha$. That is, where $\varphi_\alpha(w) = (z, v)$, we have $\varphi'_\alpha(w) = (z, f_\alpha(z)v)$. We obtain φ'_α by composing φ_α with the line bundle automorphism

$$U_\alpha \times \mathbb{C} \to U_\alpha \times \mathbb{C} : (z, v) \mapsto (z, f_\alpha(z)v),$$

and so these φ'_α define the same line bundle L, up to isomorphism. We have

$$\varphi'_\alpha \circ \varphi'^{-1}_\beta(z, v) = \left(z, \frac{f_\alpha(z)}{f_\beta(z)} g_{\alpha\beta}(z)v\right),$$

i.e. the transition functions for L relative to the trivializations φ'_α are $g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}$. Conversely, if $\{\varphi'_\alpha\}$ is any set of trivializations of L over $\{U_\alpha\}$ then, for each α, $\varphi'_\alpha \circ \varphi_\alpha^{-1}$ is a line bundle automorphism of $U_\alpha \times \mathbb{C}$, hence

$$\varphi'_\alpha \circ \varphi_\alpha^{-1}(z, v) = (z, f_\alpha(z)v)$$

for some $f_\alpha \in \mathcal{O}^*(U_\alpha)$ and

$$\varphi'_\alpha \circ \varphi'^{-1}_\beta = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}.$$

Hence two collections $\{g_{\alpha\beta}\}, \{g'_{\alpha\beta}\}$ of transition functions over $\{U_\alpha\}$ define isomorphic line bundles if and only if there exist $f_\alpha \in \mathcal{O}^*(U_\alpha)$ such that $g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}$.

Now consider

$$C^1(X, \mathcal{O}^*) = \{\{g_{\alpha\beta}\} \in \prod_{\alpha \neq \beta} \mathcal{O}^*(U_{\alpha\beta}) : g_{\beta\alpha} = g_{\alpha\beta}^{-1}\}.$$

For $g = \{g_{\alpha\beta}\}$, we have

$$(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma} \cdot g_{\alpha\gamma}^{-1} \cdot g_{\alpha\beta} = g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha},$$

so g is a cocycle if and only if it is a collection of transition functions for some line bundle. Moreover, from the discussion above, two cocycles g and g' define isomorphic line bundles if and only if there is some $f = \{f_\alpha\} \in C^0(X, \mathcal{O}^*)$ such that $g \cdot \delta f = g'$, *i.e.* g and g' are cohomologous. We conclude that the set $\mathrm{Pic}(X)$ of isomorphism classes of holomorphic line bundles over X corresponds naturally with $H^1(X, \mathcal{O}^*)$. Now if line bundles L, L' are given by transition functions $\{g_{\alpha\beta}\}, \{g'_{\alpha\beta}\}$ respectively, then $L \otimes L'$ is given by $\{g_{\alpha\beta} \cdot g'_{\alpha\beta}\}$ and L^* by $\{g_{\alpha\beta}^{-1}\}$, so the described correspondence is a group homomorphism:

Proposition 3.2. $\mathrm{Pic}(X)$ *is canonically isomorphic to* $H^1(X, \mathcal{O}^*)$.

There is a natural way of associating a line bundle to each divisor. Let D be a divisor, considered as a global section of $\mathcal{M}^*/\mathcal{O}^*$ under the isomorphism given above. Write $D \sim \{f_\alpha\}$ to indicate that D is represented by the local data $\{(U_\alpha, f_\alpha)\}$. On $U_{\alpha\beta}$ we define $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ which is nonvanishing and holomorphic, and clearly $g_{\alpha\beta} \cdot g_{\beta\alpha} = 1$ and $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ for all α, β, γ. Then these $g_{\alpha\beta}$ are transition functions for a line bundle. To see that this line bundle is well-defined up to isomorphism, we observe

that if $D \sim \{f'_\alpha\}$ then $h_\alpha := \frac{f_\alpha}{f'_\alpha} \in \mathcal{O}^*(U_\alpha)$ and so the transition functions

$$g'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta} = \frac{h_\alpha^{-1}}{h_\beta^{-1}} g_{\alpha\beta}$$

define the same line bundle as $\{g_{\alpha\beta}\}$, by observations above. The line bundle obtained in this way is the line bundle *associated* to D, denoted $[D]$. If $D \sim \{f_\alpha\}$ and $D' \sim \{f'_\alpha\}$, then $D + D' \sim \{f_\alpha \cdot f'_\alpha\}$ and so the transition functions of $[D + D']$ are $\{g_{\alpha\beta} \cdot g'_{\alpha\beta}\}$, whence $[D + D'] = [D] \otimes [D']$, and we conclude that $[\,\cdot\,] : \mathrm{Div}(X) \to \mathrm{Pic}(X)$ is a homomorphism of abelian groups. We might then ask what its kernel is. Let $D \sim \{f_\alpha\}$ be a divisor such that $[D]$ is the trivial line bundle on X. Then there exist $h_\alpha \in \mathcal{O}^*(U_\alpha)$ such that

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} = \frac{h_\alpha}{h_\beta} \cdot 1,$$

as transition functions for $X \times \mathbb{C}$ are given by the constant functions 1, and so

$$\frac{f_\alpha}{h_\alpha} = \frac{f_\beta}{h_\beta}$$

on $U_{\alpha\beta}$. Hence these $\frac{f_\alpha}{h_\alpha}$ glue together to give a global meromorphic function f on X. For any point p, $\mathrm{ord}_p(f) = \mathrm{ord}_p(f_\alpha) - \mathrm{ord}_p(h_\alpha) = \mathrm{ord}_p(f_\alpha)$, and so $D = \sum_p \mathrm{ord}_p(f_\alpha) = \sum_p \mathrm{ord}_p(f)$ is the principal divisor (f). Conversely, given a meromorphic function f on X, the principal divisor (f) is represented by local data $\{f_\alpha\}$ for $f_\alpha = f|_{U_\alpha}$, and $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} = 1$, whence $[(f)]$ is trivial. We conclude from this the

Proposition 3.3. *The kernel of* $[\,\cdot\,]$ *is precisely the set of principal divisors on* X.

Definition 18. *Two divisors* $D, D' \in \mathrm{Div}(X)$ *are* linearly equivalent *if* $[D] = [D']$.

By the above proposition, D and D' are linearly equivalent if and only if $D = D' + (f)$ for some $f \in \mathcal{M}(X)$.

A much more concise way to describe the homomorphism $[\,\cdot\,]$ is in terms of a long exact sequence in sheaf cohomology. From the short exact sequence

$$\mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^*$$

we obtain a connecting homomorphism

$$\delta_* : H^0(X, \mathcal{M}^*/\mathcal{O}^*) \to H^1(X, \mathcal{O}^*).$$

Explicitly, representing an element f of $H^0(X, \mathcal{M}^*/\mathcal{O}^*)$ by a collection $\{f_\alpha \in \mathcal{M}^*(U_\alpha)\}$ with

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_{\alpha\beta}),$$

$\delta_* f$ is represented by the collection of these functions

$$\frac{f_\alpha}{f_\beta}.$$

Under the identifications

$$\mathrm{Div}(X) = H^0(X, \mathcal{M}^*/\mathcal{O}^*)$$

and

$$\mathrm{Pic}(X) = H^1(X, \mathcal{O}^*),$$

it is clear that $[\,\cdot\,] = \delta_*$. Moreover, the homomorphism

$$H^0(X, \mathcal{M}^*) \to H^0(X, \mathcal{M}^*/\mathcal{O}^*)$$

is precisely the map that sends a global meromorphic function to the divisor associated to it, and we

conclude immediately from exactness that the kernel of δ consists of the set of principal divisors (as explicitly shown above).

3.3 Holomorphic and meromorphic sections

Let $L \to X$ be a holomorphic line bundle with local trivializations φ_α over open sets U_α, and transition functions $g_{\alpha\beta}$. A holomorphic section s of L over an open set $U \subset X$ is then defined by a collection of holomorphic functions $s_\alpha : U \cap U_\alpha \to \mathbb{C}$ satisfying $s_\alpha = g_{\alpha\beta} \cdot s_\beta$ on $U \cap U_{\alpha\beta}$, giving the sheaf $\mathcal{O}(L)$ of holomorphic sections of L.

Definition 19. *The sheaf of* meromorphic sections *of L is*

$$\mathcal{M}(L) = \mathcal{O}(L) \otimes \mathcal{M}$$

as a tensor product of $\mathcal{O}$-modules.

This naturally contains $\mathcal{O}(L)$ as a subsheaf. A meromorphic section of L over U, then, is defined by a collection of meromorphic functions $s_\alpha \in \mathcal{M}(U_\alpha)$ satisfying $s_\alpha = g_{\alpha\beta} \cdot s_\beta$ on $U \cap U_{\alpha\beta}$. Clearly the quotient of two meromorphic sections of L defines a meromorphic function on X, and the quotient of two holomorphic sections defines a holomorphic function on X. Given a meromorphic section s described by $\{s_\alpha\}$, we have $\mathrm{ord}_p(s_\alpha) = \mathrm{ord}_p(g_{\alpha\beta}) + \mathrm{ord}_p(s_\beta) = \mathrm{ord}_p(s_\beta)$ for each point p, and so we can define $\mathrm{ord}_p(s) = \mathrm{ord}_p(s_\alpha)$ for any U_α containing p, and the *divisor associated to s* is $(s) = \sum_p \mathrm{ord}_p(s) \cdot p$. Note that (s) is effective if and only if s is holomorphic, immediately from definitions.

Given a divisor D on X, considered as a section of $\mathcal{M}^*/\mathcal{O}^*$, represent D by local meromorphic functions $f_\alpha \neq 0$. By definition of the transition functions $g_{\alpha\beta}$ of $[D]$ we have $f_\alpha = g_{\alpha\beta} \cdot f_\beta$, and so these f_α define a meromorphic section s_f of $[D]$. Clearly, $(s_f) = D$. Conversely, let L be a line bundle with transition functions $g_{\alpha\beta}$ and let s be a meromorphic section of L other than 0, represented by local meromorphic functions s_α, so that $s_\alpha = g_{\alpha\beta} \cdot s_\beta$ on $U_{\alpha\beta}$. Then these s_α define a section s of $\mathcal{M}^*/\mathcal{O}^*$ since

$$s_\alpha / s_\beta = g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta}),$$

and moreover

$$L = [(s)]$$

. We summarize this by the

Proposition 3.4. *Let L be a line bundle. Then there exists a divisor D with $L = [D]$ if and only if L admits a global meromorphic section other than the zero section. Moreover, if L admits such a meromorphic section s, then $L = [(s)]$. In particular, L admits a nontrivial global holomorphic section if and only if $L = [D]$ for some effective divisor D.*

3.4 The first Chern class

The exponential sequence is a short exact sequence of sheaves on a complex manifold which is of fundamental importance. Given a complex manifold X, the *(holomorphic) exponential sequence* is

$$0 \to \underline{\mathbb{Z}} \hookrightarrow \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0$$

where exp is defined by the homomorphisms

$$\exp_U : \mathcal{O}(U) \to \mathcal{O}^*(U) : f \mapsto e^{2\pi i f}.$$

That $\underline{\mathbb{Z}}$ is the kernel of exp is immediate, and exactness of the sequence at $\mathcal{O}$ follows from the fact that X admits a cover by contractible open sets, over each of which any non-vanishing holomorphic function has a logarithm which is unique up to a constant of the form $2\pi k i, k \in \mathbb{Z}$.

Definition 20. *The first Chern class of a line bundle L is the image of its isomorphism class under the connecting homomorphism*

$$c_1 : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$$

in the long exact sequence associated to the exponential sequence.

Explicitly, if L has the transition functions $\{g_{ij}\}$ with a cover $\{U_i\}$, by refining the cover sufficiently we assume that we can choose a branch of logarithm on each U_{ij}, and define $h_{ij} = \frac{1}{2\pi i} \log g_{ij}$, so $\exp h_{ij} = g_{ij}$. Let $z_{ijk} = h_{ij} + h_{jk} - h_{ik}$. This is not necessarily zero since the chosen branches of logarithm may not agree, but because g is a cocycle we have $\exp z_{ijk} = 1$, hence z_{ijk} is a locally constant integer function. The cohomology class of the cocycle $\{z_{ijk}\}$ is precisely $c_1(L)$.

We are particularly interested in the case that X is a connected Riemann surface, in which case we identify $H^2(X, \mathbb{Z})$ with $\mathbb{Z}$ via the canonical orientation on X. In any case, we may write $c_1(L)$ to denote the image of $c_1(L)$ under the natural map $H^2(X, \mathbb{Z}) \to H^2_{dR}(X)$.

Proposition 3.5. *Let $L \to X$ be a holomorphic line bundle on a Riemann surface X, and let F be the curvature form of a connection ∇ on L. Then*

$$c_1(L) = \left[\frac{i}{2\pi} F\right] \in H^2_{dR}(X).$$

Proof. Using the notation given above, we represent $c_1(L)$ by $z_{ijk} = \frac{1}{2\pi i}(\log g_{ij} + \log g_{jk} - \log g_{ik})$. Represent ∇ by local 1-forms A_i with $A_i - A_j = d \log g_{ij}$ and $F = dA$. Since $c_1(L)$ is given by a Čech cocycle and F is given by a closed 2-form, we calculate explicitly the image of $[F]$ under the de Rham isomorphism $H^2_{dR}(X) \to H^2(X, \mathbb{C})$. From the Poincaré lemma, we have exact sequences of sheaves

$$0 \to \mathbb{C} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{Z}^1 \to 0$$
$$0 \to \mathcal{Z}^1 \to \mathcal{A}^1 \xrightarrow{d} \mathcal{Z}^2 \to 0$$

thereby giving boundary isomorphisms

$$H^2_{dR}(X) = \frac{H^0(X, \mathcal{Z}^2)}{dH^0(X, \mathcal{A}^1)} \xrightarrow{\delta_1} H^1(X, \mathcal{Z}^1)$$

and

$$H^1(X, \mathcal{Z}^1) \xrightarrow{\delta_2} H^2(X, \mathcal{C}),$$

since

$$H^1(X, \mathcal{A}^0) = H^2(X, \mathcal{A}^0) = H^1(X, \mathcal{A}^1) = 0$$

by the partition-of-unity argument given previously. The composition $\delta_2\delta_1$ is the de Rham isomorphism $H^2_{dR}(X) \to H^2(X, \mathbb{C})$. Explicitly, we have $\delta_1([F]) = \{A_j - A_i\} = \{-d \log g_{ij}\}$ and so

$$\begin{aligned} \delta_2\delta_1([F]) &= \{-(\log g_{ij} + \log g_{jk} - \log g_{ik})\} \\ &= -2\pi i \cdot c_1(L), \end{aligned}$$

completing the proof. □

Proposition 3.6. *Let $D = \sum_p a_p \cdot p$ be a divisor on a compact Riemann surface X. Then $c_1([D])$ is Poincaré-dual to the homology class of D.*

Proof. It suffices to prove the result when $D = p$ is a single point of weight 1, since the map $D \mapsto c_1([D])$ and the map taking D to the Poincaré dual of the homology class of D are both homomorphisms from $\mathrm{Div}(X)$ to $H^2(X, \mathbb{C})$. Let U_p be a disc about p, let $U_0 = X \setminus \{p\}$, and let z be a choice of coordinate on U_p with image the unit disc and $z(p) = 0$, giving the transition functions $g_{p0} = z, g_{0p} = z^{-1}$ for $[p]$. A

connection on this line bundle is given by the local 1-forms $A_p = d\log z$ and $A_0 = 0$. Where F is the curvature of this connection,

$$\int_X F \quad = \quad \int_{U_p} dA^p$$

Thus $\left[\frac{i}{2\pi}F\right]$ is Poincaré-dual to (p) and the result follows from the previous one. $\square$

Corollary 1. *For a divisor $D = \sum_p a_p \cdot p$ on a connected Riemann surface X, $c_1([D]) = \sum_p a_p$ where $H^2(X, \mathbb{Z})$ is identified with $\mathbb{Z}$ by its canonical orientation.*

Corollary 2. *Any holomorphic line bundle L with $c_1(L) < 0$ admits no nontrivial holomorphic sections.*

Corollary 3. *If L admits a nontrivial holomorphic section, then its dual L^* does not.*

Just as $H^1(M, \mathcal{O}^*)$ is naturally identified with the group of holomorphic line bundles on M when M is a complex manifold, for any real manifold X we can apply the same reasoning to identify $H^1(X, \mathcal{A}^*)$ with the group $\mathrm{Pic}^\infty(X)$ of smooth complex line bundles on X, and we have another short exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \mathcal{A} \to \mathcal{A}^* \to 0$$

given by exponentiation, the *smooth exponential sequence*. We can then define the first Chern class of a complex line bundle to be its image under the map $c_1 : H^1(X, \mathcal{A}^*) \to H^2(X, \mathbb{Z})$, and indeed this agrees with the previous map c_1 for holomorphic line bundles, on forgetting the complex structure. An important difference is that, unlike in the holomorphic case, the cohomology groups $H^p(X, \mathcal{A})$ are all zero for $p > 0$. The long exact sequence for the exponential sequence then tells us that c_1 is, in fact, an *isomorphism*, and in particular a holomorphic line bundle is determined, up to smooth equivalence, by its Chern class. We will later consider gerbes on 3-manifolds, which obviously admit no complex structure, and these will have a Chern class determined by the smooth exponential sequence.

We make one more observation about the first Chern class homomorphism, in the special case of a compact connected Riemann surface. Let X be such a surface, of genus g, and consider the long exact sequence

$$0 \to H^0(X, \mathbb{Z}) \to H^0(X, \mathcal{O}) \to H^0(X, \mathcal{O}^*) \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O})$$

arising from the exponential sheaf sequence. Since X is compact and connected, the first three nonzero terms are $\mathbb{Z}, \mathbb{C}, \mathbb{C}^*$ respectively. By Poincaré duality, we have $H^1(X, \mathbb{Z}) = \mathbb{Z}^{2g}$ and $H^2(X, \mathbb{Z}) = \mathbb{Z}$. From Serre duality, since $\mathcal{O}$ is the sheaf of holomorphic sections of the trivial bundle $\mathbb{C}$ on X, we have

$$H^1(X, \mathcal{O}) \cong H^0(X, \mathcal{O}(K \otimes \mathbb{C}))^* = H^0(X, \mathcal{O}(K))^*$$

which is isomorphic to $\mathbb{C}^g$ by a previous observation. Finally, by another previously stated result, we have $H^2(X, \mathcal{O}) = 0$ for X a Riemann surface. We then rewrite the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to \mathbb{Z}^{2g} \to \mathbb{C}^g \to \mathrm{Pic}(X) \xrightarrow{c_1} \mathbb{Z} \to 0.$$

Since $\exp : \mathbb{C} \to \mathbb{C}^*$ is surjective, by exactness the map $\mathbb{C}^* \to \mathbb{Z}^{2g}$ is zero, and hence the map $\mathbb{Z}^{2g} \to \mathbb{C}^g$ is injective. We conclude that $\ker c_1$, or more generally the set of holomorphic line bundles of a fixed first Chern class, is a g-dimensional complex torus, isomorphic to $\mathbb{C}^g/\Lambda$ for some lattice $\Lambda \cong \mathbb{Z}^{2g}$. Since the smooth type of a line bundle is determined by its first Chern class, such a torus can be interpreted as the moduli space whose points are the holomorphic structures admissible by a fixed smooth complex line bundle.

4 Currents and the Poincaré-Lelong equation

On an oriented manifold, the theory of currents can be roughly thought of as a way of uniformly treating k-forms and oriented codimension-k submanifolds (or, more generally, codimension-k singular chains) as similar objects.

4.1 Currents

Let X be an oriented n-manifold. For each $k \geq 0$, the space $A_c^k(X)$ of compactly supported k-forms on X is equipped with a natural topology: a sequence φ_i of k-forms converges to φ if and only if there is some compact set containing all of their supports, and all derivatives of all coefficients of $\varphi_i - \varphi$ converge uniformly to 0. Any $(n-m)$-form φ defines a continuous linear functional $T_\varphi : A_c^m(X) \to \mathbb{C}$ by the formula $\omega \mapsto \int_X \omega \wedge \varphi$, and this is an example of an $(n-m)$-current.

Definition 21. *A* k-current *on* X *is a continuous linear functional* $A_c^{n-k}(X) \to \mathbb{C}$. *The space of* k-*currents on* X *is denoted* $\mathcal{D}^k(X)$.

Operations on forms induce operations on currents. The de Rham derivative induces a map

$$d : \mathcal{D}^k(X) \to \mathcal{D}^{k+1}(X)$$

given by

$$(dT)\varphi = (-1)^{k+1} T(d\varphi).$$

This generalizes the usual d on forms: if ψ is a k-form and φ is an $(n-k-1)$-form, then

$$\begin{aligned}
(dT_\psi)\varphi &= (-1)^{k+1} \int_X \psi \wedge d\varphi \\
&= -\int_X d(\psi \wedge \varphi) + \int_X d\psi \wedge \varphi \\
&= \int_X d\psi \wedge \varphi \\
&= T_{d\psi}\varphi,
\end{aligned}$$

this second-last equality following from Stokes' theorem.

Clearly $d^2 = 0$ on currents, and so $(\mathcal{D}^\bullet(X), d)$ is a (co)chain complex of vector spaces (called the *complex of currents*) which then has cohomology $H^\bullet(\mathcal{D}^\bullet(X), d)$. The inclusion $\psi \mapsto T_\psi$ of smooth forms into the currents commutes with d, and therefore induces a natural map

$$H^\bullet_{dR}(X) \to H^\bullet(\mathcal{D}^\bullet(X), d).$$

It can be shown that this is an isomorphism.

We define the Hodge star $* : \mathcal{D}^k(X) \to \mathcal{D}^{n-k}(X)$ by the formula

$$(*T)(\varphi) = (-1)^{nk+n} T(*\varphi),$$

the codifferential $d^* : \mathcal{D}^k(X) \to \mathcal{D}^{k-1}(X)$ by

$$(d^*T)(\varphi) = (-1)^{n(k+1)+1}(*d*T)(\varphi),$$

and the Laplacian $\Delta : \mathcal{D}^k(X) \to \mathcal{D}^k(X)$ by

$$\Delta = dd^* + d^*d.$$

For T a current, we say that T is *closed* if $dT = 0$, *coclosed* if $d^*T = 0$, and *harmonic* if $\Delta T = 0$. If

$dS = T$ for some current S then we say T is *exact*, and if $d^*S = T$ for some S then T is *coexact*. As with forms, any current which is exact and coexact is harmonic.

The inner product of a k-current T with a k-form α is defined by

$$(T, \alpha) = (\alpha, T) = T(*\alpha).$$

Immediately from the definition of the inner product on forms, we have $(T_\alpha, \beta) = (\alpha, \beta)$. As before, we have

$$(\Delta T, \alpha) = (T, \Delta\alpha)$$

and

$$(dT, \alpha) = (T, d^*\alpha).$$

We state here some fundamental results concerning currents; proofs can be found in [3].

Theorem 4.1. *(Thm. 17') A current T is exact if and only if $T(\varphi) = 0$ for all closed forms φ of compact support.*

Theorem 4.2. *(Corollary to Thm. 23) If X is compact then each current is uniquely expressible as the sum of an exact current, a coexact current, and a harmonic form.*

Theorem 4.3. *(in section 31 of [3]) If X is compact then equation $\Delta S = T$ of k-currents has a global solution S if and only if T is orthogonal to all harmonic k-forms.*

This last theorem shows, in particular, that if $T = dS$ is an exact current on a compact Riemannian manifold then $\Delta S' = T$ for some S': if α is a harmonic k-form then it is coclosed, thus

$$(T, \alpha) = S(d * \alpha) = S(*d^*\alpha) = 0.$$

Currents arise naturally from submanifolds. Given an oriented codimension-k submanifold Y of X, the linear functional $A_c^{n-k}(X) \to \mathbb{C}$ given by integration over Y is continuous, and therefore defines a k-current on X. In particular, a divisor on a Riemann surface gives rise to a 2-current: if $D = \sum_p a_p \cdot p$ is a divisor on X, the functional $T_D : A_c^0(X) \to \mathbb{C}$ given by $T_D(f) = \sum_p a_p f(p)$ is continuous and is therefore a 2-current. (This sum is well-defined since a function of compact support will vanish on all but finitely many of the points p having $a_p \neq 0$, as these points form a discrete set.) At the same time, such a divisor gives rise to the line bundle $[D]$. Given a hermitian metric h on $[D]$, the curvature F of the Chern connection of h defines a global 2-form on X. The Poincaré-Lelong equation, discussed in the sequel, describes the relationship between the 2-currents T_F and T_D when D is effective.

4.2 The Chern connection on a line bundle and the Poincaré-Lelong equation

Let X be a Riemann surface and $L \to X$ a holomorphic line bundle equipped with a hermitian metric h (one of which always exists, as shown previously). Recall that a connection ∇ is *unitary* with respect to h if $V(h(s,t)) = h(\nabla_V s, t) + h(s, \nabla_V t)$ for all smooth vector fields V on X and all smooth sections s, t of L, and that the *Chern connection* of h is the unique unitary connection ∇ satisfying $\nabla = \nabla' + \bar{\partial}$ for some $\nabla' : \Gamma(L) \to \Gamma(T^*_{1,0} \otimes L)$. The curvature ∇^2 is a section of $\wedge^2 T^* \otimes L \otimes L^* \cong \wedge^2 T^*$, and we therefore identify it with a global 2-form F, so that $\nabla^2 s = F \otimes s$ for all sections s of L. We call this 2-form the *Chern form* of h. Recall that in the de Rham cohomology group $H^2_{dR}(X)$, we have $\left[\frac{i}{2\pi}F\right] = c_1(L)$, the Chern class of L.

First, we give a more explicit description of the Chern connection of h. Choose any smooth local trivializations s_α on open sets U_α, with transition functions $g_{\alpha\beta}$ so that $s_\alpha = g_{\alpha\beta} \cdot s_\beta$. Choosing these s_α such that $\|s_\alpha\|^2 = 1$ with respect to the metric, which we can do by possibly rescaling since we are not requiring these trivializations to be holomorphic, we have $|g_{\alpha\beta}|^2 = 1$ and so $g_{\alpha\beta}^{-1} = \overline{g_{\alpha\beta}}$. Now, we

have $\bar\partial s_\alpha = A^{0,1}_\alpha \otimes s_\alpha$ for some $(0,1)$-form $A^{0,1}_\alpha$ which is possibly nonzero as s_α may not be holomorphic. Applying $\bar\partial$ to the equation $s_\alpha = g_{\alpha\beta}\cdot s_\beta$, we have

$$\begin{aligned}\bar\partial s_\alpha &= (\bar\partial g_{\alpha\beta})\otimes s_\beta + g_{\alpha\beta}\cdot\bar\partial s_\beta\\ A^{0,1}_\alpha\otimes(g_{\alpha\beta}\cdot s_\beta) &= (\bar\partial g_{\alpha\beta})\otimes s_\beta + g_{\alpha\beta}\cdot A^{0,1}_\beta\otimes s_\beta\end{aligned}$$

which, cancelling out the factors s_β, yields

$$A^{0,1}_\alpha - A^{0,1}_\beta = g^{-1}_{\alpha\beta}\cdot\bar\partial g_{\alpha\beta}.$$

Let ∇ be the Chern connection. We then have $\nabla s_\alpha = A_\alpha\otimes s_\alpha$ for some local 1-forms A_α. Since ∇ is unitary, we have

$$\begin{aligned}d(h(s_\alpha,s_\alpha)) &= h(\nabla s_\alpha,s_\alpha)+h(s_\alpha,\nabla s_\alpha)\\ d(1) &= h(\nabla s_\alpha,s_\alpha)+\overline{h(\nabla s_\alpha,s_\alpha)}\\ 0 &= A_\alpha+\overline{A_\alpha}\end{aligned}$$

and we conclude that $i\cdot A_\alpha$ is a real 1-form. A natural candidate for A_α is $A^{0,1}_\alpha - \overline{A^{0,1}_\alpha}$. It is easy to see that $s_\alpha\mapsto(A^{0,1}_\alpha-\overline{A^{0,1}_\alpha})\otimes s_\alpha$ defines a unitary connection whose $(0,1)$-component is $\bar\partial s_\alpha = A^{0,1}_\alpha\otimes s_\alpha$, so indeed $A_\alpha = A^{0,1}_\alpha - \overline{A^{0,1}_\alpha}$.

Now suppose that L admits a nontrivial holomorphic section t, so that $L=[(t)]$. On the open set $X\setminus t^{-1}(0)$, $s:=\frac{t}{\|t\|}$ is a smooth local trivialization of unit length. As above, locally $\nabla s = A\otimes s$ for a pure-imaginary 1-form $A = A^{0,1}-\overline{A^{0,1}}$ where $\bar\partial s = A^{0,1}\otimes s$. We compute

$$\begin{aligned}\bar\partial s &= \bar\partial\frac{t}{\|t\|}\\ &= \frac{\|t\|\bar\partial t-\bar\partial\|t\|\otimes t}{\|t\|^2}\\ &= -\|t\|^{-1}\bar\partial\|t\|\otimes s\end{aligned}$$

whence $A^{0,1} = -\|t\|^{-1}\bar\partial\|t\|$ and

$$\begin{aligned}A &= \|t\|^{-1}(\partial-\bar\partial)\|t\|\\ &= (\partial-\bar\partial)\log\|t\|.\end{aligned}$$

From $\nabla s = A\otimes s$, we have

$$\begin{aligned}\nabla^2 s &= dA\otimes s - A\wedge A\otimes s\\ &= dA\otimes s\\ &= (\partial+\bar\partial)(\partial-\bar\partial)\log\|t\|\otimes s\\ &= -2\partial\bar\partial\log\|t\|\otimes s\\ &= -\partial\bar\partial\log\|t\|^2\otimes s\end{aligned}$$

which is to say that the Chern form F is equal to $-\partial\bar\partial\log\|t\|^2$ on $X\setminus t^{-1}(0)$. Then if φ is a smooth complex function on X that vanishes on a neighbourhood of $t^{-1}(0)$, we have

$$\int_X\varphi\cdot F = -\int_X\varphi\cdot\partial\bar\partial\log\|t\|^2.$$

If φ does not vanish on a neighbourhood of $t^{-1}(0)$ then this equation may not hold; the Poincaré-Lelong equation describes the correction to this equation in terms of the divisor (t):

Proposition 4.1. ***(Poincaré-Lelong equation on a Riemann surface)*** *With the notation given above, we have the equation of 2-currents*

$$T_F = 2\pi i T_{(t)} - \partial\bar{\partial} T_{\log \|t\|^2}.$$

Proof. Choose pairwise-disjoint smooth charts (U_p, z_p) around each $p \in t^{-1}(0)$ such that $\|t\|^2 = |z_p^{a_p}|^2$ on U_p, where $a_p = \mathrm{ord}_p(t)$. Since $F = -\partial\bar{\partial}\log\|t\|^2$ on $X \setminus t^{-1}(0)$, by using a partition of unity it suffices to show that, for each smooth function φ,

$$\int_{\mathbb{C}} (z_p^{-1})^*(\varphi \cdot F) = 2\pi i a_p((z_p^{-1})^*\varphi)(p) - \int_{\mathbb{C}} \log|z^{a_p}|^2 \cdot \partial\bar{\partial}(z_p^{-1})^*\varphi.$$

The form $\varphi \cdot F$ is closed, and so $(z_p^{-1})^*(\varphi \cdot F)$ is exact by the Poincaré lemma and the integral on the left-hand side is zero by Stokes' theorem. It then suffices to prove

$$2\pi i a_p (z_p^{-1})^*\varphi(p) = \int_{\mathbb{C}} \log|z^{a_p}|^2 \cdot \partial\bar{\partial}(z_p^{-1})^*\varphi$$

or, equivalently,

$$\phi(0) = \int_{\mathbb{C}} \log|z|^2 \cdot dd^c\phi$$

where $\phi = (z_p^{-1})^*\varphi$. (Recall that $d^c = (4\pi i)^{-1}(\partial - \bar{\partial})$, and $dd^c = \frac{-1}{2\pi i}\partial\bar{\partial}$.) We begin by rewriting the right-hand side of this equation as a limit,

$$\begin{aligned}
\int_{\mathbb{C}} \log|z|^2 \cdot dd^c\phi &= \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} \log|z|^2 \cdot dd^c\phi \\
&= \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} d(\log|z|^2 \cdot d^c\phi) - \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} d\log|z|^2 \wedge d^c\phi \\
&= -\lim_{\varepsilon\to 0} \int_{|z|=\varepsilon} \log|z|^2 \cdot d^c\phi - \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} d\log|z|^2 \wedge d^c\phi,
\end{aligned}$$

this last equality coming from Stokes' theorem. But we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{|z|=\varepsilon} \log|z|^2 \cdot d^c\phi &= \lim_{\varepsilon\to 0} 2\log\varepsilon \int_{|z|=\varepsilon} d^c\phi \\
&= 0
\end{aligned}$$

as it can be shown [9] that $\int_{|z|=\varepsilon} d^c\phi$ is $O(\varepsilon)$. Hence

$$\begin{aligned}
\int_{\mathbb{C}} \log|z|^2 \cdot dd^c\phi &= -\lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} d\log|z|^2 \wedge d^c\phi \\
&= \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} d^c\log|z|^2 \wedge d\phi \\
&= \lim_{\varepsilon\to 0} \int_{|z|\geq\varepsilon} \phi \cdot dd^c\log|z|^2 + \lim_{\varepsilon\to 0} \int_{|z|=\varepsilon} d^c\log|z|^2 \cdot \phi
\end{aligned}$$

again invoking Stokes' theorem. We have also used the fact that $d\log|z|^2 \wedge d^c\phi = -d^c\log|z|^2 \wedge d\phi$, which follows from the definition of d^c and the fact that $\partial\log|z|^2 \wedge \partial\phi = \bar{\partial}\log|z|^2 \wedge \bar{\partial}\phi = 0$. We observe that

$$\begin{aligned}
\partial\bar{\partial}\log|z|^2 &= \partial\bar{\partial}\log z - \bar{\partial}\partial\log\bar{z} \\
&= 0 - 0
\end{aligned}$$

on $|z| \geq \varepsilon$, hence

$$\int_{\mathbb{C}} \log|z|^2 \cdot dd^c\phi = \lim_{\varepsilon\to 0} \int_{|z|=\varepsilon} d^c\log|z|^2 \cdot \phi.$$

Now, Cauchy's integral formula for smooth functions tells us that

$$\phi(0) = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \phi \cdot \frac{1}{z}dz - \frac{1}{2\pi i}\int_{|z|\leq\varepsilon} \frac{1}{z}\bar{\partial}\phi \wedge dz$$

for each $\varepsilon > 0$; taking $\varepsilon \to 0$, the rightmost integral vanishes and we have

$$\phi(0) = \lim_{\varepsilon\to 0}\int_{|z|=\varepsilon} \phi \cdot \frac{1}{2\pi i}\frac{1}{z}dz.$$

But

$$\begin{aligned} d^c \log|z|^2 &= \frac{1}{4\pi i}\left((\partial - \bar{\partial})\log z - (\partial - \bar{\partial})\log \bar{z}\right) \\ &= \frac{1}{4\pi i}\left(\frac{1}{z}dz - \frac{1}{\bar{z}}d\bar{z}\right) \\ &= \frac{1}{2\pi(x^2+y^2)}\left(xdy - ydx\right), \end{aligned}$$

whereas

$$\begin{aligned} \frac{1}{2\pi i}\frac{1}{z}dz &= \frac{1}{2\pi(x^2+y^2)}(x-iy)(dx+idy) \\ &= \frac{1}{2\pi(x^2+y^2)}\left(xdy - ydx\right) \end{aligned}$$

since $xdx = -ydx$ on the circle $|z| = \varepsilon$, and we conclude that

$$\phi(0) = \int_{\mathbb{C}} \log|z|^2 \cdot dd^c\phi$$

as claimed. □

5 Gerbes

In what follows, assume all manifolds to be oriented.

5.1 Definition of a gerbe

For X a (real) manifold, the vanishing of the sheaf cohomology groups $H^p(X, \mathcal{A})$ for $p > 0$ gives isomorphisms $H^p(X, \mathcal{A}^*) \cong H^{p+1}(X, \mathbb{Z})$ from the long exact sequence associated to the exponential sequence $\underline{\mathbb{Z}} \to \mathcal{A} \to \mathcal{A}^*$. In particular we have the first Chern class map $c_1 : H^1(X, \mathcal{A}^*) \cong H^2(X, \mathbb{Z})$ which gives us a way to realize a cohomology class in $H^2(X, \mathbb{Z})$ as a geometric object - namely, as a complex line bundle (up to isomorphism). The notion of a *gerbe* can be thought of as a way of using the isomorphism $H^2(X, \mathcal{A}^*) \cong H^3(X, \mathbb{Z})$ to analogously realize elements of $H^3(X, \mathbb{Z})$ as geometric objects in some sense.

Explicitly, we determine a gerbe on X by an open cover $\{U_\alpha\}$ of X together with smooth $\mathbb{C}^*$-valued functions $g_{\alpha\beta\gamma}$ on each $U_{\alpha\beta\gamma}$ satisfying

$$g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = g^{-1}_{\alpha\gamma\beta} = g^{-1}_{\gamma\beta\alpha}$$

and

$$(\delta g)_{\alpha\beta\gamma\delta} = g_{\beta\gamma\delta} \cdot g^{-1}_{\alpha\gamma\delta} \cdot g_{\alpha\beta\delta} \cdot g^{-1}_{\alpha\beta\gamma} = 1$$

on each quadruple intersection $U_{\alpha\beta\gamma\delta}$.

Recall that if L is a line bundle given by transition functions $g_{\alpha\beta}$, a trivialization of L over an open set U is given by nonvanishing functions f_α on $U \cap U_\alpha$ such that $f_\alpha f_\beta^{-1} = g_{\alpha\beta}$ on $U \cap U_{\alpha\beta}$. In other

words, a trivialization over U is a representation of the cocycle $\{g_{\alpha\beta}\}$, restricted to U, as a coboundary δf. Since a gerbe $\mathcal{G}$ is given by a cocycle $\{g_{\alpha\beta\gamma}\}$, we can analogously define a *trivialization* of $\mathcal{G}$ over U to be a choice of nonvanishing functions $f_{\alpha\beta}$ on $U_{\alpha\beta}$ such that $f_{\beta\alpha} = f_{\alpha\beta}^{-1}$ and

$$g_{\alpha\beta\gamma} = f_{\alpha\beta} \cdot f_{\beta\gamma} \cdot f_{\gamma\alpha}.$$

Given two trivializations $f_{\alpha\beta}, f'_{\alpha\beta}$ of $\mathcal{G}$ over U_0, their difference $h_{\alpha\beta} = f'_{\alpha\beta}/f_{\alpha\beta}$ satisfies $h_{\beta\alpha} = h_{\alpha\beta}^{-1}$ and $h_{\alpha\beta} \cdot h_{\beta\gamma} \cdot h_{\gamma\alpha} = 1$. In other words, the difference between two trivializations of $\mathcal{G}$ over U is a line bundle over U, just as the quotient of two trivializations of a line bundle is a $\mathbb{C}^*$-valued function. As pointed out earlier, over each of the open sets U_i of the cover there is a trivialization $f^i_{\alpha\beta}$ of $\mathcal{G}$, given by setting $f^i_{\alpha\beta} = g_{i\alpha\beta}$ for $\alpha, \beta \neq i$, and $f^i_{i\alpha} = f^i_{\alpha i} = 1$. Over U_{ij}, we then have functions $h^{ij}_{\alpha\beta} = f^i_{\alpha\beta}/f^j_{\alpha\beta}$ defining a line bundle $L_{ij} = [h^{ij}_{\alpha\beta}]$ over U_{ij}. Clearly L_{ij} is dual to L_{ji}, and $L_{ij} \otimes L_{jk} \otimes L_{ki}$ (tensor product) is a trivial line bundle. With this observation, following [2], we shall describe a gerbe in terms of line bundles over double intersections of an open cover.

Definition 22. *A (locally trivialized) gerbe $\mathcal{G}(I, L, \theta)$ over a manifold X is defined by the following data:*

- *An open cover $\{U_i\}_{i\in I}$ of X*
- *For each pair $i \neq j$, a complex line bundle L_{ij} over U_{ij} such that $L_{ji} = L^*_{ij}$*
- *For each ordered triple i, j, k of distinct indices, a trivialization θ_{ijk} of $(\delta L)_{ijk} := L_{ij} \otimes L_{jk} \otimes L_{ki}$ over U_{ijk} such that $\theta_{\sigma(i)\sigma(j)\sigma(k)} = \theta_{ijk}^{\mathrm{sgn}(\sigma)}$ for each permutation $\sigma \in S_3$, and $\delta\theta = 1$.*

The duality condition $L_{ij} = L_{ji}^{-1}$ implies that the line bundles

$$(\delta^2 L)_{ijkl} = (L_{jk} \otimes L_{kl} \otimes L_{lj}) \otimes (L_{ik} \otimes L_{kl} \otimes L_{li})^{-1} \otimes (L_{ij} \otimes L_{jl} \otimes L_{li}) \otimes (L_{ij} \otimes L_{jk} \otimes L_{ki})^{-1}$$

are canonically trivial; by "$\delta\theta = 1$" we mean that $(\delta\theta)_{ijkl} = \theta_{jkl} \otimes \theta_{ikl}^{-1} \otimes \theta_{ijl} \otimes \theta_{ijk}^{-1}$ is the canonical trivialization 1 of this line bundle.

Note that we do not require the line bundles L_{ij} to be trivializable. By refining a cover and restricting the trivializations θ_{ijk}, we obtain a *refinement* of a local trivialization for a gerbe; see [2] for details. Given a locally trivialized gerbe $\mathcal{G}$, we can then refine it sufficiently so that the line bundles L_{ij} are indeed all trivializable. Having done so, take trivializations λ_{ij} for L_{ij}. Writing

$$\theta_{ijk} = t_{ijk}\lambda_{ij} \otimes \lambda_{jk} \otimes \lambda_{ki}$$

we obtain a cocycle $t \in Z^2(X, \mathcal{A}^*)$, hence a cohomology class in $H^2(X, \mathcal{A}^*)$. If $\tilde{\lambda}$ is another choice of trivializations, then $\tilde{\lambda}_{ij} = f_{ij}\lambda_{ij}$ for some smooth functions f_{ij}, and from this we see that t and $\tilde{t}$ differ by δf and therefore the cohomology class is independent of the choice of λ.

There is a natural notion of equivalence of locally trivialized gerbes.

Definition 23. *Two locally trivialized gerbes $\mathcal{G}(I, L, \theta)$ and $\mathcal{H}(J, X, \eta)$ are* equivalent *if there is a common refinement $\{V_a\}_{a\in A}$ of their covers, with isomorphisms $L^b_a \cong X^b_a$ over V_{ab} which take θ_{abc} to η_{abc}.*

Such an equivalence, on passing to the common refinement, induces an equivalence of the cocycles associated to these gerbes, and so equivalent gerbes give the same cohomology class. Conversely, given a cocycle $\{g_{ijk}\} \in Z^2(X, \mathcal{A}^*)$, we can construct a gerbe $\mathcal{G}$ by taking the line bundles L_{ij} to be trivial, with trivializations λ_{ij}, and define a gerbe by

$$\theta_{ijk} = g_{ijk}\lambda_{ij} \otimes \lambda_{jk} \otimes \lambda_{ki};$$

we have $\delta\theta = 1$ since $\delta g = 1$, and this gerbe gives rise, as above, to the cohomology class of $\{g_{ijk}\}$. This correspondence identifies the set of equivalence classes of gerbes with the cohomology group $H^2(X, \mathcal{A}^*)$. The operation of *tensor product* on gerbes is given by taking a common refinement of two local trivializations, then tensoring the corresponding pairs of line bundles. (This is just as how transition functions

for the tensor product of two line bundles are obtainable by taking a common refinement of their local trivializations and multiplying together corresponding transition functions.) The group law on the set of equivalence classes of gerbes is, as with the Picard group, induced by the tensor product. We denote by $[\mathcal{G}]$ the class in $H^2(X, \mathcal{A}^*)$ associated to a gerbe $\mathcal{G}$.

Definition 24. *A locally trivialized gerbe $\mathcal{G}(I, L, \theta)$ is* globally trivialized *if there exist trivializations λ_{ij} for L_{ij} such that $\delta\lambda = \theta$,* i.e.

$$\theta_{ijk} = \lambda_{ij} \otimes \lambda_{jk} \otimes \lambda_{ki}.$$

From the above correspondence, each such gerbe has $[\mathcal{G}] = [1_{ijk}] = 0$, and conversely $[\mathcal{G}] = 1$ if and only if $\mathcal{G}$ is equivalent to a globally trivialized gerbe.

Definition 25. *A gerbe is trivial if it is equivalent to a globally trivialized gerbe; equivalently, if its class $[\mathcal{G}]$ is $0 \in H^2(X, \mathcal{A}^*)$.*

Recall that the identification $\mathrm{Pic}^\infty(X) = H^1(X, \mathcal{A}^*)$, together with the second connecting homomorphism of the exponential sequence $\underline{\mathbb{Z}} \to \mathcal{A} \to \mathcal{A}^*$, gave us the first Chern class map $c_1 : \mathrm{Pic}^\infty(X) \xrightarrow{\cong} H^2(X, \mathbb{Z})$. Moving up one degree, we have another connecting homomorphism $\delta_* : H^2(X, \mathcal{A}^*) \to H^3(X, \mathbb{Z})$. Since the group of isomorphism classes of gerbes is naturally identified with $H^2(X, \mathcal{A}^*)$, we can then define the *Chern class* of a gerbe $\mathcal{G}$ to be $c(\mathcal{G}) := \delta_*[\mathcal{G}]$.

5.2 0-connections and 1-connections

Recall that on a Riemann surface, the Chern class of a line bundle is, up to a factor of $i/2\pi$, cohomologous to any curvature form on that line bundle, and that the Chern class of the line bundle associated to a divisor was the Poincaré dual of that divisor. We wish to formulate analogues of these results for gerbes on a 3-manifold. For this, we will need to define the notion of a curvature 3-form on a gerbe.

Recall that a connection on a line bundle with transition functions g_{ij} is given by a collection of local 1-forms A_i such that $\delta A - d \log g = 0$.

Definition 26. *A 0-connection on a locally trivialized gerbe $\mathcal{G}(I, L, \theta)$ is given by a connection ∇_{ij} on each line bundle $L_{ij} \to U_{ij}$ such that $(\nabla_{ij} + \nabla_{ji})1_{L_{ij} \otimes L_{ji}} = 0$ over U_{ij}, and over U_{ijk} we have $(\delta\nabla)_{ijk}\theta_{ijk} = 0$ where $(\delta\nabla)_{ijk} = \nabla_{ij} + \nabla_{jk} + \nabla_{ki}$. (Recall that the symbol $+$ is used to denote the induced connection, as defined previously.)*

Note that the condition $(\delta\nabla)\theta = 0$ does *not* imply that $(\delta\nabla)_{ijk} = 0$ (the zero map on sections is not a connection!) but it does describe a kind of local triviality. Since θ_{ijk} is a trivialization of $(\delta L)_{ijk}$, an arbitrary section of $(\delta L)_{ijk}$ is of the form $s = f \cdot \theta_{ijk}$, and the Leibniz rule gives $(\delta\nabla)_{ijk}s = df \otimes \theta_{ijk}$. Hence, if we identify $(\delta L)_{ijk}$ with the trivial line bundle via θ_{ijk}, the condition imposed is that the induced connection $(\delta\nabla)_{ijk}$ is thereby identified with the de Rham derivative. Similarly, $\nabla_{ij} + \nabla_{ji}$ is the de Rham derivative on the canonically trivial line bundle $L_{ij} \otimes L_{ji}$.

Proposition 5.1. *Every gerbe, expressed over a given cover, admits a 0-connection expressed over that same cover.*

Proof. On each L_{ij}, choose an arbitrary connection ∇_{ij}. We then have

$$(\delta\nabla)_{ijk}\theta_{ijk} = \eta_{ijk} \otimes \theta_{ijk}$$

for some 1-forms η_{ijk}. Since $(\delta\theta)_{ijkl}$ is the canonical trivialization 1_{ijkl} of $(\delta^2 L)_{ijkl}$, it follows that

$$(\delta^2\nabla)_{ijkl}1_{ijkl} = (\delta\eta)_{ijkl} \otimes 1_{ijkl},$$

but taking local trivializations we see that $(\delta^2\nabla)_{ijkl}1_{ijkl} = 0$, so η is a cocycle in $Z^2(X, \mathcal{A}^1)$. From the proof that $H^2(X, \mathcal{A}^1) = 0$, there exist 1-forms $\zeta_{ij} \in A^1(U_{ij})$ with $\delta\zeta = \eta$. The connections $\nabla'_{ij} := \nabla_{ij} - \zeta_{ij}$ then satisfy $(\delta\nabla')_{ijk}\theta_{ijk} = 0$ and so ∇' is a 0-connection on the gerbe. $\square$

Recall that $F(\nabla + \nabla') = F(\nabla) + F(\nabla')$ for connections ∇, ∇' on a line bundle. Let ∇ be a 0-connection on a gerbe, and let $F_{ij} = F(\nabla_{ij})$. Since $(\delta\nabla)_{ijk}$ has zero curvature, $F = \{F_{ij}\}$ is a cocycle, *i.e.* an element of $Z^2(X, \mathcal{A}^2)$. From the proof that $H^2(X, \mathcal{A}^2) = 0$, we see that there exist local 2-forms $\beta_i \in A^2(U_i)$ such that $\delta\beta = F$.

Definition 27. *With the above notation, a 1-connection compatible with ∇ is any choice of local 2-forms $\beta_i \in A^2(U_i)$ satisfying $\delta\beta = F$.*

We will refer to a gerbe with a specified 0-connection and compatible 1-connection simply as a *gerbe with 1-connection.* As noted, a 1-connection compatible with a given 0-connection always exists.

Suppose L is a line bundle with transition functions f_{ij} and equipped with a connection ∇. Recall that ∇ was then given by local 1-forms A_i with $\delta A = d \log f$, and the curvature of ∇ is $F = dA$, a closed global 2-form since $d(A_j - A_i) = d^2 \log f = 0$. Analogously, if β is a 1-connection on a gerbe, we have $\delta\beta = F$ and $dF = 0$, so that $d\beta$ is a closed (but perhaps not exact) global 3-form.

Definition 28. *The curvature of a gerbe with 1-connection β is the global 3-form $\Omega = d\beta$.*

As in the case of a line bundle, the cohomology class of the curvature of a gerbe is independent of the choice of trivialization, 0-connection, and 1-connection:

Proposition 5.2. *For any curvature 3-form Ω on a gerbe $\mathcal{G}$ with 1-connection, in $H^3_{dR}(X)$ we have*

$$\left[\frac{i}{2\pi}\Omega\right] = c(\mathcal{G}).$$

A proof of this result goes very similarly to the previous one given for its line-bundle analogue.

Given a divisor D on a compact Riemann surface, we constructed a line bundle $[D]$ whose Chern class was Poincaré-dual to D. We demonstrate a similar construction for taking a finite linear combination R of points on a closed oriented 3-manifold and constructing a gerbe $[R]$ whose Chern class is Poincaré-dual to R.

Suppose $R = \sum_p a_p p$ is a finite $\mathbb{Z}$-linear combination of points[2] on an oriented 3-manifold X. Choose an open-ball neighbourhood U_p about p for $a_p \neq 0$ such that $U_{pq} = \emptyset$ for $p \neq q$, and let $U_0 = X \setminus R$ where, by abuse of notation, $R = \{p : a_p \neq 0\}$. The only nonempty pairwise intersections of this open cover are of the form $U_{0p} \cong S^2 \times \mathbb{R}$, and all triple intersections are empty, so to construct a gerbe it suffices to define a line bundle over each U_{0p}. We have $\mathrm{Pic}^\infty(U_{0p}) = H^2(U_{0p}, \mathbb{Z}) = \mathbb{Z}$ and we choose a line bundle L_{0p} of degree (Chern class) a_p on U_{0p}; this line bundle is unique up to isomorphism, and so these give us a gerbe $\mathcal{G}(R)$ which is uniquely determined up to equivalence.

Proposition 5.3. *The Chern class of $\mathcal{G}(R)$ is Poincaré-dual to the homology class of R,* i.e.

$$c(\mathcal{G}(R)) = \sum_p a_p \in H^3(X, \mathbb{Z}).$$

Again, the proof of this result is very similar to that of its line-bundle counterpart.

5.3 Construction of a 1-connection on $\mathcal{G}(R)$

We will invoke here several of the results from [3] previously listed in section 4.1. Assume that X is a compact oriented Riemannian 3-manifold, let $R = \sum_p a_p \cdot p$ be a finite $\mathbb{Z}$-linear combination of points on X, and fix a representative 3-form $\omega \in c(\mathcal{G}(R))$. We describe here the construction of a 1-connection on $\mathcal{G}(R)$ whose curvature is $\Omega = -2\pi i\omega$. Choose an open cover $\{U_0, U_p : a_p \neq 0\}$ as before.

We know that $[\omega]$ is Poincaré-dual to $[R]$, which is to say that $T_\omega(\varphi) = T_R(\varphi)$ for all constants φ. From Theorem 17' of [3], we conclude that $T_\omega - T_R = dS$ for some S. If α is a harmonic function then

[2] We refrain from using the term "divisor" to describe such an object in the case of a 3-manifold, as this usually refers to a cocycle of the sheaf $\mathcal{M}^*/\mathcal{O}^*$ on a complex manifold in this context.

$(dS, \alpha) = S(d * \alpha) = S(*d^* \alpha) = 0$, and by another of the results from [3] we conclude that there exists a 3-current γ with

$$\frac{i}{2\pi}\Delta\gamma = T_\omega - T_R.$$

Moreover this γ is given by a smooth 3-form on U_0, which we also denote by γ.

This equation is similar to the Poincaré-Lelong equation: since $\Delta = dd^*$ on 3-forms (X being 3-dimensional, all 3-forms are closed), the above formula rearranges to $T_\Omega = -2\pi i T_R + dd^* T_\gamma$.

Without loss of generality, we assume that the neighbourhoods U_p are sufficiently small that there exist smooth local 3-forms σ_p such that

$$\frac{i}{2\pi}\Delta T_{\sigma_p} = T_R$$

over U_p. (Take $\sigma_0 = 0$ on U_0.) Define the smooth 2-form

$$\beta_i = d^*(\gamma + \sigma_i)$$

on U_i, for $i = 0$ or $i = p$. We then have

$$d\beta_i = \Delta\gamma + \Delta\sigma_i = \Omega$$

over U_i. The goal is then to construct a 0-connection with line-bundle curvatures $F_{ij} := \beta_i - \beta_j$. Note that

$$F_{ij} = \beta_i - \beta_j = d^*(\sigma_i - \sigma_j)$$

so that $\beta_i - \beta_j$ is coclosed, and therefore harmonic since it is automatically closed.

Proposition 5.4. *There exists a 0-connection for $\mathcal{G}(R)$ whose line-bundle curvatures are the harmonic 2-forms F_{ij} defined above, hence the forms β_i give a 1-connection on $\mathcal{G}(R)$ with curvature Ω.*

Proof. Start by choosing an arbitrary 0-connection ∇ over the given cover, and let f_{ij} denote its line-bundle curvatures. We define a 1-connection b by choosing a partition of unity ρ_k subordinate to the given cover and setting

$$b_i = \sum_{k \neq i} \rho_k f_{ik}.$$

Indeed, on U_{0p} we have $\rho_q = 0$ for $q \neq p$ (since $U_{pq} = \emptyset$) and so

$$b_0 - b_p = \rho_p f_{0p} - \rho_0 f_{p0} = f_{0p}$$

as $f_{p0} = -f_{0p}$ and $\rho_0 + \rho_p = 1$ on U_{0p}. The curvature 3-form of this 1-connection is

$$o := db_i = \sum_{k \neq i} d\rho_k \wedge f_{ik}.$$

On U_0 we have $o = \sum_{a_p \neq 0} d\rho_p \wedge f_{0p}$ and on U_p, since $\rho_q = 0$ for $q \neq p$, we have $o = d\rho_0 \wedge f_{p0}$.

Since Ω and o have the same de Rham class (namely $-2\pi i c(\mathcal{G}(R))$), there exists a global 2-form B with $\Omega = o + dB$, and we can take $b_i + B$ as a different 1-connection for ∇. As U_p is contractible and

$$d(\beta_p - b_p - B) = \Omega - o - d\beta = 0$$

on U_p, there exist local 1-forms ψ_p with $d\psi_p = \beta_p - b_p - B$. Next, we wish to show that there is a 1-form ψ_0 on $U_0 = X \setminus R$ with $d\psi_0 = \beta_0 - b_0 - B$, *i.e.* $[\beta_0 - b_0 - B] = 0$ in $H^2_{dR}(X \setminus R)$.

First, suppose that R, considered as a 0-cycle, is the boundary of a 1-chain. As $c(\mathcal{G}(R))$ is the Poincaré dual of $[R] = 0$, there exist global 2-forms B_Ω, B_o with $\Omega = dB_\Omega, o = dB_o$, hence

$$[\beta_0 - B_\Omega] = [b_o - B_o] \in H^2_{dR}(X \setminus R).$$

We can then specifically choose our B above to be equal to $B_\Omega - B_o$, giving $[\beta_0 - b_0 - B] = 0$ in $H^2_{dR}(X \setminus R)$ as desired.

Now suppose that R is not the boundary of a 1-chain. The pair (X, A) give us the long exact sequence

$$\cdots \to 0 = H_1(R) \to H_1(X) \to H_1(X, R) \xrightarrow{\partial_*} H_0(R) \to H_0(X) \to H_0(X, R) \to 0,$$

say with complex coefficients. The connecting homomorphism ∂_* is the zero map since R is not a boundary, hence the map $H_1(X) \to H_1(X, R)$ is an isomorphism, and through de Rham's theorem this means that the map $H^2_{dR}(X) \to H^2_{dR}(X \setminus R)$ induced by restriction of forms is an isomorphism. Hence there is a closed globally defined 2-form B' such that

$$[\beta_0 - b_0 - B] = [B']$$

in $H^2_{dR}(X \setminus R)$. Noting that we can vary B by any closed 2-form and still retain its desired properties, we then replace B by $B + B'$, giving $[\beta_0 - b_0 - B] = 0$ in $H^2_{dR}(X \setminus R)$ as desired.

Set $\nabla'_{ij} = \nabla_{ij} + \psi_i - \psi_j$; this is another 0-connection, and

$$\beta_i - \beta_j = b_i - b_j + d\psi_i - d\psi_j = f_{ij} + d(\psi_i - \psi_j) = F(\nabla'_{ij})$$

but on the other hand $\beta_i - \beta_j = F_{ij}$, and so we have our desired 0-connection ∇'. □

6 Additional remarks

We focused primarily on holomorphic line bundles on Riemann surfaces, and then on smooth gerbes on 3-manifolds. This was done for the sake of simplicity; it is clear that by replacing the sheaf $\mathcal{A}$ by $\mathcal{O}$ we have a notion of holomorphic gerbe on a complex manifold, with equivalence class in $H^2(X, \mathcal{O}^*)$, although again the Chern class map $H^2(X, \mathcal{O}^*) \to H^3(X, \mathbb{Z})$ may not be an isomorphism in this case.

A complex line bundle can be thought of as a cocycle in $Z^1(X, \mathbb{Z}^*)$ or as a sheaf of sections (as a module over the sheaf of smooth complex functions on the base manifold). Similarly, a gerbe can be described as a cocycle in $Z^2(X, \mathbb{Z}^*)$ or defined as a kind of sheaf of *categories*; particularly, as a sheaf of groupoids satisfying some certain nice properties. For more details, see Brylinski [1] where the kind of gerbe we discuss here is a gerbe with *band* $\mathcal{A}^*$ (in his notation, $\underline{\mathbb{C}^*}$).

A line bundle was described locally by smooth $\mathbb{C}^*$-valued functions, and in turn a gerbe was described locally by line bundles. There are higher-order objects, called *n-gerbes*, described locally by $(n-1)$-gerbes. The gerbes described here are 1-gerbes, in this terminology, and line bundles may be considered 0-gerbes. As may be expected, the equivalence classes of an n-gerbe correspond naturally to classes in $H^{n+1}(X, \mathcal{A}^*)$. Line bundles admit connections, which are given by local 1-forms, and these give rise to a global curvature 2-form; similarly we have seen that there is a notion of 1-connection on a gerbe, each of which has a global curvature 3-form, and in both cases the curvature is determined up to an exact form by the Chern class of the line bundle or gerbe. On an n-gerbe there exist n-connections, constructed from lower-order connections just as a 1-connection was constructed from 0-connections, and these have curvature $(n+2)$-forms. Of course for any $n \geq 0$ we have connecting isomorphisms $\delta_* : H^{n+1}(X, \mathcal{A}^*) \to H^{n+2}(X, \mathbb{Z})$ and so we have a Chern class for an n-gerbe, and again this is equal, up to a factor of $i/2\pi$, to the de Rham class of any curvature form.

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